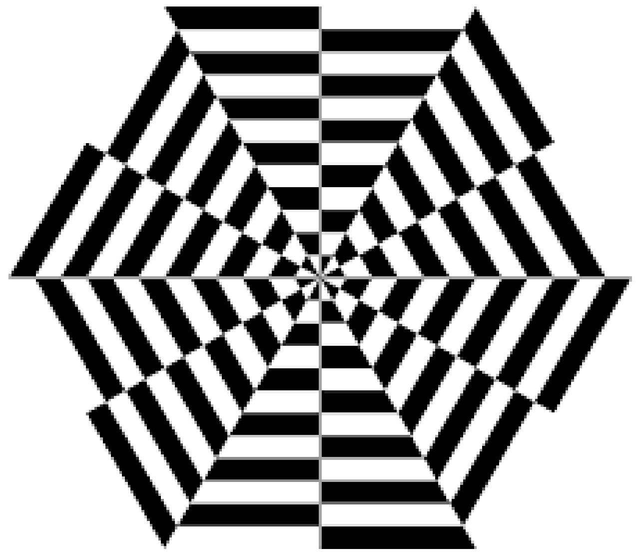
Hexagon Scattering Amplitude at the Origin



Lance Dixon (SLAC)

B. Basso, LD, G. Papathanasiou, 2001.05460

ICTP seminar, 28 April 2020

image:10binary

SLAC NATIONAL ACCELERATOR LABORATORY

Why are multi-loop QCD scattering amplitudes so hard to compute?

- Primarily because multi-loop integrals are intricate, transcendental, multi-variate functions
- In contrast, at one loop all integrals are reducible to scalar box integrals + simpler

→ combinations of dilogarithms

$$\mathrm{Li}_2(x) \;=\; -\int_0^x \frac{dt}{t}\,\ln(1-t)$$

+ logarithms and rational terms

't Hooft, Veltman (1974)

Planar N=4 SYM, toy model for QCD amplitudes

- QCD's maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, in the large N_c (planar) limit
- Structure very rigid:

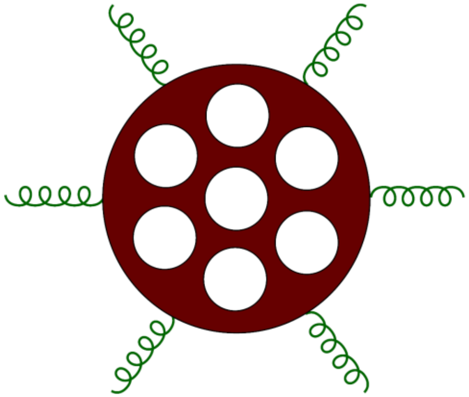
  Amplitudes = $\sum_i rational_i \times transcendental_i$
- For planar N=4 SYM, we understand rational structure quite well, focus on the transcendental functions.
- Space of functions is so restrictive, and physical constraints are so powerful, one can write L loop answer as linear combination of known weight $2L$ polylogarithms.
- Unknown coefficients found by solving (a large number of) linear constraints

Hexagon function bootstrap

Loops

3	LD, Drummond, Henn, 1108.4461, 1111.1704;
4,5	Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
6,7	Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890, 1906.07116; LD, Dulat, 20mm.nnnnn (NMHV 7 loop)

- Use analytical properties of perturbative (six) point amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops
- Step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

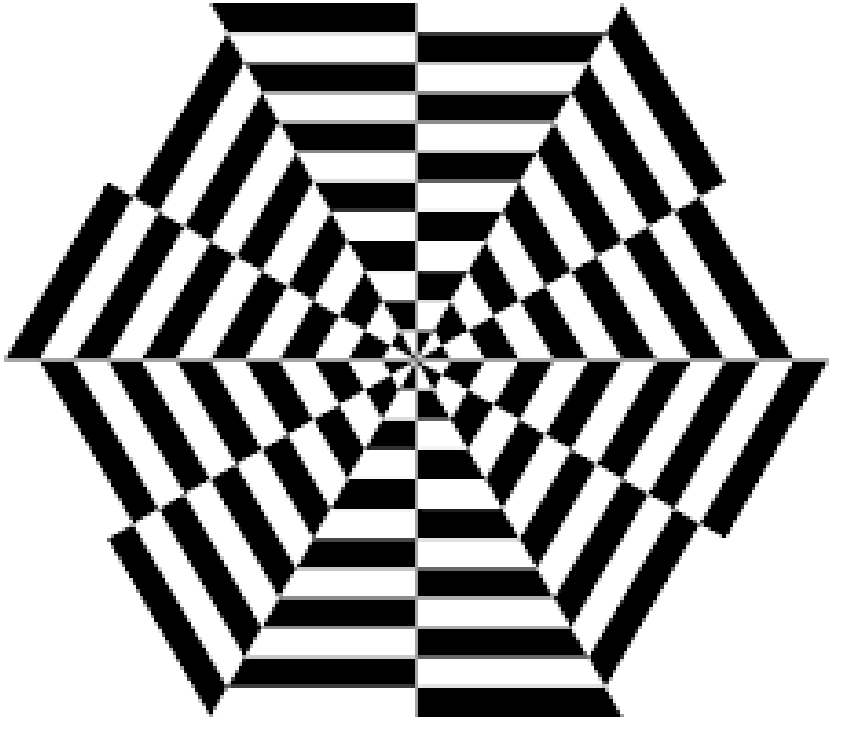


Today, we'll mainly study a kinematical limit, the origin, where we understand the amplitude nonperturbatively in terms of a "tilted BES kernel"

Outline

1. Symmetries of planar N=4 SYM
2. Other properties of (6-point) amplitudes
3. Dive to the origin & tilted BES proposal
4. Origin of results & validation
5. Conclusions & outlook

Symmetries

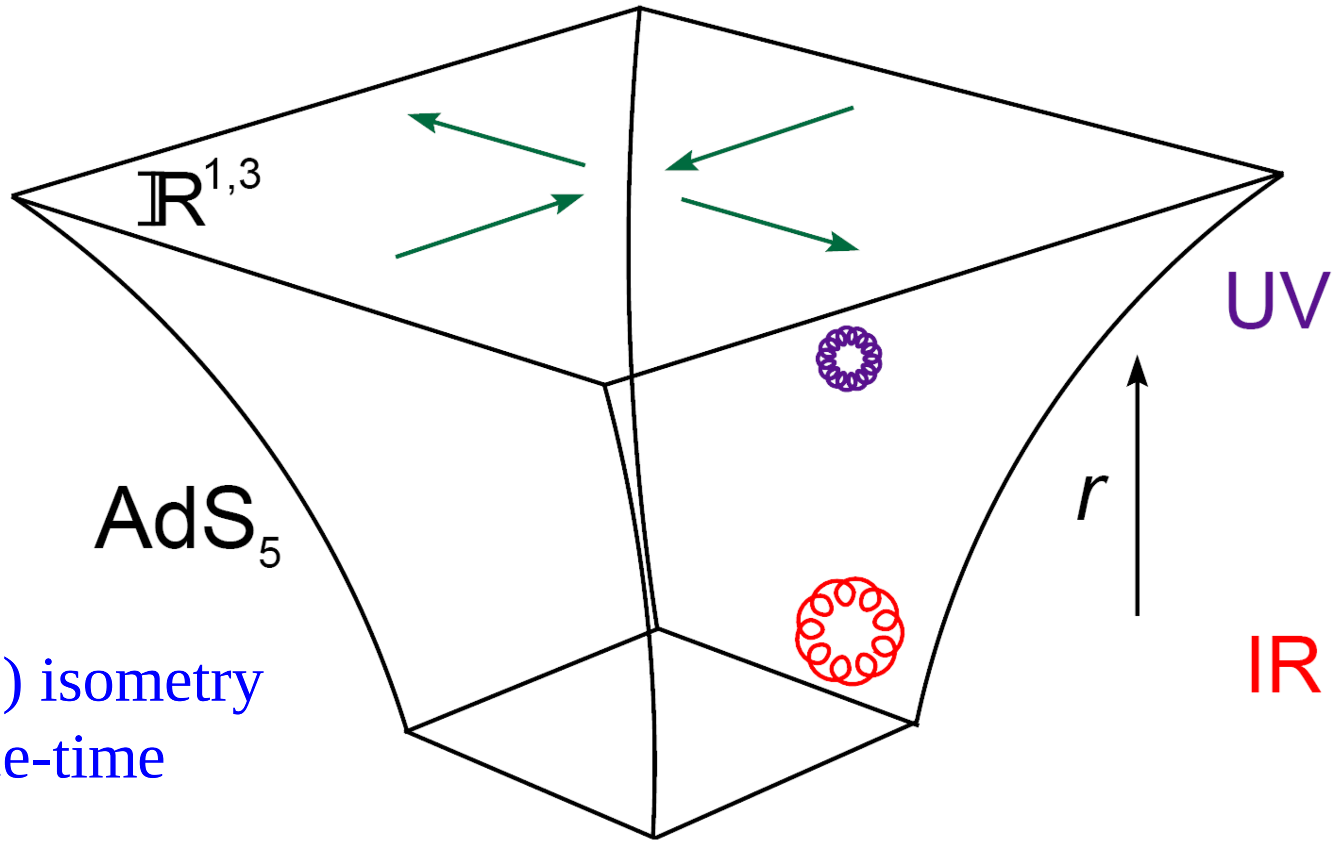


Quantum Symmetries

- Massless QCD has classical scale + conformal symmetry: $SO(3,1) \rightarrow SO(4,2)$
- Spoiled at quantum level by nonvanishing β function (asymptotic freedom).
- N=4 SYM has $\beta = 0$ → full (position space) $SO(4,2)$, actually full N=4 superconformal algebra, $PSU(2,2|4)$
- Planar N=4 SYM also has momentum-space version of $SO(4,2)$ $[PSU(2,2|4)]$
→ dual N=4 superconformal invariance

