Quantum Speed Limits:
Background and Basics

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Uncertainty principle

Captures the inherent “fuzziness” of quantum mechanics

$$\Delta p \Delta x \gtrsim \hbar \qquad \Delta E \Delta t \gtrsim \hbar$$

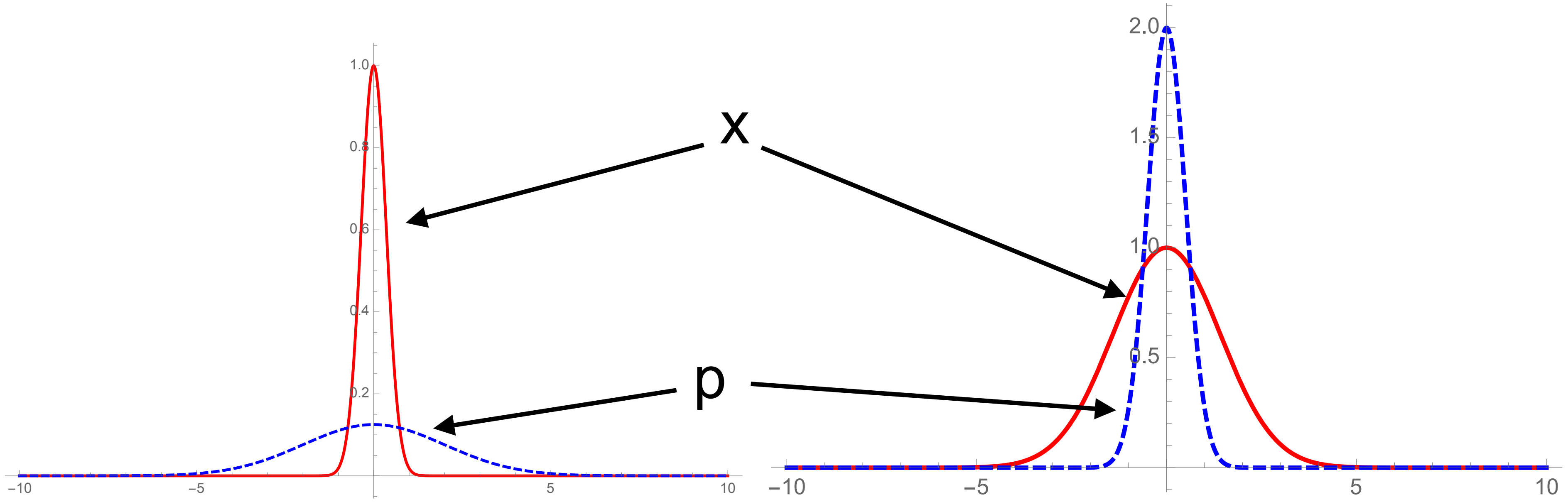
Impossibility of accurately measuring conjugate observables

While the position-momentum uncertainty relation was given a solid physical grounding in short order, the energy-time relation was much more subtle….

Uncertainty principle

Position-momentum relation can be understood from Fourier analysis of wave packets.

$$\Delta p \Delta x \gtrsim \hbar$$



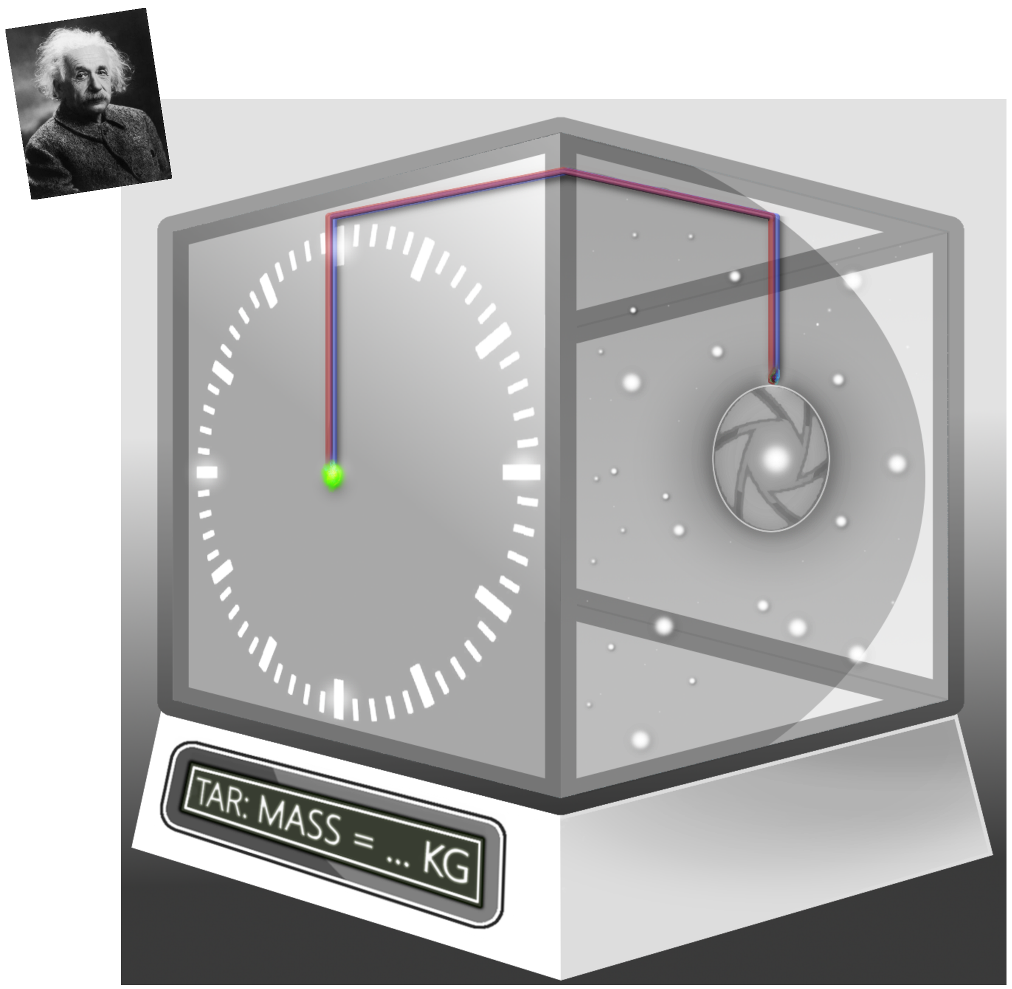
The product of their uncertainties is maintained

