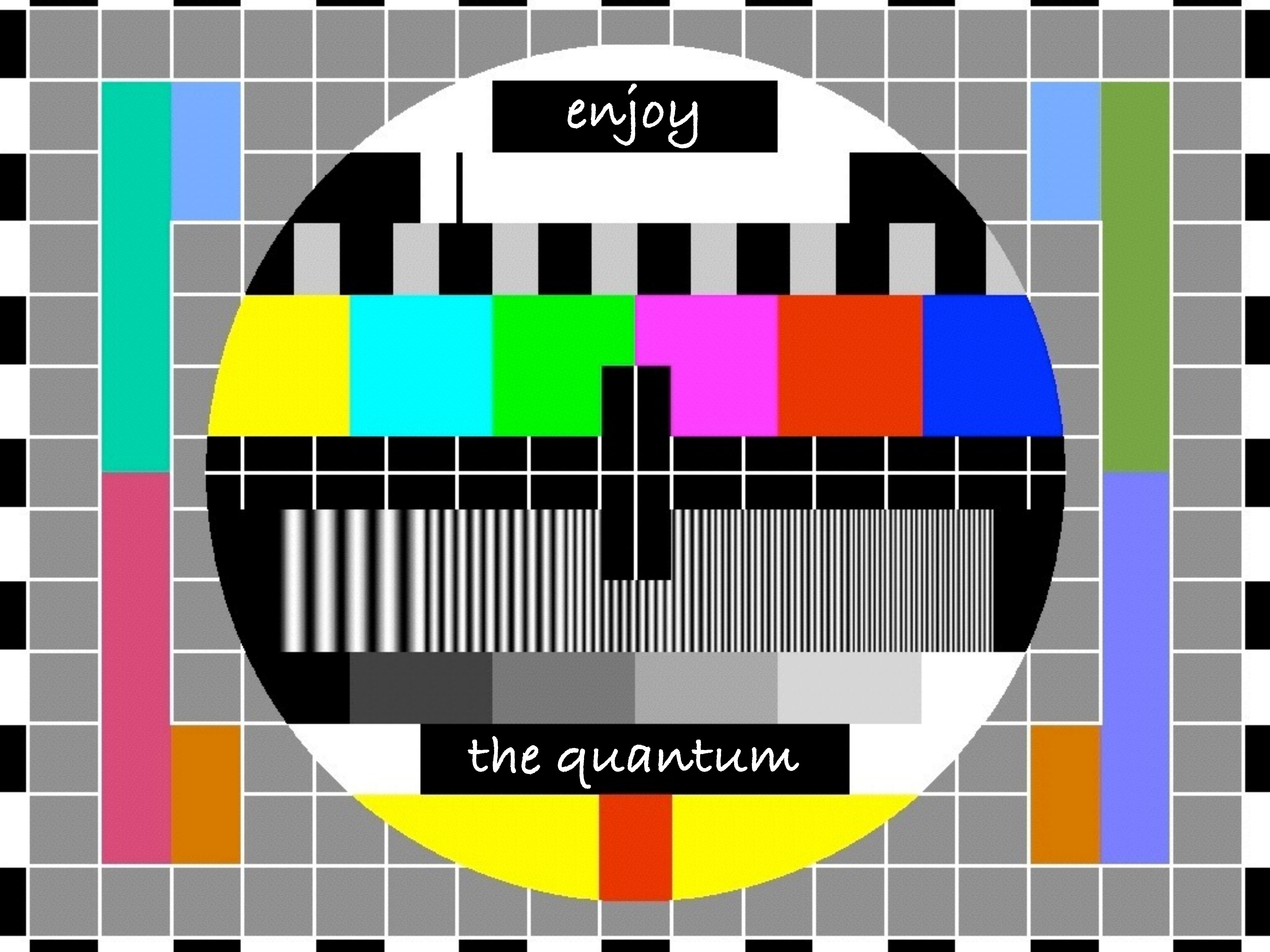
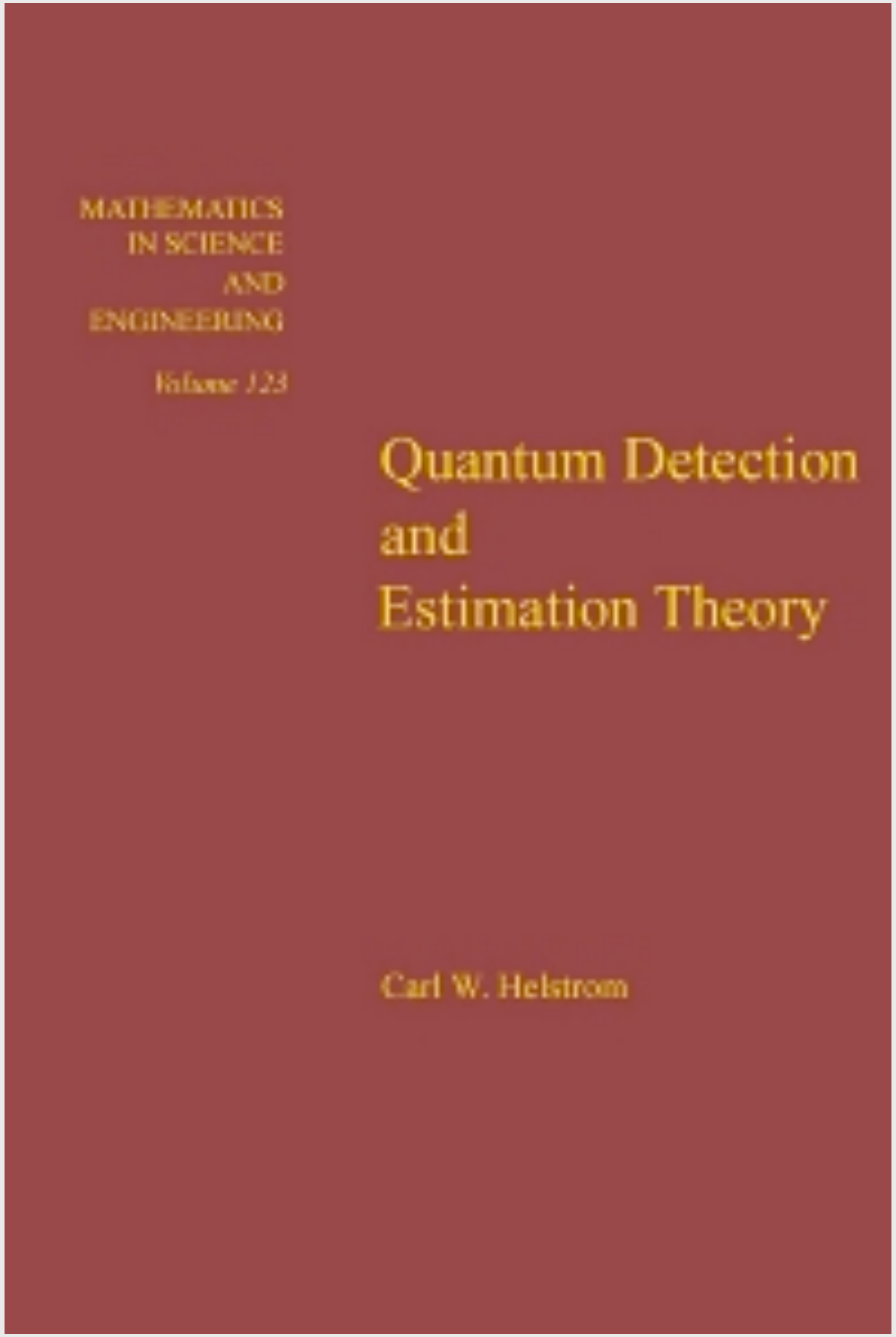
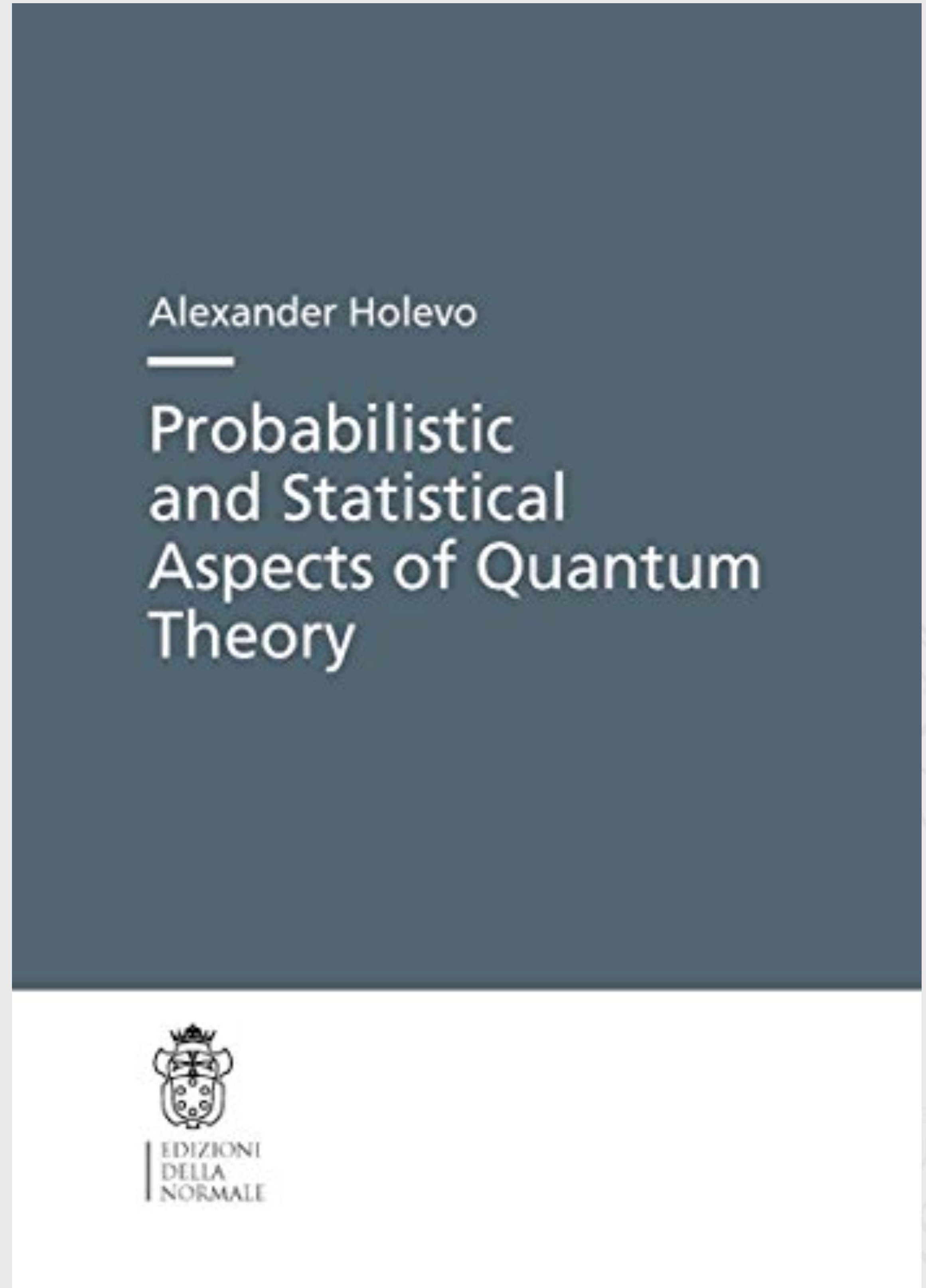