Dual conformal invariance is geometric: from AdS/CFT + T-duality

Alday, Maldacena, 0705.0303



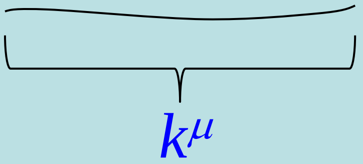
SO(4,2) isometry
of space-time

T-duality symmetry of string theory

Alday, Maldacena, 0705.0303

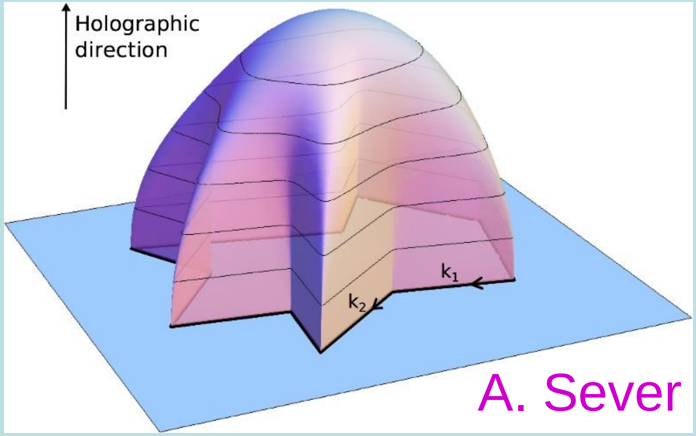
- Exchanges string world-sheet variables σ, τ
- $X^\mu(\tau,\sigma) = x^\mu + k^\mu \tau$ + oscillators

$\rightarrow$ $X^\mu(\tau,\sigma) = x^\mu + k^\mu \sigma$ + oscillators



- Strong coupling limit of planar gauge theory

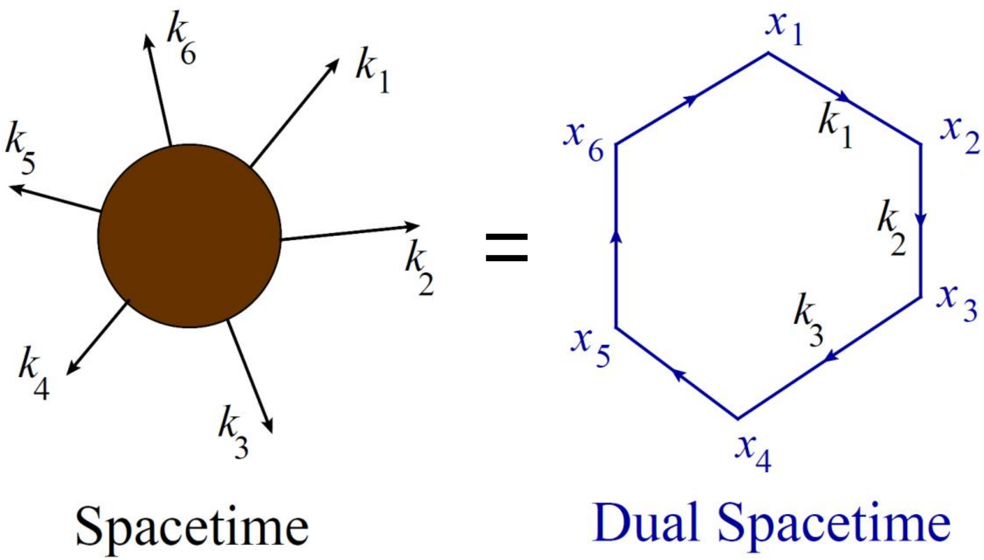
is semi-classical limit of string theory: world-sheet stretches tight around minimal area surface in AdS.



A. Sever

- Boundary determined by momenta of external states: light-like polygon with null edges = momenta k^μ

Amplitudes = Wilson loops



- Polygon vertices x_i are not positions but dual momenta,
  $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Duality verified to hold at weak coupling too!

The [Dual] Conformal Group

SO(4,2) $\supset$ SO(3,1) [rotations+boosts] + translations+dilatations + special-conformal

15 = 3 + 3 + 4 + 1 + 4

- Nontrivial generators are special conformal K^μ
- Correspond to inversion · translation · inversion
- → $f(x_{ij}^2)$ is [dual] conformally invariant if it's invariant under inversion,

$$x_i^\mu \to x_i^\mu / x_i^2$$

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}\,, \qquad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2\, x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

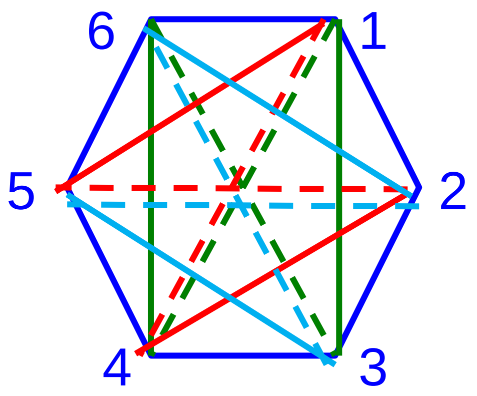
- Fixed, up to functions of invariant cross ratios:

$$\boxed{u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}}$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for n = 4,5

$n = 6$ → precisely 3 ratios:

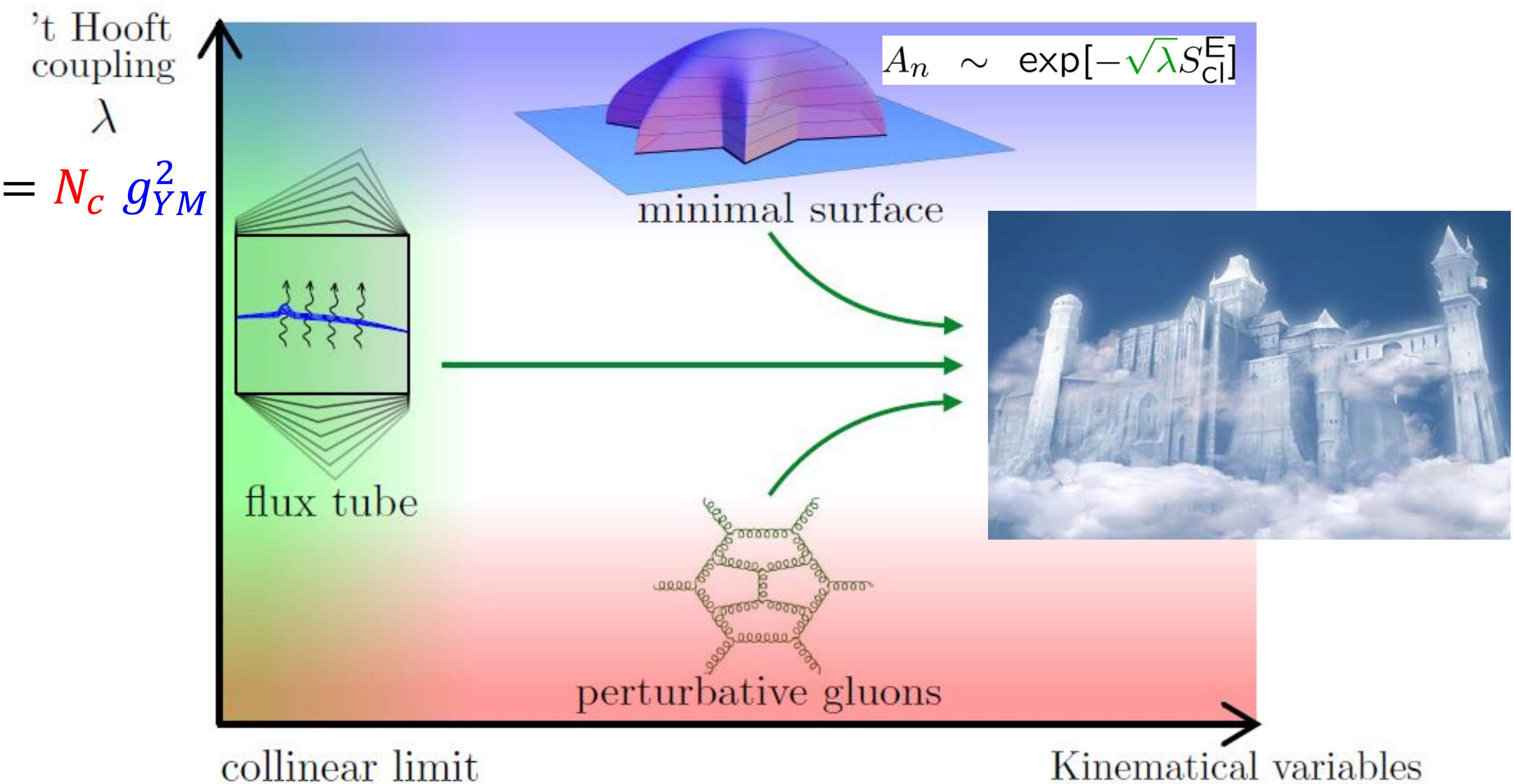
$$\begin{cases} u_1 = u = \dfrac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \dfrac{s_{12}s_{45}}{s_{123}s_{345}} \\[2ex] u_2 = v = \dfrac{s_{23}s_{56}}{s_{234}s_{123}} \\[2ex] u_3 = w = \dfrac{s_{34}s_{61}}{s_{345}s_{234}} \end{cases}$$