Often considered in relation to simultaneous measurements

Which lead to the following thought experiment….

Uncertainty principle

Einstein considered a box with photons, a controllable shutter, and a classical clock which, at a preset time, will allow photons to escape.



Since

$$E = mc^2$$

we can workout the change in energy precisely

Seemingly negating the existence of the E-t uncertainty relation

Bohr had a counter argument…

A naïve derivation

Consider a wave packet moving with some velocity v

The time it passes a particular point has a given uncertainty directly related to the uncertainty in x

$$\Delta t \approx \frac{\Delta x}{v}$$

And we know there is an associated uncertainty in its momentum which gives a corresponding uncertainty in the energy

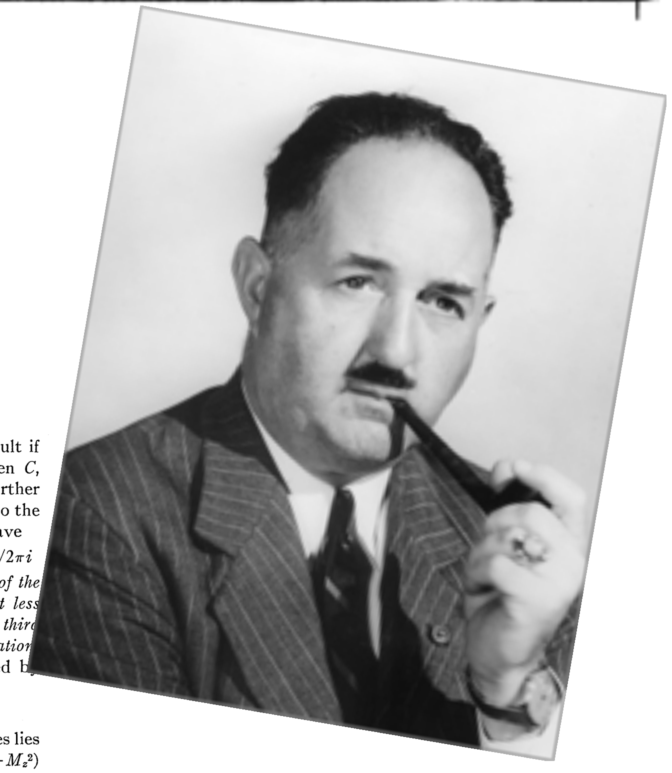
$$\Delta E \approx \frac{\partial E}{\partial p}\Delta p = v\Delta p$$

From which we “get” the E-t uncertainty principle

But

The uncertainty relations

In 1929 Robertson clarified things - at least for x and p



The uncertainty principle is one of the most characteristic and important consequences of the new quantum mechanics. This principle, as formulated by Heisenberg for two conjugate quantum-mechanical variables, states that the accuracy with which two such variables can be measured simultaneously is subject to the restriction that the product of the uncertainties in the two measurements is at least of order h (Planck's constant). Condon* has remarked that an uncertainty relation of this type can not hold in the general case where the two variables under consideration are not conjugate, and has stressed the desirability of obtaining a general formulation of the principle. It is the purpose of the present letter to give such a general formulation, and to apply it in particular to the case of angular momentum.

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* E. U. Condon "Remarks on Uncertainty Principles" Science LXIX, p. 573 (May 31, 1929), and in conversations with the writer on this topic.