enjoy
the quantum


MATHEMATICS
IN SCIENCE
AND
ENGINEERING
Volume 123
Quantum Detection
and
Estimation Theory
Carl W. Helstrom


Alexander Holevo
Probabilistic
and Statistical
Aspects of Quantum
Theory
EDIZIONI
DELLA
NORMALE

▲ 1 Observable quantities are associated to POVMs, i.e. decompositions of identity $\sum_x \Pi_x = \mathbb{I}$ in terms of positive $\Pi_x \geq 0$ operators.

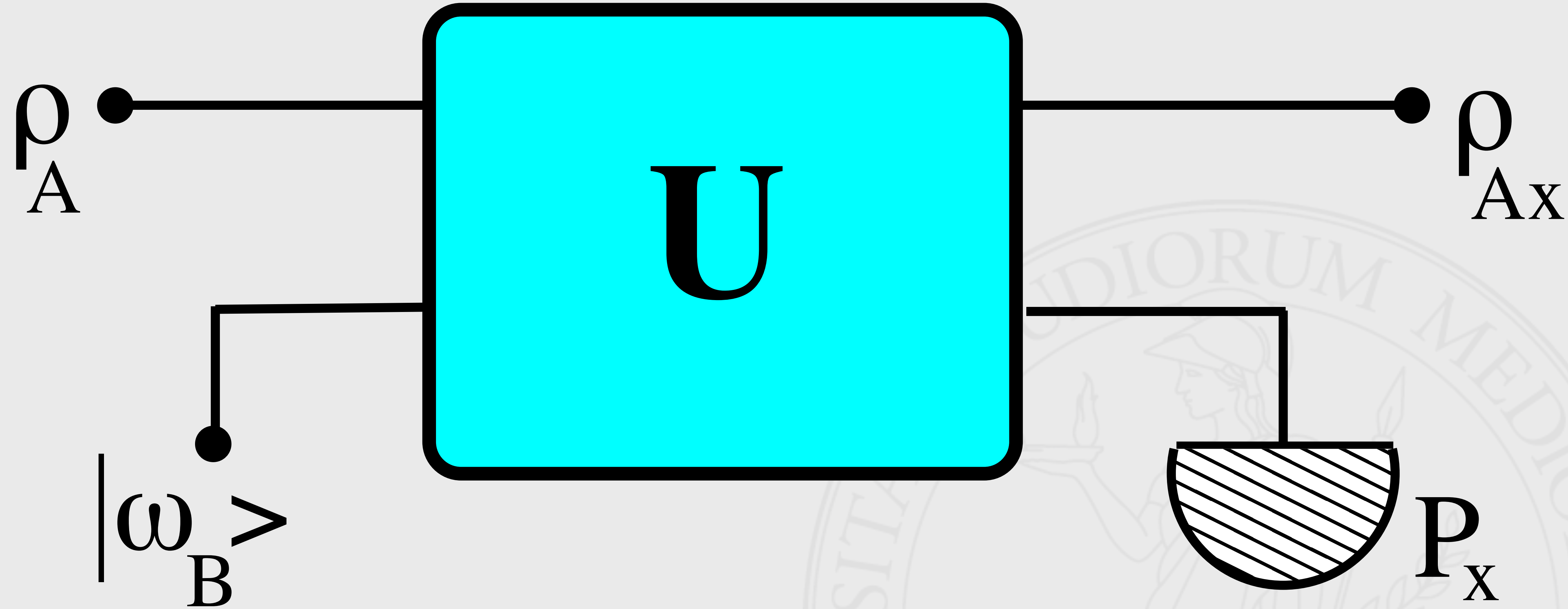
▲ 2 The elements of a POVM are positive operators expressible as $\Pi_x = M_x^\dagger M_x$ where the detection operators M_x are generic operators with the only constraint $\sum_x M_x^\dagger M_x = \mathbb{I}$.