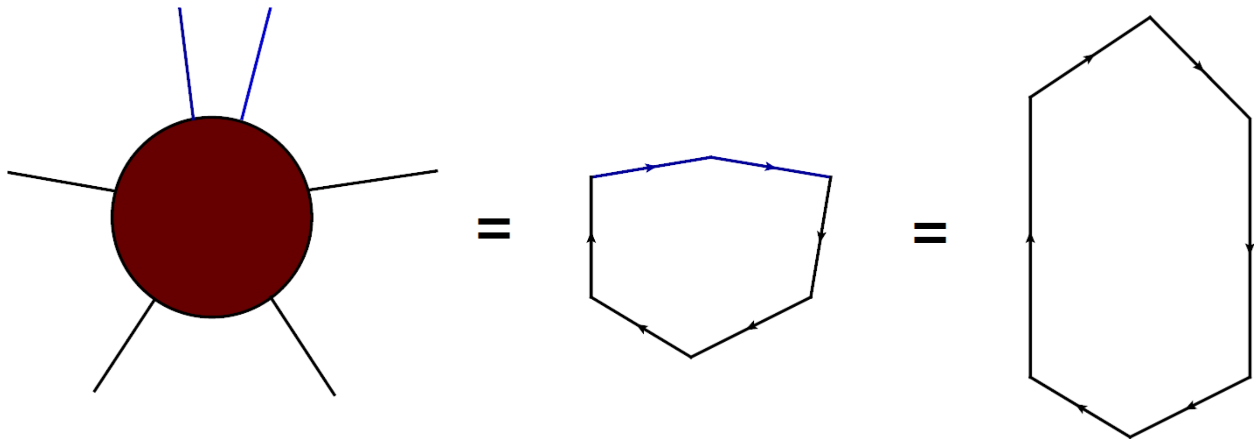
Other properties

Solving Planar N=4 SYM Scattering

Images: A. Sever, N. Arkani-Hamed

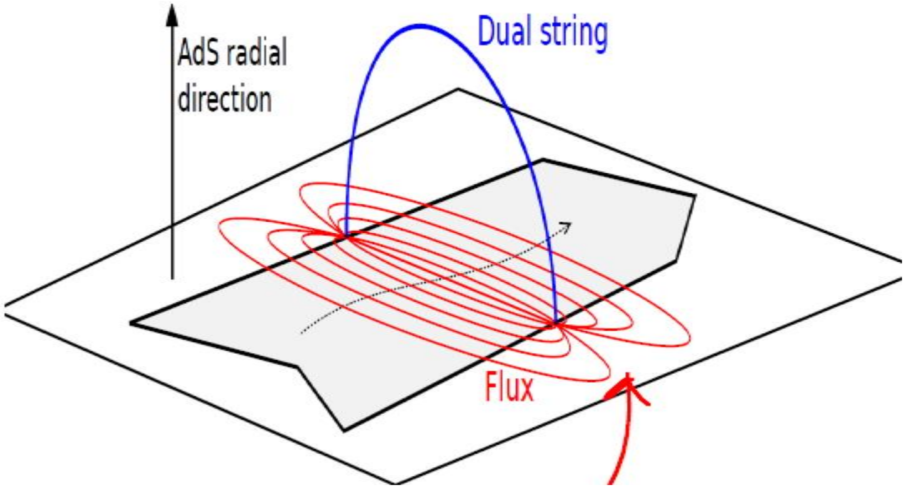


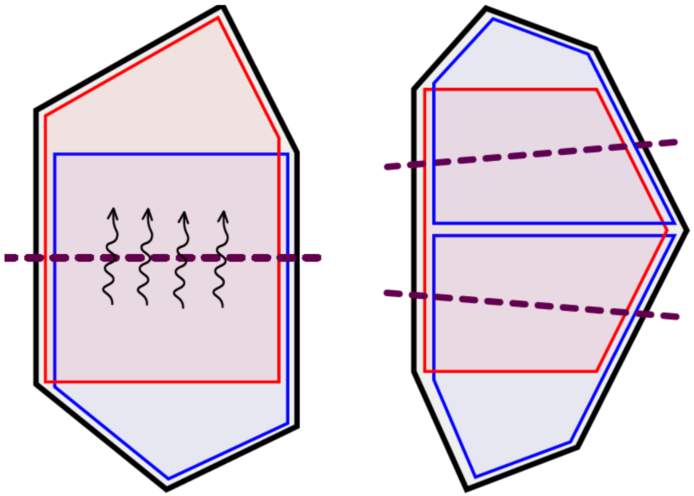
(Near) collinear (OPE) limit



Flux tubes at finite coupling

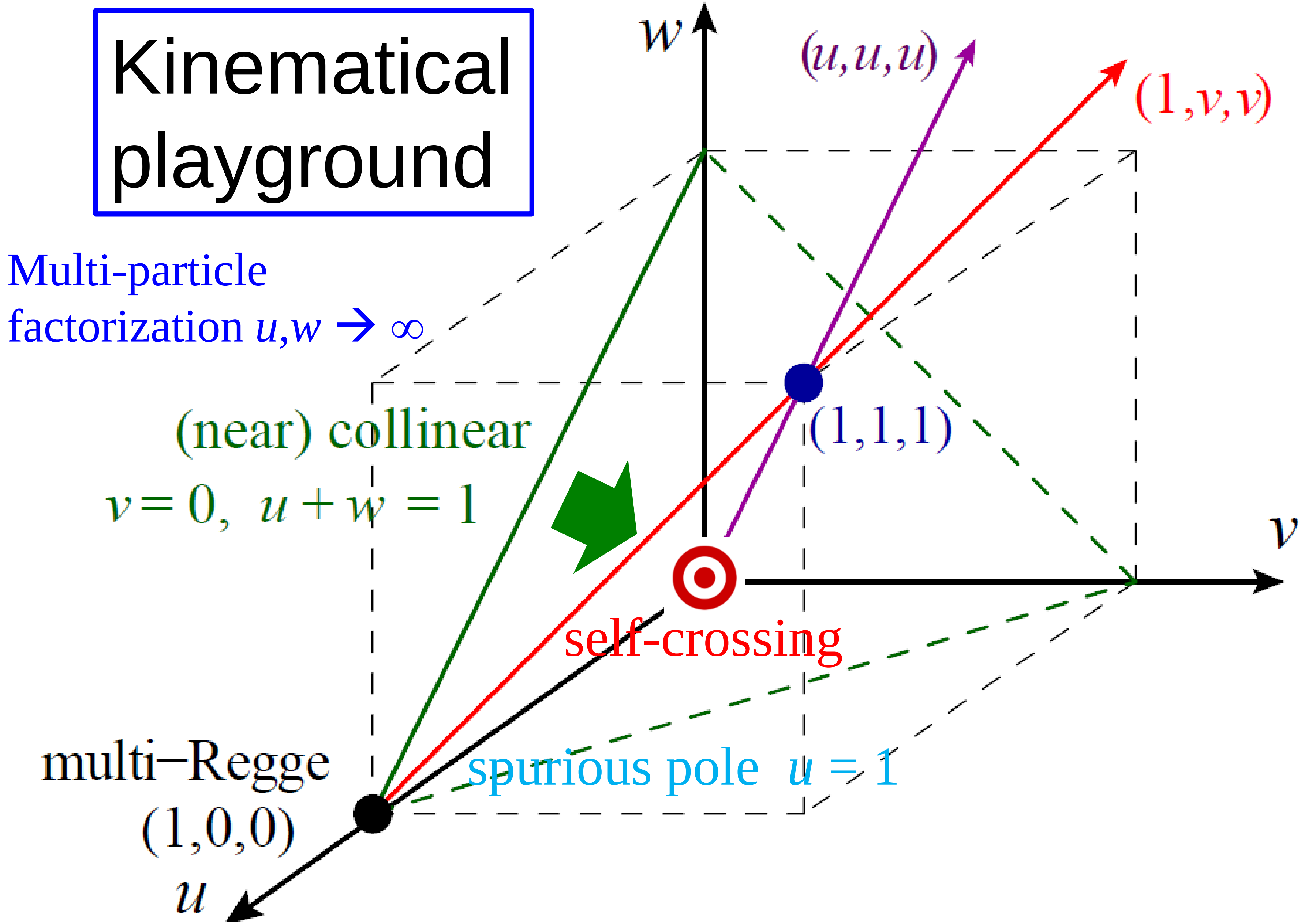
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





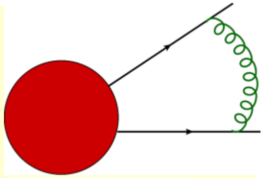
- Tile n-gon with pentagon transitions
- Quantum integrability → compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

Kinematical playground

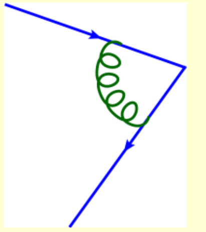


Removing Divergences

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\rm cusp}$



– known to all orders in planar N=4 SYM:
Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708

- Normalized [MHV] amplitude is finite, dual conformal invariant.
- BDS-like also maintains important relation due to causality (Steinmann).