...value" A_0 of an ... a system whose state is described by the (normal) function ψ as

$$A_0=\int\bar{\psi}A\psi d\tau$$

where the integral is extended over the entire coordinate space. The Hermitean character of A (i.e.

$$\int\bar{\phi}A\psi d\tau=\int\bar{\psi}A\phi d\tau$$

The uncertainty principle for two such variables A, B, whose commutator $AB-BA=hC/2\pi i$, is expressed by

$$\Delta A\cdot\Delta B\geqq h|C_0|/4\pi$$

i.e. the product of the uncertainties in A, B is not less than half the absolute value of the mean of their commutator.

We here confine ourselves to sketching the proof of this principle for a one-particle system and for quantum mechanical variables $A(q, p)$, $B(q, p)$ which are linear in the momenta (p_x, p_y, p_z).[1] (The proof for the general case in which the operators can be expanded in powers of the momenta can be made along exactly the same lines.) Writing

$$A=a+a_xp_x+a_yp_y+a_zp_z$$

where $p_x=(h/2\pi i)\partial/\partial x$, etc. and the a's are functions of position, the Hermitean character of A requires that these functions be real and that div $(a_x, a_y, a_z)=0$. The expression for $(\Delta A)^2$ may be written, on integrating once by parts, using the fact that div $(a)=0$ and discarding the resulting surface integral, in the form

$$(\Delta A)^2=\int|(A-A_0)\psi|^2d\tau.$$

We are now in a position to apply the Schwarzian inequality[2]

$$\left[\int(f_1\bar{f}_1+f_2\bar{f}_2)d\tau\right]\left[\int(g_1\bar{g}_1+g_2\bar{g}_2)d\tau\right]\geqq\left|\int(f_1g_1+f_2g_2)d\tau\right|^2$$

Taking

$$\bar{f}_1=(A-A_0)\psi=f_2,\quad g_1=(B-B_0)\psi=-\bar{g}_2$$

and reducing the integral on the right hand side by integration by parts we find

$$\Delta A\cdot\Delta B\geqq\tfrac{1}{2}\left|\int\bar{\psi}(AB-BA)\psi d\tau\right|,$$

the required result.

[1] Cf. proof of special case $A=p$, $B=q$ in H. Weyl "Gruppentheorie und Quantenmechanik" pp. 66, 272.

[2] Weyl, l. c. p. 272.

We obviously obtain Heisenberg's result if the two variables are conjugate, for then C, and consequently C_0, are ±1. As a further illustration of the principle, we apply it to the case of angular momentum. Here we have

$$M_x=yp_z-zp_y,\quad M_xM_y-M_yM_x=-hM_z/2\pi i$$

so *the product of the uncertainties in two of the components of angular momentum is not less than $h/4\pi$ times the mean value of the third component in the state under consideration.* Consider in particular the state, treated by Condon, defined by

$$\psi=f(r)e^{im\phi}P_l^m(\cos\theta)$$

where the pole of the spherical coordinates lies on the z-axis. Then M_z, $M^2(=M_x^2+M_y^2+M_z^2)$ have the definite values

$$M_z=M_{z0}=mh/2\pi,\quad M^2=l(l+1)(h/2\pi)^2$$

the mean values of M_x, M_y are zero and the uncertainties are given by

$$(\Delta M_x)^2=(\Delta M_y)^2=\tfrac{1}{2}[l(l+1)-m^2](h/2\pi)^2,\quad \Delta M_z=0.$$

Now from the uncertainty principle for M_x, M_y we find

$$l(l+1)\geqq m(m+1)$$

which is in fact the case. This example shows that for $m=l$ the equality holds; the inequality is consequently the most restrictive one that can be deduced for angular momenta, for we have here a case in which the ultimate limit has (in principle) been reached.

H. P. Robertson
Palmer Physical Laboratory,
Princeton, N. J.,
June 18, 1929.

The Emission of Positive Ions from Metals

...initially but these disappeared after a few minutes' heating. When the temperature of the metals was increased to the point where vaporization became appreciable, positive ion emissions were again observed. The atomic weights of these ions check within the limits of error with the atomic weights of the respective metals.

It is hoped to extend these results to other metals as well as to study the emission as a function of the temperature.