▲ 3 A measurement yields one of the alternatives corresponding to an element of the POVM. eigenvalues x as possible outcomes.

▲ 4 The probability that a particular outcome is found as the measurement result is (Born rule) $p_x = \text{Tr}\left[M_x \varrho M_x^\dagger\right] = \text{Tr}\left[\varrho M_x^\dagger M_x\right] = \text{Tr}\left[\varrho \Pi_x\right]$.

▲ 5 The state after the measurement (reduction rule) is $\varrho_x = \frac{1}{p_x} M_x \varrho M_x^\dagger$ if the outcome is x.

▲ 6 If we perform a measurement but we do not record the results, the post-measurement state is given by $\widetilde{\varrho} = \sum_x p_x \, \varrho_x = \sum_x M_x \varrho M_x^\dagger$.


ρ
A
U
ρ
Ax
|ω
B
>
P
x

Journal of Modern Optics
Vol. 57, No. 3, 10 February 2010, 160–180

TUTORIAL REVIEW

Discrimination of quantum states

János A. Bergou*

Department of Physics and Astronomy, Hunter College of the City University of New York, 695 Park Avenue, New York, NY 10021, USA

(*Received 1 August 2009; final version received 2 November 2009*)

In quantum information processing and quantum computing protocols the carrier of information is a quantum system and information is encoded in the state of a quantum system. After processing the information it has to be read out what is equivalent to determining the final state of the system. When the possible final states are not orthogonal this is a highly nontrivial task that constitutes the general area of what is known as quantum state discrimination. It consists in finding measurement schemes that, according to some figure of merit, will determine the state of the system. Optimized measurement schemes often lead to generalized measurements (Positive Operator Valued Measures [POVMs]). In this tutorial review we illustrate the power of the POVM concept on examples relevant to applications in quantum cryptography. In order to keep the flow of the presentation we give a brief introduction to the quantum theory of measurements, including generalized measurements (POVMs), in the Appendices.

Keywords: quantum measurements; measurement optimization; state discrimination

J.A. Bergou, U. Herzog, M. Hillery, Discrimination of Quantum States, Lect. Notes Phys. **649**, 417–465 (2004)
http://www.springerlink.com/

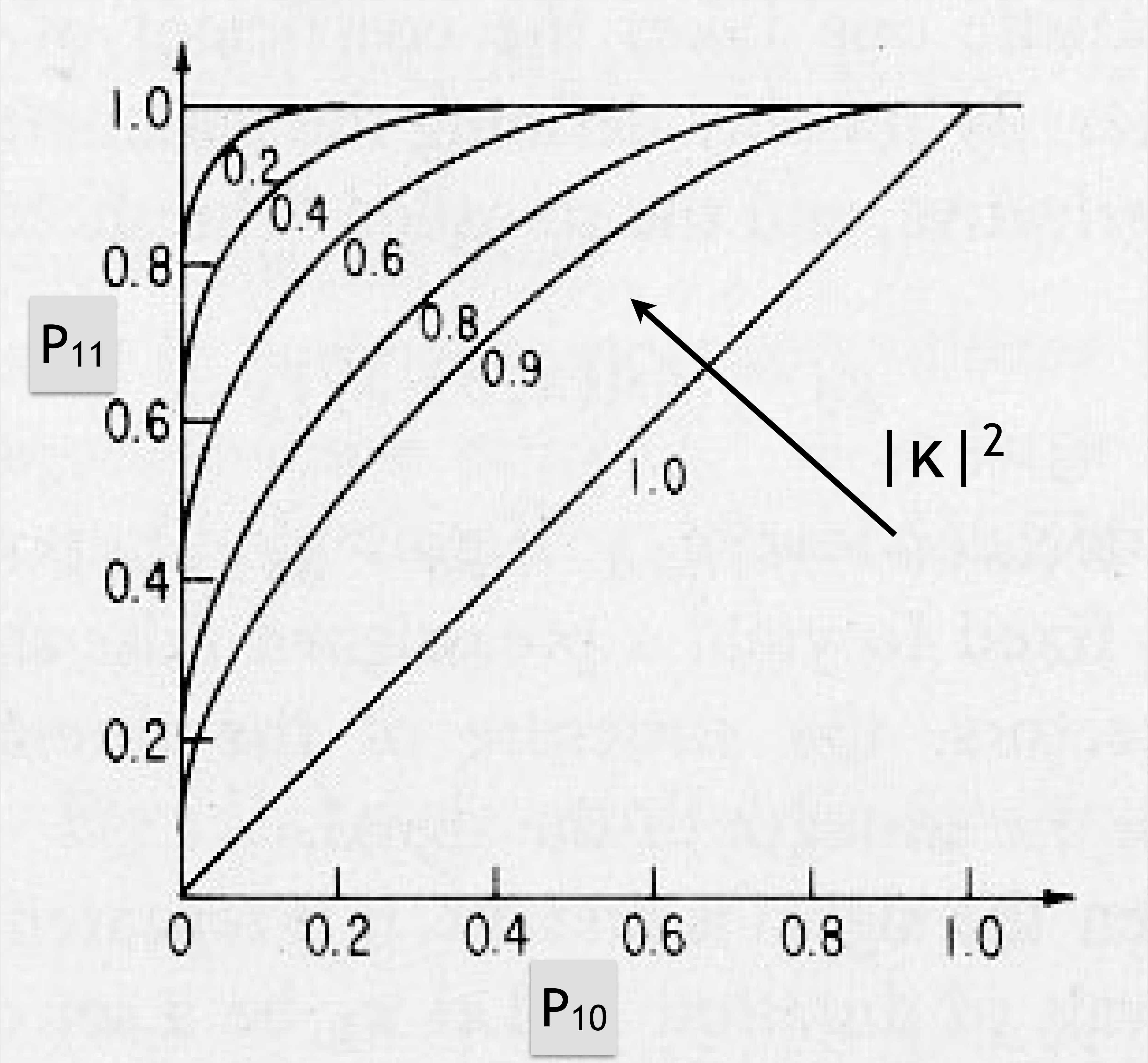
11 Discrimination of Quantum States

János A. Bergou[1], Ulrike Herzog[2], and Mark Hillery[1]

[1] Department of Physics and Astronomy, Hunter College of The City University of New York, 695 Park Avenue, New York, NY 10021, jbergou@hunter.cuny.edu, mhillery@hunter.cuny.edu

[2] Institut für Physik, Humboldt-Universität Berlin, Newtonstrasse 15, 12489 Berlin, Germany, ulrike.herzog@physik.hu-berlin.de