$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\rm BDS-like}(s_{i,i+1},\epsilon)} = \exp[\mathcal{R}_6 + \frac{\Gamma_{\rm cusp}}{4}\mathcal{E}^{(1)}]$$

Properties of Amplitudes

- Having determined 6-point amplitudes to 7 loops, study their properties:
- Analytic behavior in various factorization limits.
- Simple "bulk" lines like $(u,u,1)$, $(u,1,1)$, (u,u,u).
- Singular line $(u,0,0)$, then take $u \to 0$ to approach origin.
- Planar N=4 SYM has finite radius of convergence of perturbative expansion (unlike QCD, QED, whose perturbative series are asymptotic).
- For BES solution to cusp anomalous dimension, using coupling $g^2 = \frac{\lambda}{16\pi^2}$, radius is $\frac{1}{16}$
- Ratio of successive terms $\frac{\Gamma_{\rm cusp}^{(L)}}{\Gamma_{\rm cusp}^{(L-1)}} \to -16$

At $(u,v,w) = (u,u,1)$, amplitudes → HPLs

MHV

$$\mathcal{E}^{(1)} = 2H_2 = 2\mathrm{Li}_2\left(1-\frac{1}{u}\right)$$

$$\mathcal{E}^{(2)} = -4H_4 - 8H_{2,1,1} - 4\zeta_2 H_2 - 9\zeta_4$$

$$\begin{aligned}
\mathcal{E}^{(3)} = {} & 24H_6 + 4H_{4,2} + 4H_{2,4} + 16H_{4,1,1} + 4H_{3,3} + 12H_{2,2,2} + 8H_{3,2,1} + 8H_{3,1,2} \\
& + 8H_{2,3,1} + 8H_{2,1,3} + 16H_{2,2,1,1} + 16H_{2,1,2,1} + 16H_{2,1,1,2} + 96H_{2,1,1,1,1} \\
& + \zeta_2(8H_4 + 8H_{2,2} + 48H_{2,1,1}) + 42\zeta_4 H_2 + 121\zeta_6
\end{aligned}$$

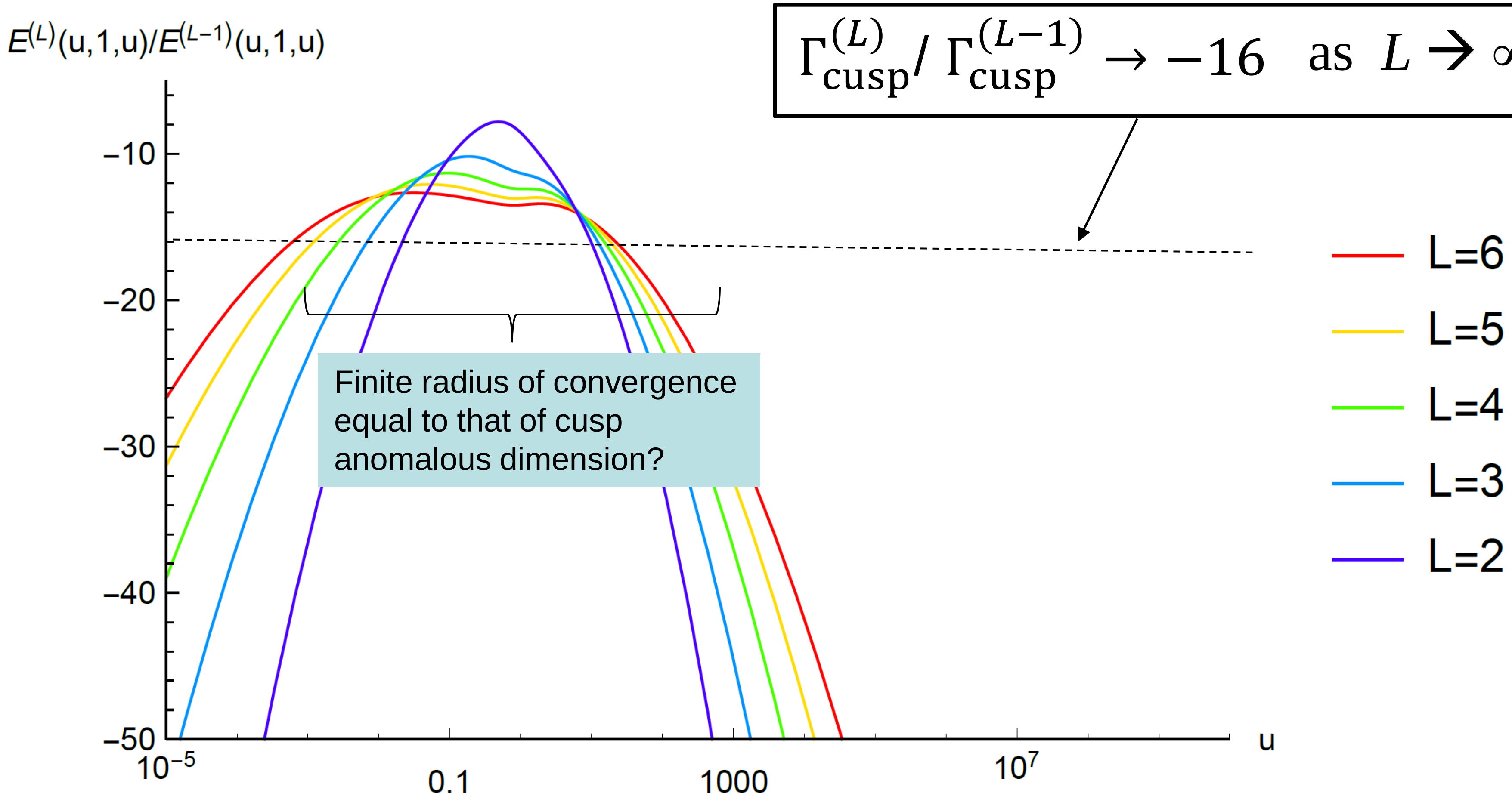
$$\begin{aligned}
\mathcal{E}^{(4)} = {} & -240H_8 - 40H_{6,2} - 40H_{5,3} - 40H_{4,4} - 48H_{3,5} - 48H_{2,6} - 64H_{6,1,1} - 32H_{5,2,1} \\
& - 32H_{5,1,2} - 40H_{2,1,5} - 32H_{2,5,1} - 32H_{4,3,1} - 32H_{3,4,1} - 32H_{3,1,4} - 32H_{4,1,3} \\
& - 40H_{4,2,2} - 40H_{2,2,4} - 32H_{2,4,2} - 48H_{4,2,1,1} - 48H_{4,1,2,1} - 48H_{4,1,1,2} - 16H_{2,4,1,1} \\
& - 16H_{2,1,4,1} - 24H_{2,1,1,4} - 192H_{4,1,1,1,1} - 32H_{3,3,2} - 32H_{3,2,3} - 32H_{2,3,3} \\
& - 32H_{3,1,2,2} - 32H_{2,3,2,1} - 32H_{2,3,1,2} - 32H_{3,2,2,1} - 32H_{3,2,1,2} - 16H_{2,2,3,1} \\
& - 16H_{2,2,1,3} - 24H_{2,1,3,2} - 8H_{2,1,2,3} - 16H_{3,3,1,1} - 16H_{3,1,3,1} - 16H_{3,1,1,3} \\
& - 96H_{3,2,1,1,1} - 96H_{3,1,2,1,1} - 96H_{3,1,1,2,1} - 96H_{3,1,1,1,2} - 96H_{2,3,1,1,1} - 64H_{2,1,3,1,1} \\
& - 48H_{2,1,1,3,1} - 32H_{2,1,1,1,3} - 40H_{2,2,2,2} - 144H_{2,2,2,1,1} - 144H_{2,2,1,2,1} \\
& - 144H_{2,2,1,1,2} - 128H_{2,1,2,2,1} - 128H_{2,1,2,1,2} - 128H_{2,1,1,2,2} - 384H_{2,2,1,1,1,1} \\
& - 384H_{2,1,2,1,1,1} - 384H_{2,1,1,2,1,1} - 384H_{2,1,1,1,2,1} - 384H_{2,1,1,1,1,2} - 1920H_{2,1,1,1,1,1,1} \\
& - \zeta_2(48H_{3,1,2} + 48H_{2,3,1} + 32H_{2,1,3} + 192H_{2,2,1,1} + 192H_{2,1,2,1} + 192H_{2,1,1,2} + 960H_{2,1,1,1,1} \\
& \qquad + 32H_6 + 24H_{4,2} + 8H_{2,4} + 96H_{4,1,1} + 8H_{3,3} + 72H_{2,2,2} + 48H_{3,2,1}) \\
& - \zeta_4(84H_4 + 168H_{2,2} + 816H_{2,1,1}) - 8(5\zeta_5 - 2\zeta_2\zeta_3)H_{2,1} - 425\zeta_6 H_2 \\
& - \frac{6381}{4}\zeta_8 + 24(\zeta_{5,3} + 5\zeta_3\zeta_5 - \zeta_2\zeta_3^2)
\end{aligned}$$

$$H_{\vec{w}} \equiv H_{\vec{w}}\left(1-\frac{1}{u}\right)$$

+ 5, 6, 7, loop results...