H. B. Wahlin
University of Wisconsin,
June 24, 1929.

The uncertainty relations

For any two operators, through the Cauchy-Schwarz inequality

$$\Delta A \Delta B \geq \frac{1}{2} |\langle (AB - BA) \rangle|$$

where $\Delta O = \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$

From the canonical commutation relation

$$[x, p] = i\hbar$$

It readily follows that

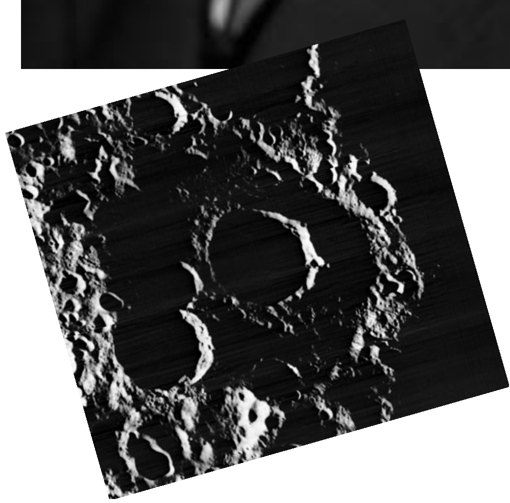
$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

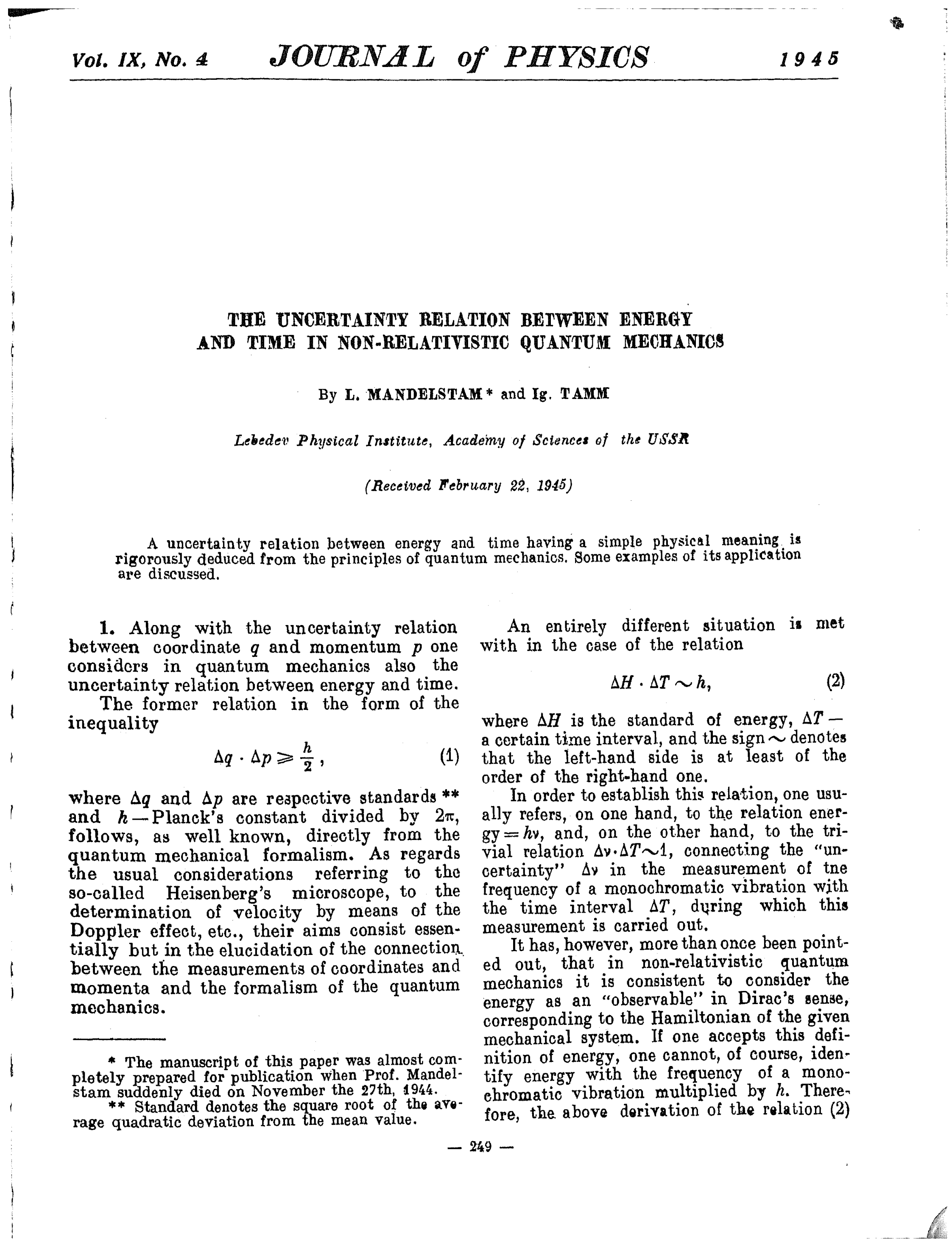
But we do not have a time operator!

The Mandelstam-Tamm Bound

It was some 20 years before a conceptually solid interpretation was established

Leonid Mandelstam





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THE UNCERTAINTY RELATION BETWEEN ENERGY AND TIME IN NON-RELATIVISTIC QUANTUM MECHANICS

By L. MANDELSTAM * and Ig. TAMM

Lebedev Physical Institute, Academy of Sciences of the USSR

(Received February 22, 1945)

A uncertainty relation between energy and time having a simple physical meaning is rigorously deduced from the principles of quantum mechanics. Some examples of its application are discussed.