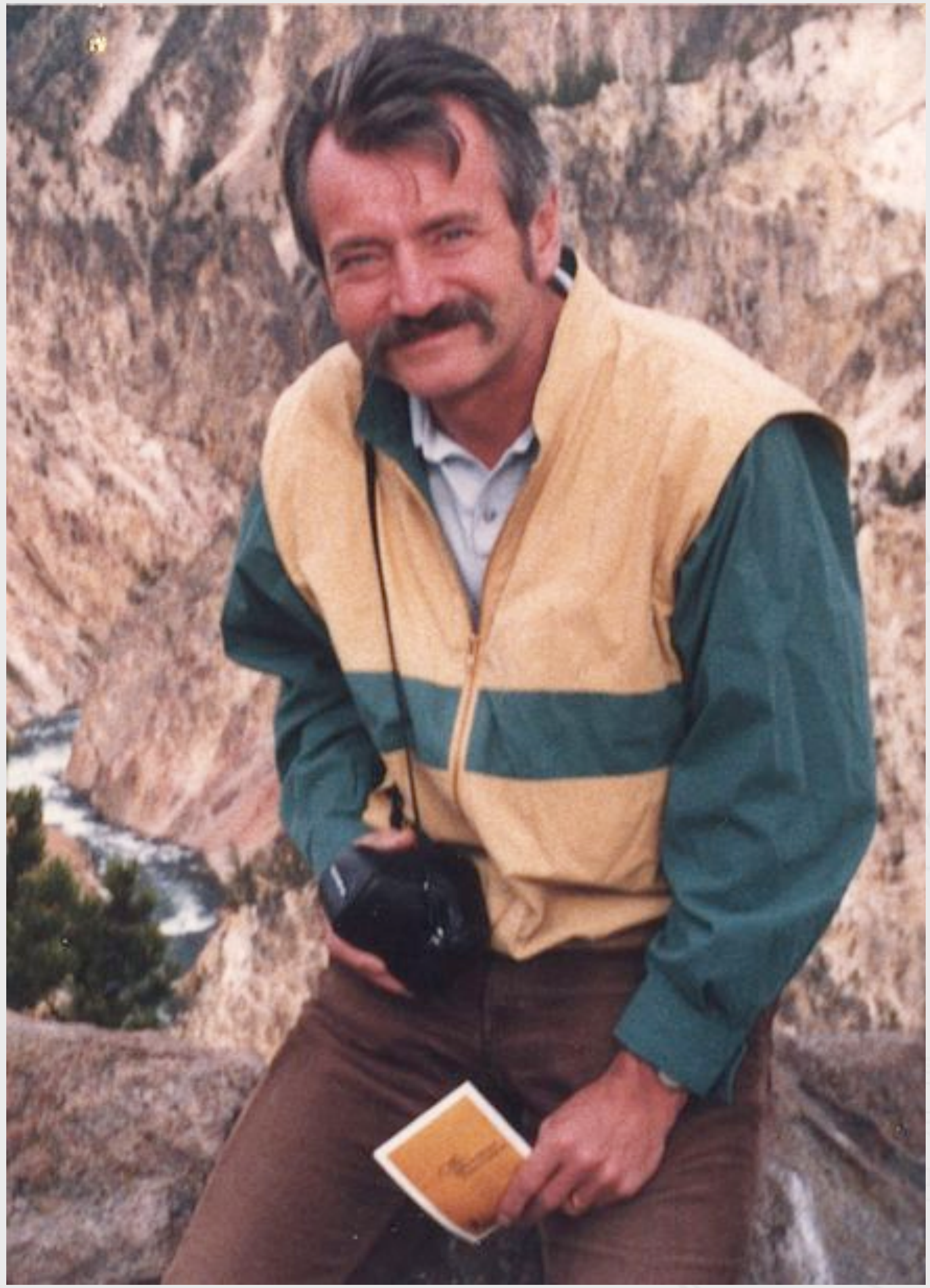
Abstract. The problem of discriminating among given nonorthogonal quantum states is underlying many of the schemes that have been suggested for quantum communication and quantum computing. However, quantum mechanics puts severe limitations on our ability to determine the state of a quantum system. In particular, nonorthogonal states cannot be discriminated perfectly, even if they are known, and various strategies for optimum discrimination with respect to some appropriately chosen criteria have been developed. In this article we review recent theoretical progress regarding the two most important optimum discrimination strategies. We also give a detailed introduction with emphasis on the relevant concepts of the quantum theory of measurement. After a brief introduction into the field, the second chapter deals with optimum unambiguous, i. e error-free, discrimination. Ambiguous discrimination with minimum error is the subject of the third chapter. The fourth chapter is devoted to an overview of the recently emerging subfield of discriminating multiparticle states. We conclude with a brief outlook where we attempt to outline directions of research for the immediate future.



1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1.0
0.2
0.4
0.6
0.8
0.9
1.0
P_{11}
P_{10}
$|\kappa|^2$

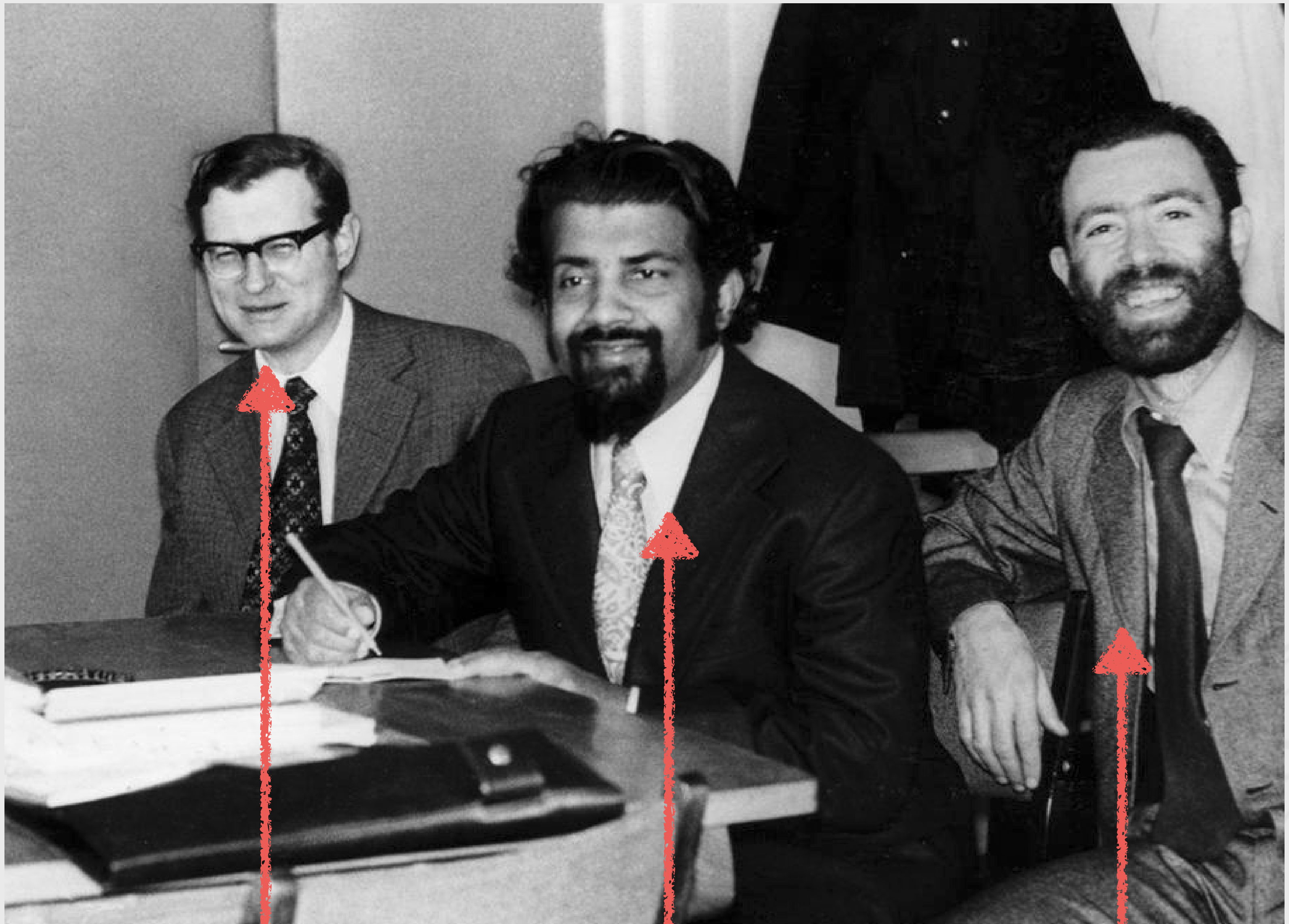
Mark Naimark
(Марк Аронóвич Наймарк)