NMHV Amplitude on $(u,u,1)$

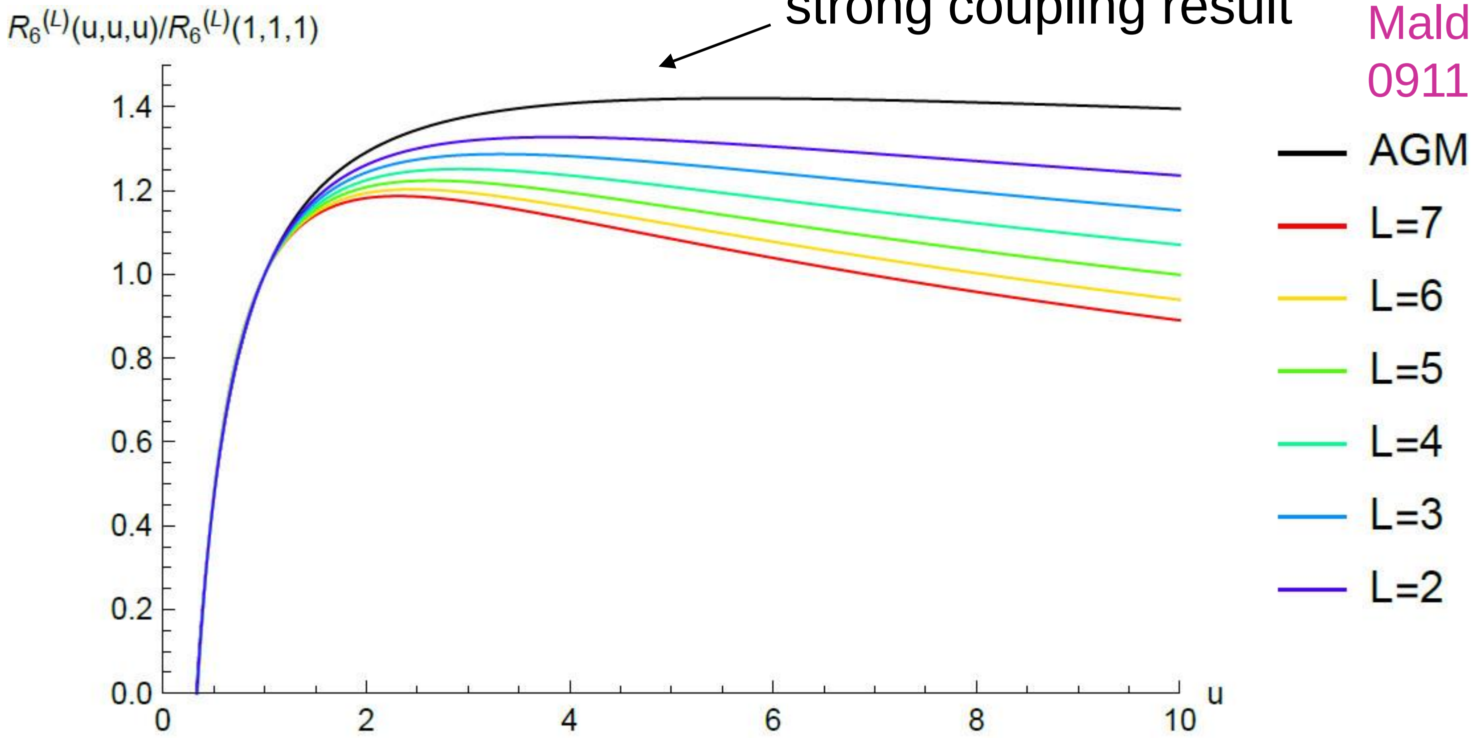
$\Gamma_{\rm cusp}^{(L)} / \Gamma_{\rm cusp}^{(L-1)} \to -16$ as $L \to \infty$



Remainder function on (u,u,u)

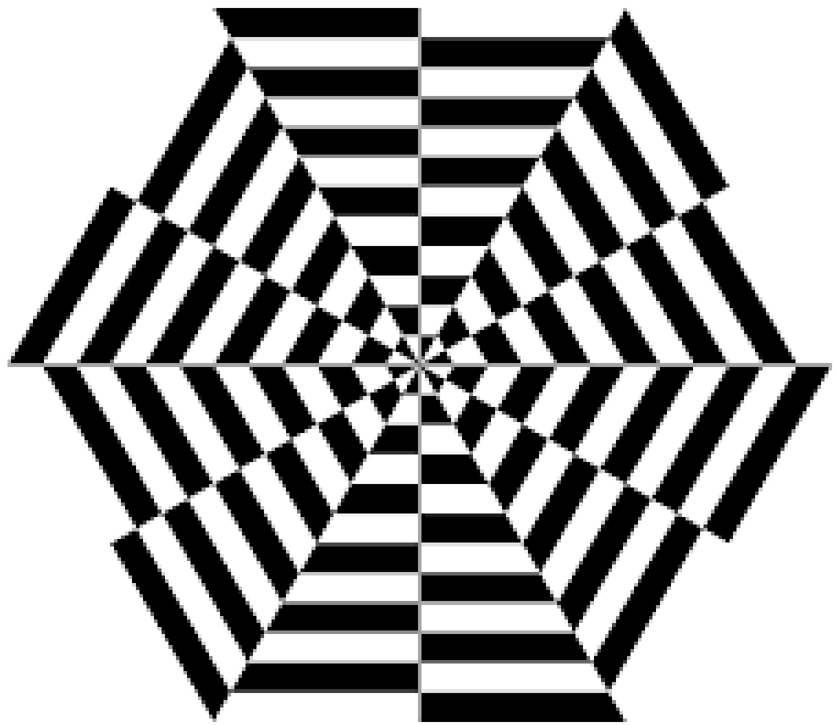
strong coupling result

Alday, Gaiotto, Maldacena, 0911.4708



- **Amazing proportionality** of **each** perturbative coefficient at small u, also with the strong coupling result.
- Suggests we should take all $u_i \rightarrow 0$

Dive to the origin



Weak coupling at origin

- Remarkably, $\ln \mathcal{E}$ is quadratic in logarithms through 7 loops CDDvHMP, 1903.10890
- Previously observed through 2 loops, and at strong coupling, on the diagonal (u,u,u) AGM, 0911.4708

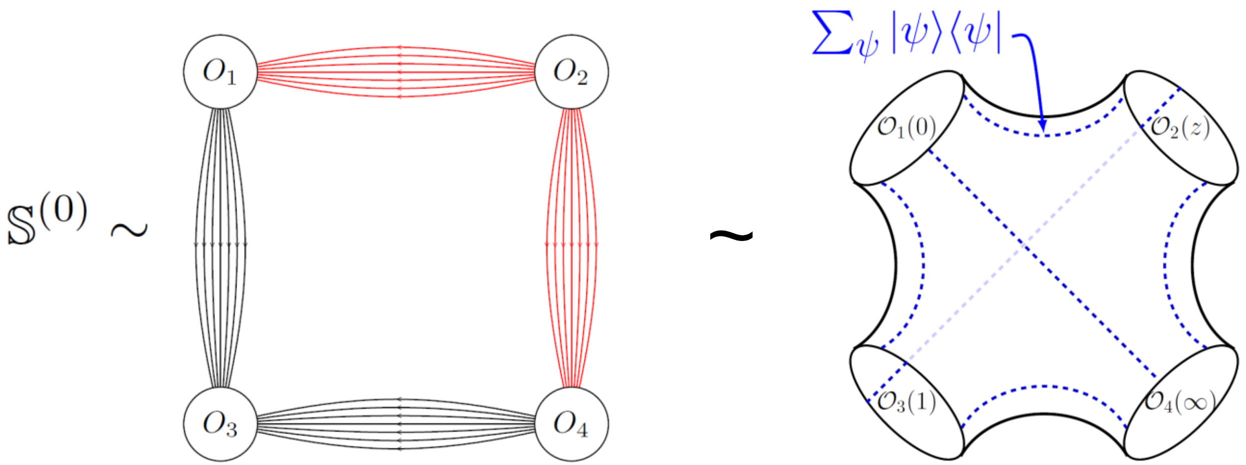
$$\boxed{\ln \mathcal{E}(u_i) \approx -\frac{\Gamma_{\rm oct}}{24}\ln^2(u_1 u_2 u_3) - \frac{\Gamma_{\rm hex}}{24}\sum_{i=1}^{3}\ln^2\frac{u_i}{u_{i+1}} + C_0}$$

	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
$\Gamma_{\rm oct}$	4	$-16\zeta_2$	$256\zeta_4$	$-3264\zeta_6$	$\frac{126976}{3}\zeta_8$
$\Gamma_{\rm cusp}$	4	$-8\zeta_2$	$88\zeta_4$	$-876\zeta_6 - 32\zeta_3^2$	$\frac{28384}{3}\zeta_8 + 128\zeta_2\zeta_3^2 + 640\zeta_3\zeta_5$
$\Gamma_{\rm hex}$	4	$-4\zeta_2$	$34\zeta_4$	$-\frac{603}{2}\zeta_6 - 24\zeta_3^2$	$\frac{18287}{6}\zeta_8 + 48\zeta_2\zeta_3^2 + 480\zeta_3\zeta_5$
C_0	$-3\zeta_2$	$\frac{77}{4}\zeta_4$	$-\frac{4463}{24}\zeta_6 + 2\zeta_3^2$	$\frac{67645}{32}\zeta_8 + 6\zeta_2\zeta_3^2 - 40\zeta_3\zeta_5$	$-\frac{4184281}{160}\zeta_{10} - 65\zeta_4\zeta_3^2 - 120\zeta_2\zeta_3\zeta_5 + 228\zeta_5^2 + 420\zeta_3\zeta_7$