1. Along with the uncertainty relation between coordinate q and momentum p one considers in quantum mechanics also the uncertainty relation between energy and time.

The former relation in the form of the inequality

$$\Delta q \cdot \Delta p \geqslant \frac{h}{2}, \quad (1)$$

where Δq and Δp are respective standards ** and h — Planck's constant divided by 2π, follows, as well known, directly from the quantum mechanical formalism. As regards the usual considerations referring to the so-called Heisenberg's microscope, to the determination of velocity by means of the Doppler effect, etc., their aims consist essentially but in the elucidation of the connection between the measurements of coordinates and momenta and the formalism of the quantum mechanics.

An entirely different situation is met with in the case of the relation

$$\Delta H \cdot \Delta T \sim h, \quad (2)$$

where ΔH is the standard of energy, ΔT — a certain time interval, and the sign $\sim$ denotes that the left-hand side is at least of the order of the right-hand one.

In order to establish this relation, one usually refers, on one hand, to the relation energy $= h\nu$, and, on the other hand, to the trivial relation $\Delta\nu \cdot \Delta T \sim 1$, connecting the "uncertainty" $\Delta\nu$ in the measurement of tne frequency of a monochromatic vibration with the time interval ΔT, during which this measurement is carried out.

It has, however, more than once been pointed out, that in non-relativistic quantum mechanics it is consistent to consider the energy as an "observable" in Dirac's sense, corresponding to the Hamiltonian of the given mechanical system. If one accepts this definition of energy, one cannot, of course, identify energy with the frequency of a monochromatic vibration multiplied by h. Therefore, the above derivation of the relation (2)

* The manuscript of this paper was almost completely prepared for publication when Prof. Mandelstam suddenly died on November the 27th, 1944.

** Standard denotes the square root of the average quadratic deviation from the mean value.

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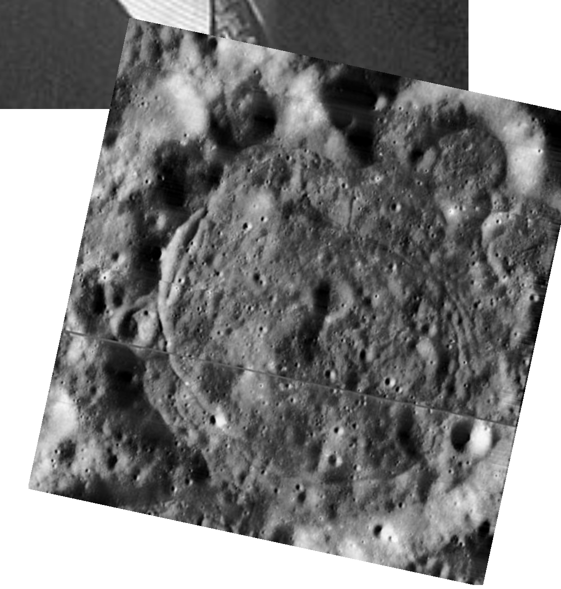
Igor Tamm



The Mandelstam-Tamm Bound

Consider some observable *A* which evolves according to the Liouville-von Neumann equation

$$\frac{\partial A}{\partial t} = \frac{i}{\hbar}[H, A]$$

Robertson's uncertainty relation tells us

$$\Delta H \Delta A \geq \frac{\hbar}{2}\left|\left\langle \frac{\partial A}{\partial t} \right\rangle\right|$$

If *A* is our initial state then

$$A = |\psi(0)\rangle\langle\psi(0)| \qquad \Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} = \sqrt{\langle A\rangle - \langle A\rangle^2}$$

Robertson's inequality can be "easily" integrated

> This inequality contains only one variable quantity $\bar{L}(t)$ and its derivative and can easily be integrated.

$$\frac{1}{\hbar}\Delta H t \geq \frac{\pi}{2} - \arcsin\sqrt{\langle A\rangle_t}$$

The Mandelstam-Tamm Bound

$$\frac{1}{\hbar}\Delta H t \geq \frac{\pi}{2} - \arcsin\sqrt{\langle A\rangle_t}$$

From here we arrive at the first **Quantum Speed Limit Time**.