Karl Kraus



Andrzej Jamiolkowski

Man-Duen Choi



Andrzej Kossakowski

Ennackal Chandy George Sudarshan

Vittorio Gorini

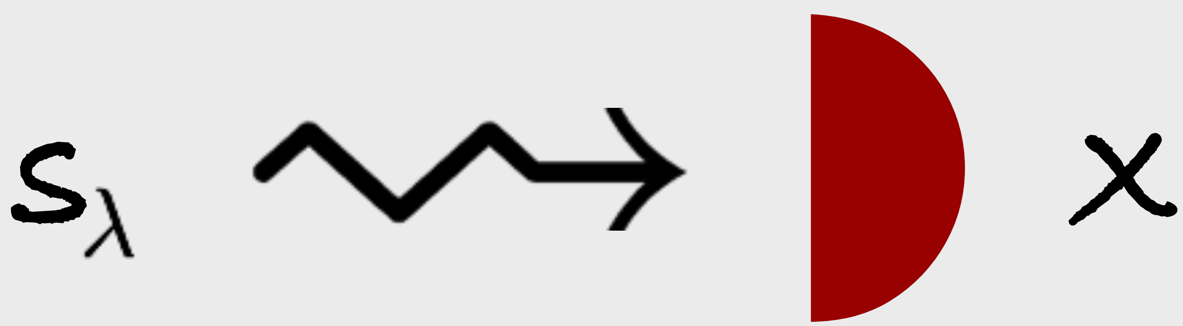
Göran Lindblad

Alexander S. Holevo
(Алекса́ндр Семе́нович Хо́лево)

Carl W. Helstrom

- ~~direct measurements~~
- indirect measurements

→ influence on a different quantity

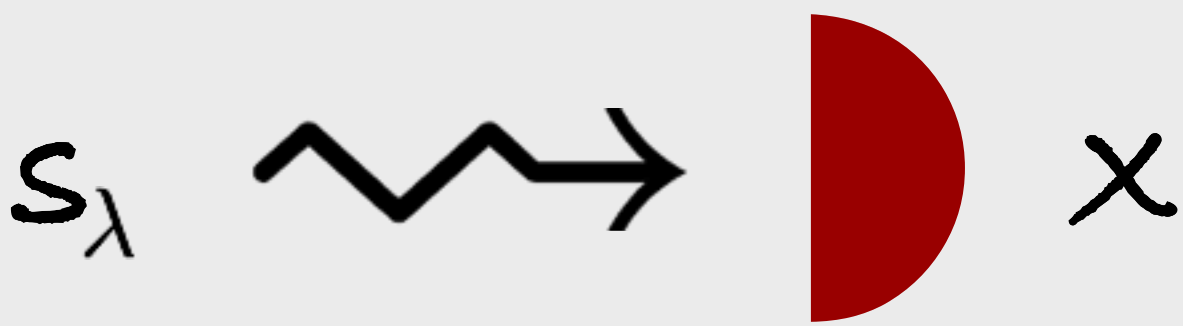


$$\chi = (x_1, x_2, \ldots)$$

Measurement and estimation

- ~~direct measurements~~
- indirect measurements

↪ influence on a different quantity



$$\boldsymbol{\chi} = (x_1, x_2, \ldots)$$

- choice of the measurement $\quad p(x|\lambda)$
- choice of the estimator $\quad \boldsymbol{\chi} \mapsto \hat{\lambda} = f(\boldsymbol{\chi})$

Measurement and estimation

- global estimation theory
  (when you have no a priori information)
  look for a measurement which is optimal in average
  (over the possible values of the parameter)

- local estimation theory
  (when you have some a priori information)
  look for a measurement which is optimal for a
  specific value of the parameter (better, but...)

local estimation theory: Cramer - Rao bound

- variance of unbiased estimators

$$\mathrm{Var}_\lambda[\widehat{\lambda}] \geq \frac{1}{M F(\lambda)}$$

- M -> number of measurements
- F -> Fisher Information

$$F(\lambda) = \int dx\, p(x|\lambda) \left[\partial_\lambda \log p(x|\lambda)\right]^2$$

$$= \int dx\, \frac{\left[\partial_\lambda p(x|\lambda)\right]^2}{p(x|\lambda)}$$

local estimation theory: Cramer - Rao bound

The proof of the Cramer-Rao bound is obtained by observing that given two functions $f_1(x)$ and $f_2(2)$ the average

$$\langle f_1, f_2 \rangle = \int dx\, p(x|\lambda)\, f_1(x)\, f_2(x)$$

defines a scalar product. Upon chosing $f_1(x) = \hat{\lambda}(x) - \lambda$ and $f_2(x) = \partial_\lambda \ln p(x|\lambda)$ we have

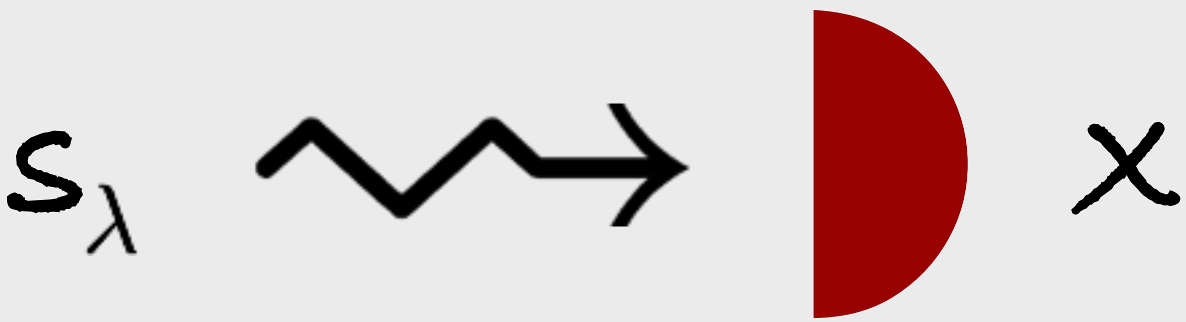
$$||f_1||^2 = \mathrm{Var}(\lambda)$$
$$||f_2||^2 = F(\lambda)$$
$$\langle f_1, f_2 \rangle = 1$$

$$\int dx\, \partial_\lambda p(x|\lambda) = 0$$

$x_1, x_2, ..., x_M$ independent we have $p(x_1, x_2, ..., x_M|\lambda) = \prod_{k=1}^{M} \log p(x_k|\lambda)$ and, in turn,

$$F_M(\lambda) = \int dx_1 ... dx_M\, p(x_1, x_2, ..., x_M|\lambda)\, [\partial_\lambda \ln p(x_1, x_2, ..., x_M|\lambda)]^2$$
$$= M \int dx\, p(x|\lambda)\, [\partial_\lambda \ln p(x|\lambda)]^2 = M F(\lambda)\,.$$

Optimal estimation scheme (classical)



$$\boldsymbol{x} = (x_1, x_2, \ldots)$$

$$\boldsymbol{x} \mapsto \hat{\lambda} = f(\boldsymbol{x})$$

- Optimal measurement -> maximum Fisher
  (no recipes on how to find it)

- Optimal estimator -> saturation of CR inequality
  (e.g. Bayesian or MaxLik asymptotically)