Mysterious octagon connection

- Remarkably, $\Gamma_{\text{oct}} = \frac{2}{\pi^2} \ln\cosh(2\pi g)$

  recently appeared in light-like limit of correlator of 4 large R-charge operators, dubbed the octagon
  Coronado, 1811.00467, 1811.03282; Kostov, Petkova, Serban, 1903.05038; Belitsky, Korchemsky, 1907.13131, 2003.01121; Bargheer, Coronado, Vieira, 1904.00965, 1909.04077



BES Kernel

Beisert, Eden, Staudacher, hep-th/0610251

- Plays a critical role in describing a spinning string, or equivalently, twist two operators in planar N=4 SYM.
  Basso, 1010.5237
- Integral equation for spin fluctuation density $\sigma(t)$ with magic kernel $K(t,t')$:

$$\frac{e^t-1}{t}\sigma(t) = K(2gt,0) - 4g^2\int_0^\infty dt' K(2gt,2gt')\sigma(t')$$

- Solution provides $\Gamma_{\mathrm{cusp}}(g^2) = 8g^2\sigma(0)$
- Expanding in Bessel functions,

Benna, Benvenuti, Klebanov, Scardicchio, hep-th/0611135

equivalent to inverting a semi-infinite matrix,

$$\Gamma_{\mathrm{cusp}}(g^2) = 4g^2\left[\frac{1}{1+\mathbb{K}}\right]_{11} \qquad \mathbb{K}_{ij} = 2j(-1)^{ij+j}\int_0^\infty \frac{dt}{t}\frac{J_i(2gt)J_j(2gt)}{e^t-1}$$

Our Tilted BES Proposal

- Write $\mathbb{K}_{ij}$ in 2 x 2 block form, according to whether i, j are odd/even:

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{\circ\circ} & \mathbb{K}_{\circ\star} \\ \mathbb{K}_{\star\circ} & \mathbb{K}_{\star\star} \end{bmatrix}$$

- Introduce "tilt angle" $\alpha = 0, \ \frac{\pi}{4}, \ \frac{\pi}{3}$

for oct, cusp, hex

$$\boxed{\mathbb{K}(\alpha) \ = 2\cos\alpha \begin{bmatrix} \cos\alpha \ \mathbb{K}_{\circ\circ} & \sin\alpha \ \mathbb{K}_{\circ\star} \\ \sin\alpha \ \mathbb{K}_{\star\circ} & \cos\alpha \ \mathbb{K}_{\star\star} \end{bmatrix}}$$

- Then $\Gamma_\alpha(g^2) = 4g^2 [\frac{1}{1 + \mathbb{K}(\alpha)}]_{11}$

Constants are determinants

- We also find that

$$C_0 = -D\left(\frac{\pi}{3}\right) - \frac{1}{2} D(0) + \cancel{D\left(\frac{\pi}{4}\right) - \frac{\zeta_2}{2}\Gamma_{\mathrm{cusp}}}$$

where

$$D(\alpha) = \mathrm{lndet}[1 + \mathbb{K}(\alpha)]$$

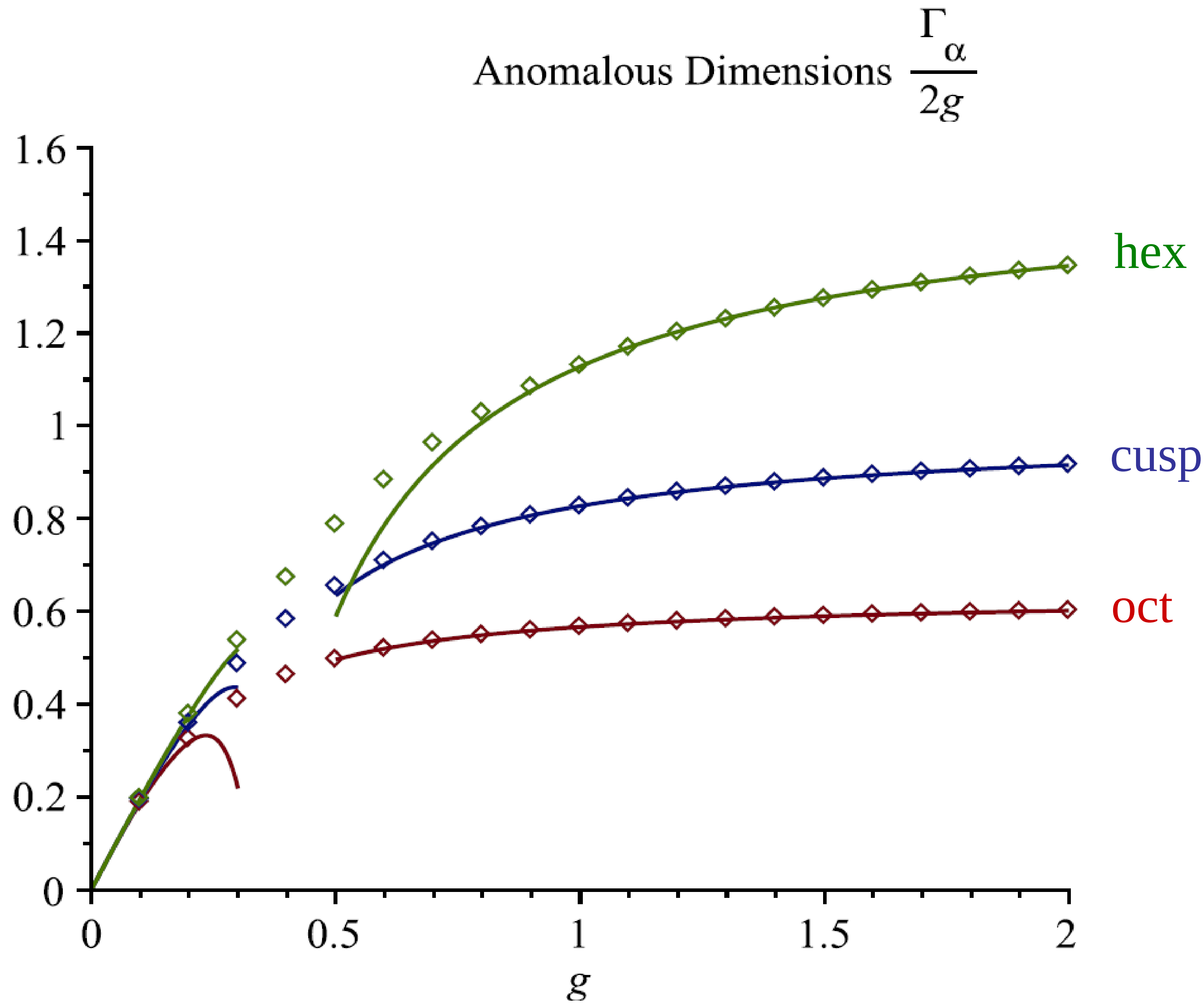
- A number-theoretic "coaction principle" Schnetz, 1302.6445; Panzer, Schnetz, 1603.04289; Brown, 1512.06409

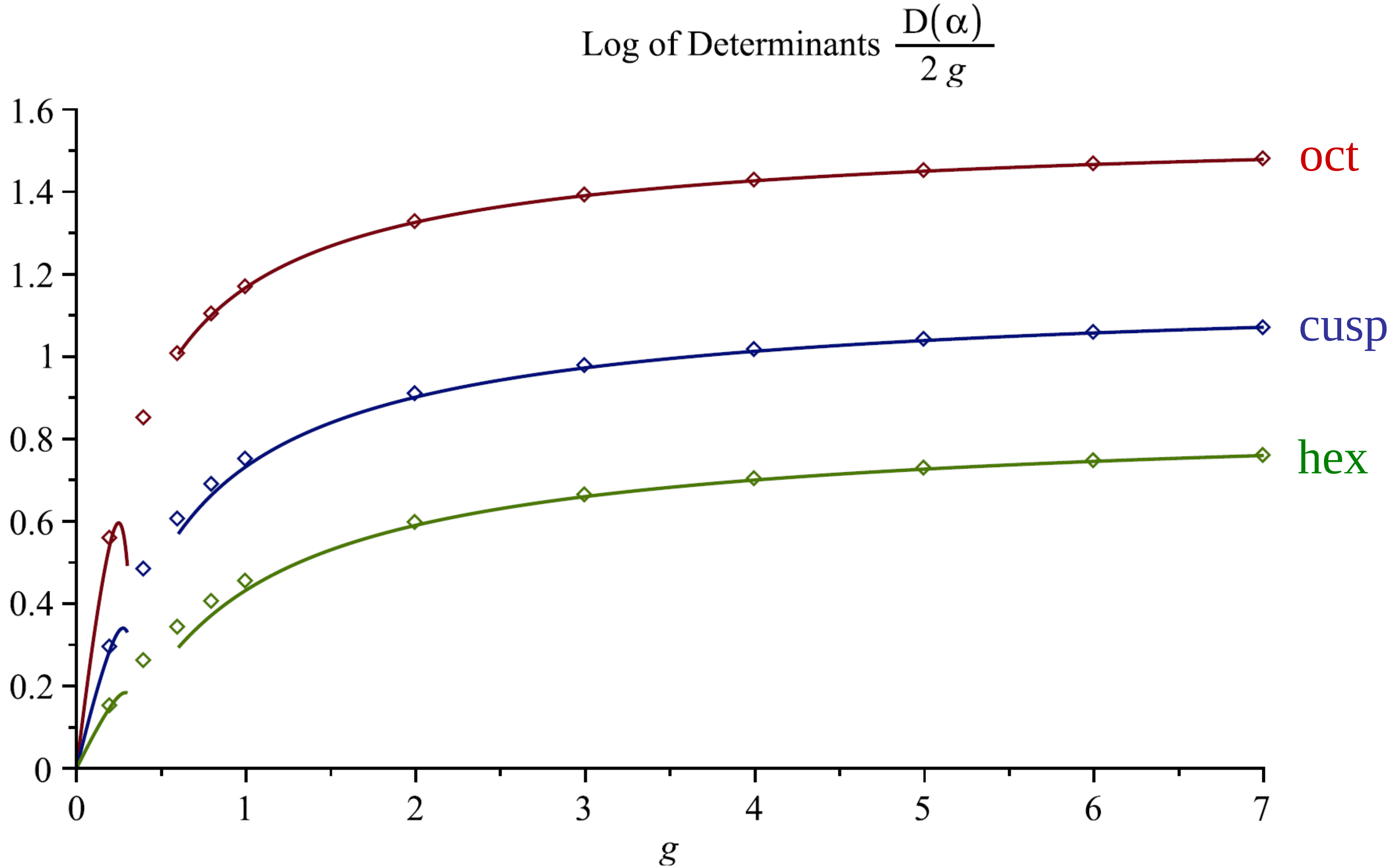
suggests a best ("cosmic") normalization for amplitude:

$\ln \mathcal{E} \to \ln \mathcal{E} - \ln \rho$, and through 7 loops [CDDvHMP, 1906.07116]

$$\ln \rho^{\mathrm{new}} = D\left(\frac{\pi}{4}\right) - \frac{\zeta_2}{2}\Gamma_{\mathrm{cusp}} = \ln \rho^{\mathrm{old}} - \zeta_4 g^4 + \frac{50}{3}\zeta_6 g^6 - \frac{483}{2}\zeta_8 g^8 + \cdots$$

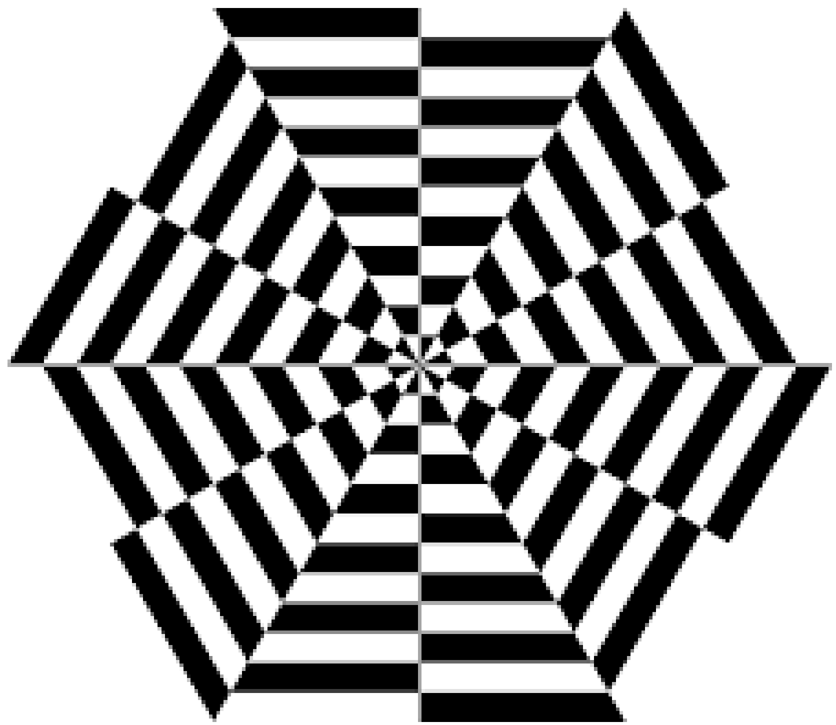
- In this normalization, only $\alpha = 0, \frac{\pi}{3}$ enter hexagon!


Anomalous Dimensions $\frac{\Gamma_\alpha}{2g}$
hex
cusp
oct
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
g

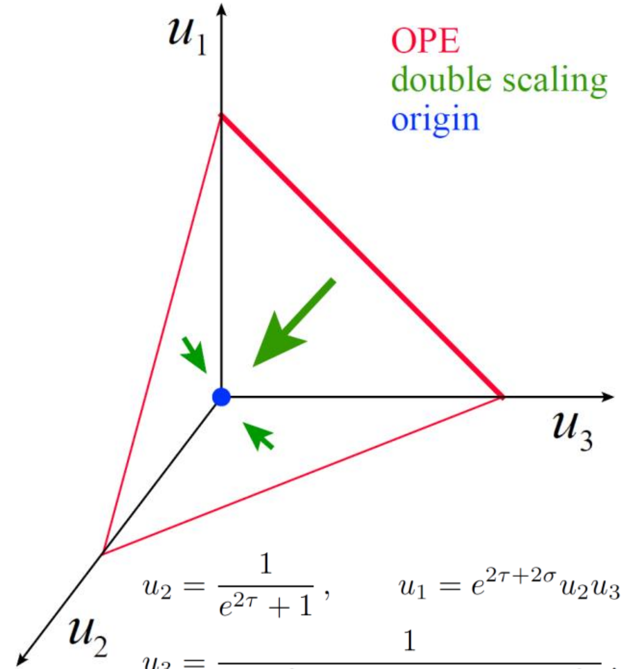


Log of Determinants $\frac{D(\alpha)}{2g}$
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
g
oct
cusp
hex

Origin of results & validation



Origin of the results



$$u_2 = \frac{1}{e^{2\tau}+1}, \qquad u_1 = e^{2\tau+2\sigma} u_2 u_3$$

$$u_3 = \frac{1}{1+e^{2\sigma}+2e^{\sigma-\tau}\cos\phi + e^{-2\tau}}.$$

- To approach origin via pentagon OPE, must sum over large number N of large helicity a_k gluonic bound state flux tube excitations.
- Framed Wilson loop:

$$\tau \to \infty$$

$$\varphi \equiv i\phi \to \infty$$

$$\mathcal{W}_6 = \mathcal{E} \times \exp[\frac{\Gamma_{\rm cusp}}{2}(\sigma^2+\tau^2+\zeta_2)]$$

gluonic contribution:

rapidity, energy, momentum, measure

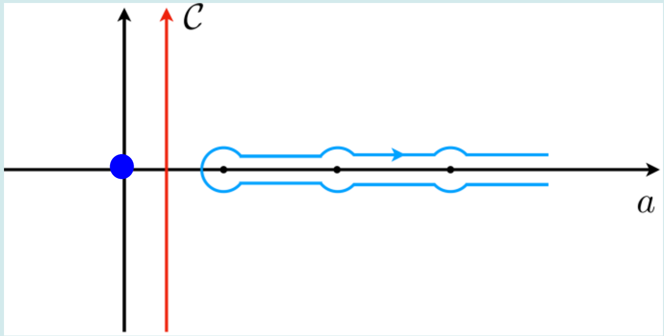
$$\mathcal{W}_6 = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\mathbf{a}} e^{i\phi \sum_{k=1}^{N} a_k} \int \frac{d\mathbf{u}}{(2\pi)^N} \frac{e^{-\tau E + i\sigma P} \prod_k \mu_k}{\prod_{k<l} P_{kl} P_{lk}}$$

pentagon transition

Weak coupling

- Expand E, P, μ_k, P_{kl} in g.
- N excitation contribution only starts at N^2 loops, so can get to 8 loops (9 loops for log) with only 2 excitations.
- Large a_k → Sommerfeld-Watson transform

$$\sum_{a \geqslant 1} (-1)^a f(a) \to \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{i f(a) da}{2 \sin(\pi a)}$$



- Deform a integral to $a = 0$ residue (after doing u integrals)
- Agrees with full amplitude limit, goes to 8 (9) loops

Finite coupling

- To simplify E, p, μ_k, P_{kl} , analytically continue $\mathbf{u}$ to "Goldstone sheet" Basso, Sever, Vieira, 1407.1736

… → $$\mathcal{E} = \mathcal{N} \int \prod_{i=1}^{\infty} d\xi_i^+ d\xi_i^- F_\varphi(\vec{\xi}\,)\, e^{-\vec{\xi}\cdot M\cdot\vec{\xi}}$$ $M \sim 1 + \mathbb{K}$

- $\vec{\xi}$ is conjugate to charges $\vec{Q} = \sum_{k=1}^{N} \vec{q}\,(u_k, a_k)$
- F_φ is Fredholm determinant,

$$\ln F_\varphi = -\sum_{N\geqslant 1} \frac{1}{N} \sum_{\mathbf{a}} \oint \frac{d\mathbf{u}}{(2\pi)^N} \prod_{k=1}^{N} \frac{\hat{\mu}_k e^{\varphi a_k}}{x_k^+ - x_{k+1}^-}\, e^{2i\vec{Q}\cdot\vec{\xi}}$$

A Secretly Gaussian Integral

- At weak coupling, $Q_i \sim g^i$, and can expand

$$\ln F_\varphi(\vec{\xi}) = \langle 1\rangle + 2i\langle Q_i^m\rangle \xi_i^m - 2\langle Q_i^m Q_j^n\rangle \xi_i^m \xi_j^n + \ldots$$

- All moments > 2 vanish as $\varphi \to \infty$ (!):

$$\lim_{\varphi\to\infty} \langle Q_i^m Q_j^n\, Q_k^p \ldots\rangle = 0$$

- Also compute $\langle 1\rangle$, $\langle\vec{Q}\rangle$, $\langle\overleftrightarrow{QQ}\rangle$ explicitly through 4 loops, extrapolate by writing in terms of $\mathbb{K}(\alpha)$
- Leads to our finite-coupling proposals.