Consider if we are interested in orthogonal states

$$\langle\psi(0)|\psi(\tau)\rangle = 0$$

It follows then

$$\tau \geq \tau_{\mathrm{QSL}} = \frac{\pi}{2}\frac{\hbar}{\Delta H}$$

The QSL sets an intrinsic timescale for the dynamics to occur on

The Mandelstam-Tamm Bound

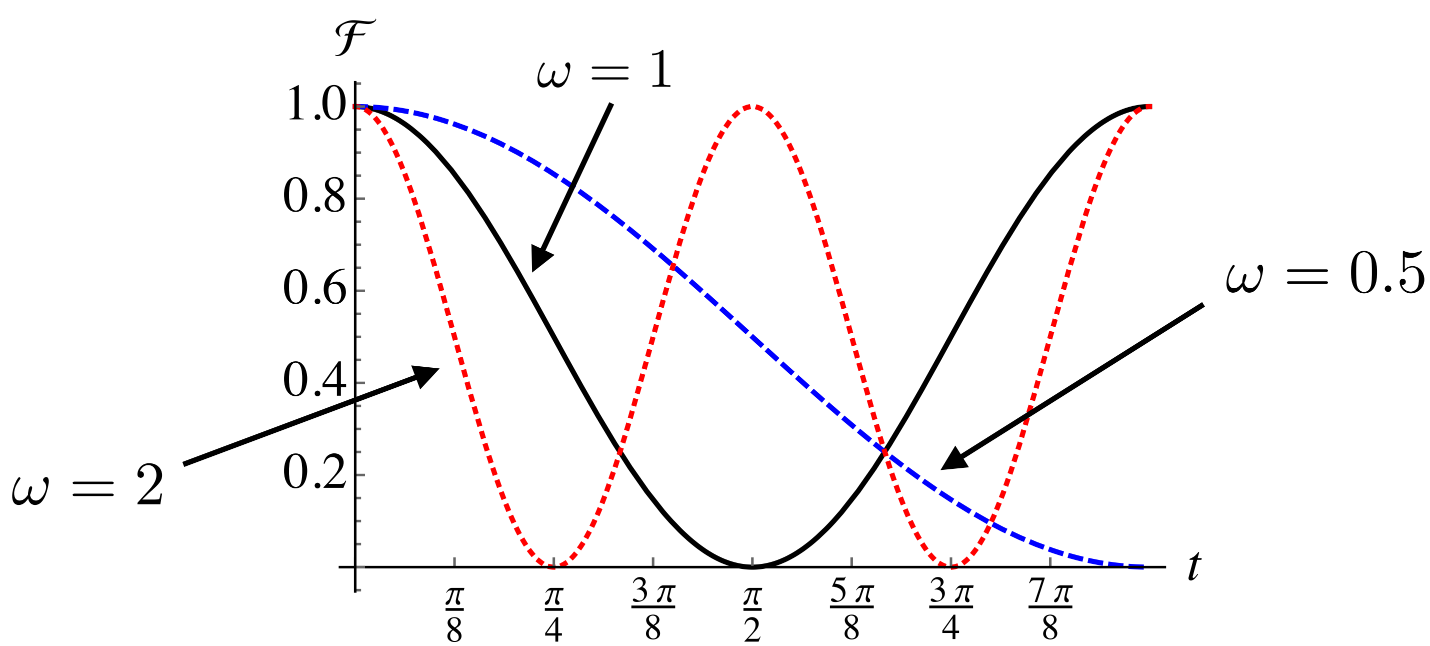
$$\tau \geq \tau_{\mathrm{QSL}} = \frac{\pi}{2}\frac{\hbar}{\Delta H}$$

The relationship is clear for the simple example

$$H = \hbar\omega\sigma_z \qquad |\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

The variance is simply

$$\Delta H = \hbar\omega$$



The QSL sets an intrinsic timescale for the dynamics to occur on

The Margolus-Levitin Bound

(Quantum) computation brought a renewed interest to the question of minimal evolution times

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PHYSICA D

The maximum speed of dynamical evolution

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Abstract

We discuss the problem of counting the maximum number of distinct states that an isolated physical system can pass through in a given period of time – its *maximum speed of dynamical evolution*. Previous analyses have given bounds in terms of ΔE, the standard deviation of the energy of the system; here we give a strict bound that depends only on $E - E_0$, the system's average energy minus its ground state energy. We also discuss bounds on information processing rates implied by our bound on the speed of dynamical evolution. For example, adding 1 J of energy to a given computer can never increase its processing rate by more than about 3×10^{33} operations per second. © 1998 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

In the realm of computation, the first two quantitative questions that one is likely to ask about a machine are: (i) *How much memory does it have*? and (ii) *How fast does it run*? In exploring the computational limits of physical dynamics, one might try to ask the same questions about an arbitrary physical system.

Question (i) essentially asks, "How many distinct states can my system be put into, subject to whatever physical constraints I have?" This is really a very old question: the correct counting of physical states is the problem that led to the introduction of Planck's constant into physics [17], and is the basis of all of quantum statistical mechanics. This question can be answered by a detailed quantum mechanical counting of distinct (mutually orthogonal) states. It can also be well approximated in the macroscopic limit [9,21] by simply computing the volume of phase space accessible to the system, in units where Planck's constant is 1.

Question (ii) will be the focus of this paper. This question can be asked with various levels of sophistication. Here we will discuss a particularly simple measure of speed: the maximum number of distinct states that the system can pass through, per unit of time. For a classical computer, this would correspond to the maximum number of operations per second.