Bayesian estimators (1)

- Bayes theorem $$p(x|\lambda)p(\lambda) = p(\lambda|x)p(x)$$

- M indipendent events: a posteriori distribution

$$p(\lambda|\{x\}) = \frac{1}{N}\prod_{k=1}^{M} p(x_k|\lambda) \qquad N = \int d\lambda \prod_{k=1}^{M} p(x_k|\lambda)$$

- Bayesian estimator: $$\lambda_B = \int d\lambda\, \lambda\, p(\lambda|\{x\})$$

mean of the a posteriori distribution

Bayesian estimators (2)

- Laplace - Bernstein - von Mises theorem

$$p(\lambda|\{x\}) \xrightarrow{M \gg 1} G(\lambda^*, \sigma^2)$$

- Bayes estimator is asymptotically efficient

$$\sigma^2 = \frac{1}{MF(\lambda^*)}$$

MaxLik estimation

- Probability distribution $p(x|\lambda)$
- Random sample $x_1, x_2, ..., x_M$
- Joint probability of the sample

$$\mathcal{L}(x_1, x_2, ... x_M|\lambda) = \Pi_{k=1}^{M} p(x_k|\lambda)$$

Maxlik estimation ➞ take the value of the parameters which maximize the likelihood of the observed data

Quantum estimation

- What about time and temperature in quantum mechanics?
- The "resources" involved in quantum-enahnced technology are entanglement, nonlocality, entropy, interferometric phase-shift, etc.. In general they are not observable quantities in strict sense (do not correspond to a selfadjoint operator)

  - No correspondence principle
  - No uncertainty relations

Quantum estimation

- What about time and temperature in quantum mechanics?
- The "resources" involved in quantum-enahnced technology are entanglement, nonlocality, entropy, interferometric phase-shift, etc.. In general they are not observable quantities in strict sense (do not correspond to a selfadjoint operator)

Quantum estimation theory

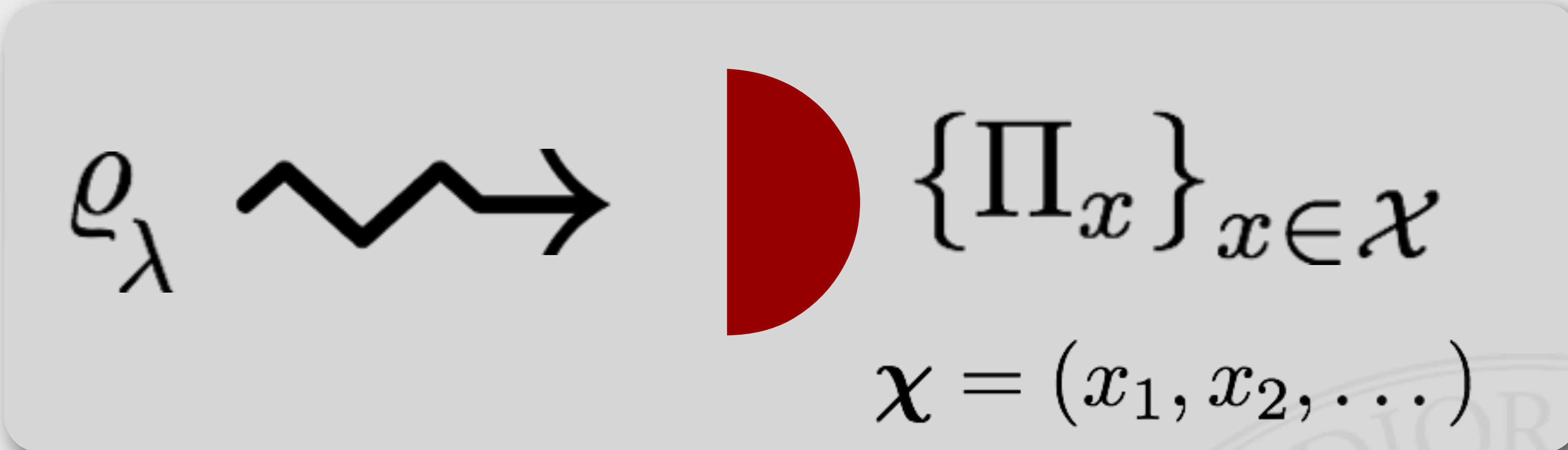
Quantum estimation

$$\varrho_\lambda \rightsquigarrow \quad \{\Pi_x\}_{x\in\mathcal{X}}$$

$$\chi = (x_1, x_2, \ldots)$$

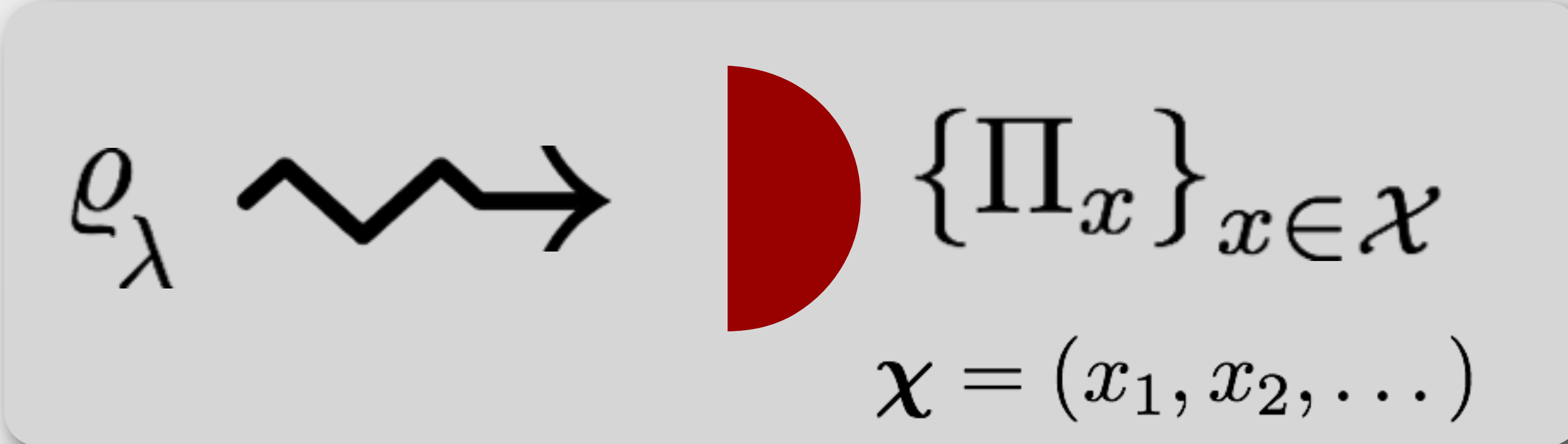
- Optimal measurements
- Ultimate bounds to precision

Quantum estimation



- Probability density $\quad p(x|\lambda) = \mathrm{Tr}\left[\varrho_\lambda\, \Pi_x\right]$

Let's go quantum (local) (1)



- probability density $p(x|\lambda) = \mathrm{Tr}\left[\varrho_\lambda \Pi_x\right]$

- symm. log. derivative (SLD) $\dfrac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \dfrac{\partial \varrho_\lambda}{\partial \lambda}$

  selfadjoint, zero mean $\mathrm{Tr}\left[\varrho_\lambda L_\lambda\right] = 0$

- Fisher Information $F(\lambda) = \displaystyle\int dx \, \frac{\mathrm{Re}\left(\mathrm{Tr}\left[\varrho_\lambda \Pi_x L_\lambda\right]\right)^2}{\mathrm{Tr}[\varrho_\lambda \Pi_x]}$