Validation

- All formulas agree with weak coupling expansions through 8 or 9 loops
- Strong coupling limit tested against string theory: area of minimal surface Alday, Maldacena, 0705.0303
  plus constant from determinant of scalars for S^5 in $\mathrm{AdS}_5 \times \mathrm{S}^5$ Basso, Sever, Vieira, 1405.6350
- On diagonal, area can be computed analytically:
  Alday, Gaiotto, Maldacena, 0911.4708

$$\left.\frac{\ln \mathcal{E}(u,u,u)}{\Gamma_{\text{cusp}}}\right|_{g\to\infty} = -\frac{3}{4\pi}\ln^2 u - \frac{\pi^2}{12} - \frac{\pi}{6} + \frac{\pi}{72}$$

- Agrees perfectly with strong coupling limit of C_0

Conclusions & Outlook

- Planar N=4 SYM scattering amplitudes/Wilson Loops determined to high loop order by writing linear combination of right functions and imposing boundary constraints
- Rich information about many different kinematic limits
- Along with pentagon OPE, leads to understanding of finite-coupling behavior at **origin**, $u,v,w \sim 0$, in terms of tilted BES kernel
- Three anomalous dimensions and three determinants, all with similar analytic structure and behavior.
- Next origin challenges:
  - analogous kinematics for 7 gluons (new α values?)
  - NMHV at origin
  - interpolation between origin and near-collinear limits

Extra Slides

Bootstrappers' Master Table

(MHV,NMHV): parameters left in $(\mathcal{E}^{(L)},\ E^{(L)} \& \tilde{E}^{(L)})$

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. All functions	(6,6)	(25,27)	(92,105)	(313,372)	(991,1214)	(2951,3692?)
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear limit	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^*)$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. all MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. $T^2F^2\ln^4 T$ OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
12. all T^2F^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

(0,0) → amplitude uniquely determined

Also $L=7$

or use origin

BDS-like ansatz

$$\frac{\mathcal{A}_6^{\text{BDS-like}}}{\mathcal{A}_6^{\text{MHV}\,(0)}} = \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon) \frac{1}{2} \hat{M}_6(L\epsilon) + C^{(L)} \right)\right]$$

where $f^{(L)}(\epsilon) = \frac{1}{4}\gamma_K^{(L)} + \epsilon\, \frac{L}{2}\mathcal{G}_0^{(L)} + \epsilon^2\, f_2^{(L)}$ are constants, and

$$\begin{aligned} \hat{M}_6(\epsilon) &= M_6^{1-\text{loop}}(\epsilon) + Y(u,v,w) \\ &= \sum_{i=1}^{6} \left[-\frac{1}{\epsilon^2}(1 - \epsilon \ln s_{i,i+1}) - \ln s_{i,i+1} \ln s_{i+1,i+2} + \frac{1}{2} \ln s_{i,i+1} \ln s_{i+3,i+4} \right] + 6\zeta_2 \end{aligned}$$

- Y is dual conformally invariant part of one-loop amplitude $M_6^{1-\text{loop}}$ containing all 3-particle invariants:

$$Y(u,v,w) = -\mathcal{E}^{(1)} = -\text{Li}_2\left(1 - \frac{1}{u}\right) - \text{Li}_2\left(1 - \frac{1}{v}\right) - \text{Li}_2\left(1 - \frac{1}{w}\right)$$

- More minimal BDS-like ansatz contains all IR poles, but **no 3-particle invariants**.

A little number theory

- Classical polylogs

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

evaluate to Riemann zeta values

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.
  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$
- At the origin, no MZV's

At $(u,v,w) = (1,1,1)$, amplitude $\rightarrow$ MZVs

Allowed MZV's obey a Galois "co-action" principle, restricting the combinations that can appear
Brown, Panzer, Schnetz

MHV

$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] - 96\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

NMHV

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = -\frac{36271}{9}\,\zeta_8 - 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] + 132\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

Cosmic normalization

- To fit amplitudes into the minimal space of functions requires, starting at 3 loops, redefinining the BDS-like ansatz, by a multi-loop constant ρ :

$$\mathcal{A}_6^{\mathrm{BDS-like}'} = \mathcal{A}_6^{\mathrm{BDS-like}} \times \rho$$

$$\begin{aligned}
\rho(g^2) = 1 + 8(\zeta_3)^2\, g^6 - 160\zeta_3\zeta_5\, g^8 + \Big[1680\zeta_3\zeta_7 + 912(\zeta_5)^2 - 32\zeta_4(\zeta_3)^2\Big]\, g^{10} \\
- \Big[18816\zeta_3\zeta_9 + 20832\zeta_5\zeta_7 - 448\zeta_4\zeta_3\zeta_5 - 400\zeta_6(\zeta_3)^2\Big]\, g^{12} \\
+ \Big[221760\zeta_3\zeta_{11} + 247296\zeta_5\zeta_9 + 126240(\zeta_7)^2 - 3360\zeta_4\zeta_3\zeta_7 - 1824\zeta_4(\zeta_5)^2 \\
- 5440\zeta_6\zeta_3\zeta_5 - 4480\zeta_8(\zeta_3)^2\Big]\, g^{14} \; + \; \mathcal{O}(g^{16}).
\end{aligned}$$

- Now we have a flux tube interpretation for ρ !

Branch cut condition

• All massless particles → all branch cuts start at origin in

$$s_{i,i+1}, \quad s_{i,i+1,i+2}$$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad \text{or } v \text{ or } w$$

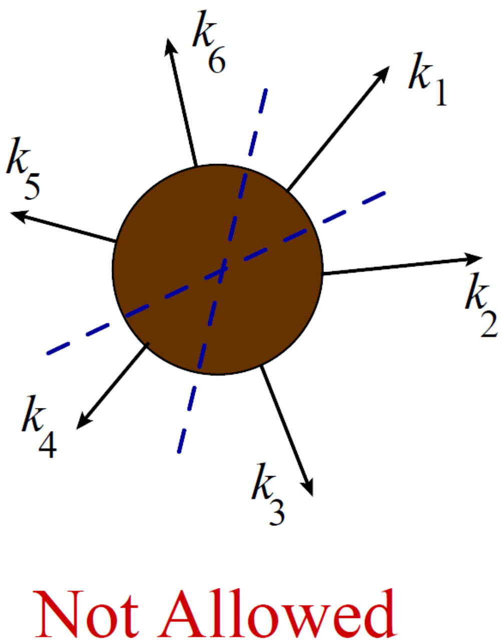
→ Only 3 weight 1 functions, not 9: $\{ \ln u, \ln v, \ln w \}$

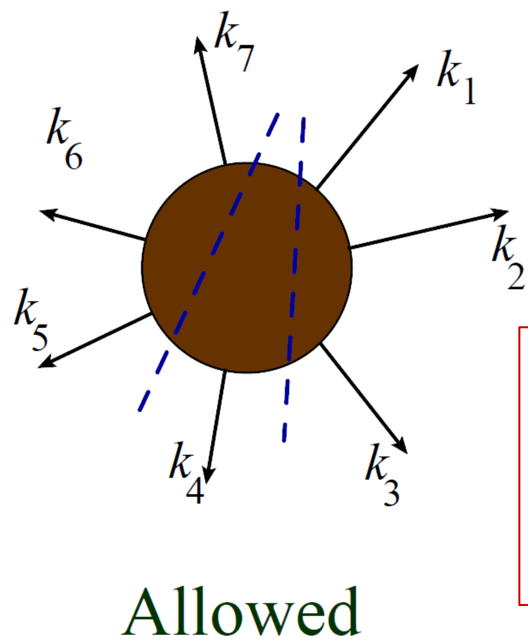
- Discontinuities commute with branch cuts
- Require derivatives of higher weight functions to obey branch-cut condition too.
- Powerful constraint: At weight 8 (four loops) would have 1,675,553 functions without it; exactly 6,916 with it.
- But almost all of the 6,916 functions are still unphysical.

Steinmann relations

Steinmann, Helv. Phys. Acta (1960) Bartels, Lipatov, Sabio Vera, 0802.2065

- Amplitudes should not have overlapping branch cuts:





can't apply to 2 particle cuts in massless case because they are not independent

$$\boxed{\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0}$$

Violated by ABDK and BDS ansatz!

Steinmann relations

S. Caron-Huot, LD, M. von Hippel, A. McLeod, 1609.00669

$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0$$

+ cyclic conditions

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \qquad w = \frac{s_{61}s_{34}}{s_{345}s_{234}}$$

$$\frac{uv}{w} = \frac{s_{12}s_{23}s_{45}s_{56}}{s_{34}s_{61}s_{123}^2}$$

$\ln^2 u$	$\ln^2 \frac{uv}{w}$
NO	**OK**

Weight 2 functions restricted to 6 out of 9:

$$\mathrm{Li}_2(1-1/u) \quad \mathrm{Li}_2(1-1/v) \quad \mathrm{Li}_2(1-1/w)$$

$$\ln^2\frac{uv}{w} \quad \ln^2\frac{vw}{u} \quad \ln^2\frac{wu}{v}$$

Analogous constraints for $n=7$

LD, J. Drummond, T. Harrington, A. McLeod, G. Papathanasiou, M. Spradlin, 1612.08976