For a quantum system, the notion of distinct states is well-defined: two states are distinct if they are orthogonal. The connection between orthogonality and rate of information processing has previously been discussed [3,4,7,10,12,13], but no universal bound on computation rate was proposed. The minimum time needed for a quantum system to pass from one orthogonal state to another has also previously been characterized, in terms of the standard deviation of the energy ΔE [5,11,15,16,20]. This bound places no limit, however, on how fast a system with bounded average energy can

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The Margolus-Levitin Bound

Consider a system with Hamiltonian

$$H = \sum_n E_n |E_n\rangle\langle E_n|$$

Any state can then be written and evolves as

$$|\psi_0\rangle = \sum_n c_n |E_n\rangle \qquad |\psi_t\rangle = \sum_n c_n \exp(-iE_n t/\hbar)|E_n\rangle$$

Now consider the overlap

$$S(t) \equiv \langle\psi_0|\psi_t\rangle = \sum_n |c_n|^2 \exp(-iE_n t/\hbar)$$

Margolus+Levitin asked what is the smallest t such that S(t)=0 ?

$$\Re[S(t)] = \sum_n |c_n|^2 \cos(-E_n t/\hbar)$$

The Margolus-Levitin Bound

Some mathematical manoeuvres

$$\Re[S(t)] = \sum_n |c_n|^2 \cos(-E_n t/\hbar)$$

$$\geq \sum_n |c_n|^2 \left[1 - \frac{2}{\pi}\left(\frac{E_n t}{\hbar} + \sin\left(\frac{E_n t}{\hbar}\right)\right)\right]$$

$\cos x \geq 1 - \frac{2}{\pi}(x + \sin x)$

$\langle H \rangle$

$\Im[S(t)]$

If we again restrict to evolving to orthogonal states then

$$\Re[S(t)] = \Im[S(t)] = 0$$

And we achieve the second **Quantum Speed Limit Time**.

$$\tau \geq \tau_{\mathrm{QSL}} = \frac{\pi}{2}\frac{\hbar}{\langle H \rangle}$$

(Average energy above the ground state!)

M-T vs M-L

$\tau \geq \tau_{\mathrm{QSL}} = \frac{\pi}{2}\frac{\hbar}{\langle H \rangle}$

$\tau \geq \tau_{\mathrm{QSL}} = \frac{\pi}{2}\frac{\hbar}{\Delta H}$

But how can we have two seemingly independent bounds based on two different physical properties of the same quantum state?

In our simple example there are no issues

$$H = \hbar\omega\sigma_z \qquad |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

and

$$\langle H \rangle = \Delta H \implies \tau_{\mathrm{QSL}}^{\mathrm{MT}} = \tau_{\mathrm{QSL}}^{\mathrm{ML}}$$

In general though one will find that the variance and the average energy will be different!

There are also cases where Eq. (4) gives a much better bound than Eq. (5). Consider, for example, the state

$$|\psi_0\rangle = a(|0\rangle + |\varepsilon\rangle) + b(|n\varepsilon\rangle + |(n+1)\varepsilon\rangle), \qquad (12)$$

which evolves into an orthogonal state in a time $\tau_\perp = h/2\varepsilon$. Given E, as long as we choose $\varepsilon < 2E$ (i.e., $\tau_\perp > h/4E$) the average energy of the first pair of kets will be less than E. Given ε, for large enough n the average energy of the second pair of kets will be greater than E. Then we can always find coefficients a and b that make the average energy of $|\psi_0\rangle$ to be E and also normalize the state. But this state has a ΔE that depends on our choice of n: in fact $\Delta E = \Theta(\sqrt{n})$. With fixed E, ΔE can be as large as we like. Thus in this case, Eq. (5) is not a useful bound and Eq. (4) is optimal.

The Unified Bound

It became generally assumed that the minimal time was given by

$$\tau_{\mathrm{QSL}} = \max\left\{\frac{\pi}{2}\frac{\hbar}{\Delta H}, \frac{\pi}{2}\frac{\hbar}{\langle H\rangle}\right\}$$

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Quantum limits to dynamical evolution

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Fundamental Limit on the Rate of Quantum Dynamics: The Unified Bound Is Tight

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(Received 16 June 2009; published 13 October 2009)

How fast a quantum state can evolve has attracted considerable attention in connection with quantum measurement and information processing. A lower bound on the orthogonalization time, based on the energy spread ΔE, was found by Mandelstam and Tamm. Another bound, based on the average energy E, was established by Margolus and Levitin. The bounds coincide and can be attained by certain initial states if $\Delta E = E$. Yet, the problem remained open when $\Delta E \neq E$. We consider the unified bound that involves both ΔE and E. We prove that there exist no initial states that saturate the bound if $\Delta E \neq E$. However, the bound remains tight: for any values of ΔE and E, there exists a one-parameter family of initial states that can approach the bound arbitrarily close when the parameter approaches its limit. These results establish the fundamental limit of the operation rate of any information processing system.