Let's go quantum (local) (2)

$$F(\lambda) \leq \int dx \left| \frac{\mathrm{Tr}\,[\varrho_\lambda \Pi_x L_\lambda]}{\sqrt{\mathrm{Tr}[\varrho_\lambda \Pi_x]}} \right|^2$$

$$= \int dx \left| \mathrm{Tr} \left[\frac{\sqrt{\varrho_\lambda}\sqrt{\Pi_x}}{\sqrt{\mathrm{Tr}\,[\varrho_\lambda \Pi_x]}} \sqrt{\Pi_x} L_\lambda \sqrt{\varrho_\lambda} \right] \right|^2$$

$$\leq \int dx\, \mathrm{Tr}\,[\Pi_x L_\lambda \varrho_\lambda L_\lambda]$$

$$= \mathrm{Tr}[L_\lambda \varrho_\lambda L_\lambda] = \mathrm{Tr}[\varrho_\lambda L_\lambda^2]$$

Helstrom 1976
Braunstein & Caves 1994

- Fisher vs Quantum Fisher

$$F(\lambda) \leq H(\lambda) \equiv \mathrm{Tr}[\varrho_\lambda L_\lambda^2] = \mathrm{Tr}[\partial_\lambda \varrho_\lambda L_\lambda]$$

- ultimate bound on precision $\quad \mathrm{Var}(\lambda) \geq \dfrac{1}{M H(\lambda)}$

Optimal estimation scheme (quantum, local)

ϱ_λ  $\{\Pi_x\}_{x\in\mathcal{X}}$

$$\boldsymbol{\chi} = (x_1, x_2, \dots)$$

Optimal measurement -> Fisher = quantum Fisher

It is projective! The spectral measure of the SLD

Optimal estimator -> saturation of CR inequality (classical postprocessing, e.g. Bayesian or MaxLix)

$$\boldsymbol{\chi} \mapsto \widehat{\lambda} = f(\boldsymbol{\chi})$$

General formulas (basis indepedent)

$$\frac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \frac{\partial \varrho_\lambda}{\partial \lambda}$$

Lyapunov equation

- Symmetric logarithmic derivative

$$L_\lambda = 2\int_0^\infty dt\, \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}$$

- Quantum Fisher Information

$$H(\lambda) = 2\int_0^\infty dt\, \mathrm{Tr}\left[\partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}\right]$$

General formulas

- **Family of quantum states**

$$\varrho_\lambda = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$$

- **Symmetric logarithmic derivative**

$$L_\lambda = \sum_p \frac{\partial_\lambda \varrho_p}{\varrho_p} |\psi_p\rangle\langle\psi_p| + 2\sum_{n\neq m} \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} \langle\psi_m|\partial_\lambda\psi_n\rangle\, |\psi_m\rangle\langle\psi_n|$$

- **Quantum Fisher Information**

$$H(\lambda) = \sum_p \frac{(\partial_\lambda \varrho_p)^2}{\varrho_p} + 2\sum_{n\neq m} \frac{(\varrho_n - \varrho_m)^2}{\varrho_n + \varrho_m} |\langle\psi_m|\partial_\lambda\psi_n\rangle|^2$$

PHYSICAL REVIEW A **91**, 042104 (2015)

Extended convexity of quantum Fisher information in quantum metrology

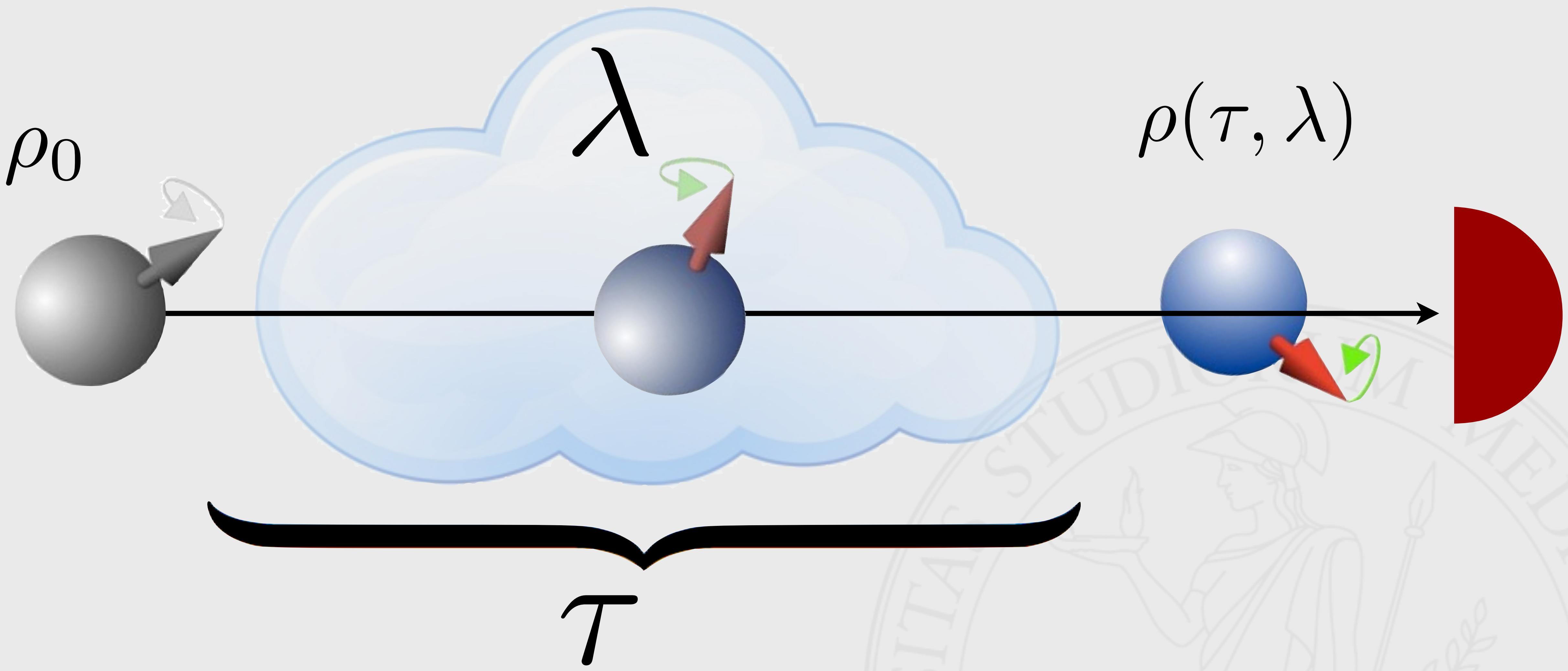
S. Alipour and A. T. Rezakhani
Department of Physics, Sharif University of Technology, Tehran 14588-89694, Iran

(Received 8 July 2014; revised manuscript received 16 November 2014; published 7 April 2015)

We prove an extended convexity for quantum Fisher information of a mixed state with a given convex decomposition. This convexity introduces a bound which has two parts: (i) The *classical* part associated with the Fisher information of the probability distribution of the states contributing to the decomposition, and (ii) the *quantum* part given by the average quantum Fisher information of the states in this decomposition. Next we use a non-Hermitian extension of a symmetric logarithmic derivative in order to obtain another upper bound on quantum Fisher information, which helps to derive a closed form for the bound in evolutions having the semigroup property. We enhance the extended convexity with this concept of a non-Hermitian symmetric logarithmic derivative (which we show is computable) to lay out a general metrology framework where the dynamics is described by a quantum channel and derive the ultimate precision limit for open-system quantum metrology. We illustrate our results and their applications through two examples where we also demonstrate how the extended convexity allows identifying a crossover between quantum and classical behaviors for metrology.