The Unified Bound

As stated by Levitin+Toffoli

However, what remained unnoticed is the paradoxical situation of the existence of two bounds based on two different characteristics of the quantum state, seemingly independent of one another. Since the average energy E and the energy uncertainty ΔE play the most determinative role in quantum evolution, it is important to have a unified bound that would take into account both of these characteristics.

$$\tau_{\mathrm{QSL}} = \max\left\{\frac{\pi}{2}\frac{\hbar}{\Delta H}, \frac{\pi}{2}\frac{\hbar}{\langle H\rangle}\right\} = \frac{\pi\hbar}{\langle H\rangle + \Delta H - |\langle H\rangle - \Delta H|}$$

$$\frac{\Delta H}{\langle H\rangle} = 1$$

We see both bounds are identical and achievable

$$\frac{\Delta H}{\langle H\rangle} \neq 1$$

L+T show that the unified bound is asymptotically attainable and therefore tight

What states actually achieve the bound(s)?

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|E_0\rangle + |E_1\rangle\right)$$

The Unified Bound

Following a similar approach to M-L they establish the M-T bound

To prove statement (i) we shall use the trigonometric inequality

$$\cos x \geq 1 - \frac{4}{\pi^2} x \sin x - \frac{2}{\pi^2} x^2, \quad (5)$$

which is valid for any real x. Note that (5) turns into an equality iff $x = 0$ or $x = \pm\pi$. Let the initial state be $|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n |E_n\rangle$, where the $|E_n\rangle$ are energy eigenstates of the system and $\sum_{n=0}^{\infty} |c_n|^2 = 1$. Then

$$\begin{aligned} |S(t)|^2 &= |\langle\psi(0)|\psi(t)\rangle|^2 \\ &= \sum_{n,n'=0}^{\infty} |c_n|^2 |c_{n'}|^2 e^{-i(E_n - E_{n'})t/\hbar} \\ &= \sum_{n,n'=0}^{\infty} |c_n|^2 |c_{n'}|^2 \cos\frac{E_n - E_{n'}}{\hbar/t}. \end{aligned} \quad (6)$$

Using inequality (5), we obtain

$$\begin{aligned} |S(t)|^2 &\geq 1 - \frac{4}{\pi^2} \sum_{n,n'=0}^{\infty} |c_n|^2 |c_{n'}|^2 \frac{E_n - E_{n'}}{\hbar/t} \sin\frac{E_n - E_{n'}}{\hbar/t} \\ &\quad - \frac{2}{\pi^2} \sum_{n,n'=0}^{\infty} |c_n|^2 |c_{n'}|^2 \left(\frac{E_n - E_{n'}}{\hbar/t}\right)^2 \\ &= 1 + \frac{4t}{\pi^2} \frac{d|S(t)|^2}{dt} - \frac{1}{\pi^2} \left(\frac{\Delta E}{\hbar/2t}\right)^2. \end{aligned} \quad (7)$$

Since $|S(t)|^2 \geq 0$, it follows that $\frac{d|S(t)|^2}{dt} = 0$ whenever $S(t) = 0$. Thus, at a time τ such that $S(\tau) = 0$, the second term in (7) vanishes, and we obtain

$$0 \geq 1 - \frac{4\tau^2}{\pi^2 \hbar^2} (\Delta E)^2, \quad (8)$$

which yields inequality (1); this is just another way to derive that bound. Yet, for (8) to turn into an equality it is necessary that inequality (5) turn into an equality for every term of the double summation (6). Hence, either $x_{nn'} = \frac{E_n - E_{n'}}{\hbar/\tau} = 0$ or $x_{nn'} = \frac{E_n - E_{n'}}{\hbar/\tau} = \pm\pi$ for all n, n' such that $c_n \neq 0$, $c_{n'} \neq 0$. It follows that, to attain bound (1), $|\psi(0)\rangle$ must be a superposition of only two energy eigenstates with energies $E_0 = 0$ and E_1.

Notice that this does not explicitly involve any uncertainty relation argument

References

S. Deffner, SC,
Quantum speed limits: from Heisenberg's uncertainty principle to optimal quantum control
J. Phys. A: Math. Theor. **50**, 453001 (2017)

L. Mandelstam and Ig. Tamm
The uncertainty relation between energy and time in nonrelativistic quantum mechanics
J. Phys. USSR **9**, 249 (1945) / I. E. Tamm Selected Papers Springer

H. P. Robertson
The uncertainty principle
Phys. Rev. **34**, 163 (1929)

N. Margolus and L. B. Levitin
The maximum speed of dynamical evolution
Physica D **120**, 188 (1998)

L. B. Levitin and T. Toffoli
Fundamental limit on the rate of quantum dynamics: The unified bound is tight
Phys. Rev. Lett. **103**, 160502 2009