DOI: 10.1103/PhysRevA.91.042104 PACS number(s): 03.65.Yz, 03.67.Lx, 06.20.−f, 03.65.Ta

Quantum probes for complex systems



Maximizing the extraction of information by optimizing the preparation of the probe, the interaction time and the measurement at the output.

Quantum probes for complex systems

Quantum probes for the cutoff frequency of Ohmic environments

Claudia Benedetti,[1] Fahimeh Salari Sehdaran,[2] Mohammad H. Zandi,[2] and Matteo G. A. Paris[1]
[1]*Quantum Technology Lab, Physics Department, Università degli Studi di Milano, Milano, Italy*
[2]*Faculty of Physics, Shahid Bahonar University of Kerman, Kerman, Iran*

PRA **97**, 012126 (2018)

Quantum thermometry by single-qubit dephasing

Sholeh Razavian,[1] Claudia Benedetti,[2] Matteo Bina,[2] Yahya Akbari-Kourbolagh,[3] and Matteo G. A. Paris[2,4]
[1]*Faculty of Physics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[2]*Quantum Technology Lab, Dipartimento di Fisica, Università degli Studi di Milano, I-20133 Milano, Italy*
[3]*Faculty of Physics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[4]*INFN, Sezione di Milano, I-20133 Milano, Italy*

arXiv:1807.11810

Quantum metrology out of equilibrium

Sholeh Razavian[1,2], Matteo G. A. Paris[1,3]
[1]*QCSE School, Villa del Grumello, I-22100, Como, Italy*
[2]*Faculty of Physics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[3]*Quantum Technology Lab, Dipartimento di Fisica 'Aldo Pontremoli', Università degli Studi di Milano, I-20133 Milano, Italy*

arXiv:1808.07180

Universal Quantum Magnetometry with Spin States at Equilibrium

Filippo Troiani[1,*] and Matteo G. A. Paris[2,3,†]
[1]*Centro S3, CNR-Istituto di Nanoscienze, I-41125 Modena, Italy*
[2]*Quantum Technology Lab, Dipartimento di Fisica dell'Università degli Studi di Milano, I-20133 Milano, Italy*
[3]*INFN, Sezione di Milano, I-20133 Milano, Italy*

(Received 2 November 2017; published 29 June 2018)

PRL **120**, 260503 (2018)

Quantum probes for complex systems

Continuous-variable quantum probes for structured environments

Matteo Bina,[1,*] Federico Grasselli,[2,1] and Matteo G. A. Paris[1,3]
[1]*Quantum Technology Lab, Dipartimento di Fisica, Università degli Studi di Milano, I-20133 Milano, Italy*
[2]*Institut für Theoretische Physik III, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany*
[3]*INFN, Sezione di Milano, I-20133 Milano, Italy*

PRA **97**, 012125 (2018)

The walker speaks its graph: global and nearly-local probing of the tunnelling amplitude in continuous-time quantum walks

J. Phys. A: Math. Theor. **52** (2019) 10530

Luigi Seveso, Claudia Benedetti and Matteo G A Paris
Quantum Technology Lab, Dipartimento di Fisica Aldo Pontremoli, Università degli Studi di Milano, I-20133 Milano, Italy

Quantum walker as a probe for its coin parameter

PRA **99**, 052117 (2019)

Shivani Singh[*] and C. M. Chandrashekar[†]
The Institute of Mathematical Sciences, CIT campus, Taramani, Chennai 600113, India
and Homi Bhabha National Institute, Training School Complex, Anushakti Nagar, Mumbai 400094, India

Matteo G. A. Paris[‡]
Quantum Technology Lab, Dipartimento di Fisica "Aldo Pontremoli," Università degli Studi di Milano, I-20133 Milano, Italy

Quantum probes for complex systems

Journal of Magnetism and Magnetic Materials 491 (2019) 165534

Contents lists available at ScienceDirect

Journal of Magnetism and Magnetic Materials

journal homepage: www.elsevier.com/locate/jmmm

ELSEVIER

Towards quantum sensing with molecular spins

F. Troiani[a], A. Ghirri[a], M.G.A. Paris[b], C. Bonizzoni[c,a], M. Affronte[c,a,*]

[a] *CNR, Instituto Nanoscienze, via G. Campi 213A, 41125 Modena, Italy*
[b] *Quantum Technology Lab, Dipartimento di Fisica Aldo Pontremoli, Università degli Studi di Milano, I-20133 Milano, Italy*
[c] *Dipartimento di Scienze Fisiche Informatiche e Matematiche, Universita di Modena e Reggio Emilia, via G. Campi 213A, 41125 Modena Italy*

PHYSICAL REVIEW A **94**, 042129 (2016)

Characterization of qubit chains by Feynman probes

Dario Tamascelli,[1,2,*] Claudia Benedetti,[1,†] Stefano Olivares,[1,3,‡] and Matteo G. A. Paris[1,3,§]
[1]*Quantum Technology Lab, Dipartimento di Fisica, Università degli Studi di Milano, I-20133 Milano, Italy*
[2]*Institut für Theoretische Physik and IQST, Universität Ulm, Albert-Einstein-Allee 11, Ulm, Germany*
[3]*INFN, Sezione di Milano, I-20133 Milano, Italy*
(Received 25 July 2016; published 26 October 2016)

In the search of new physics

International Journal of Theoretical Physics (2019) 58:2914–2935
https://doi.org/10.1007/s10773-019-04174-9

Quantum Sensing of Curvature

Daniele Bonalda[1] · Luigi Seveso[1] · Matteo G. A. Paris[1]

Received: 16 March 2019 / Accepted: 27 May 2019 / Published online: 3 July 2019

Annals of Physics 380 (2017) 213–223

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Can quantum probes satisfy the weak equivalence principle?

Luigi Seveso[a,*], Matteo G.A. Paris[a,b]

[a] Quantum Technology Lab, Dipartimento di Fisica, Università degli Studi di Milano, I-20133 Milano, Italy
[b] INFN, Sezione di Milano, I-20133 Milano, Italy

PHYSICAL REVIEW D **94,** 024014 (2016)

Probing deformed quantum commutators

Matteo A. C. Rossi[*]
Quantum Technology Lab, Dipartimento di Fisica, Università degli Studi di Milano, 20133 Milano, Italy

Tommaso Giani[†]
Dipartimento di Fisica, Università degli Studi di Milano, 20133 Milano, Italy

Matteo G. A. Paris[‡]
Quantum Technology Lab, Dipartimento di Fisica, Università degli Studi di Milano, 20133 Milano, Italy and INFN, Sezione di Milano, I-20133 Milano, Italy
(Received 21 June 2016; published 6 July 2016)

Current topics (not covered in the lectures)

- Multiparameter quantum estimation, see e.g.

  arXiv:1911.12067
  A perspective on multiparameter quantum metrology: from theoretical tools to applications in quantum imaging
  Francesco Albarelli, Marco Barbieri, Marco G. Genoni, Ilaria Gianani
  (Physics Letters A, in press)

- Q metrology beyond the QCR bound

  Quantum metrology beyond the Quantum Cramér-Rao theorem
  Luigi Seveso, Matteo A. C. Rossi, Matteo G. A. Paris
  Phys. Rev. A 95, 012111 (2017)   arXiv:1605.08653

  Estimation of general Hamiltonian parameters via…
  Luigi Seveso, Matteo G. A. Paris
  Phys. Rev. A 98, 032114 (2018)   arXiv:1712.07858