Mechanisms for anomalous transport in one-dimensional systems

Emergent Hydrodynamics in Integrable Quantum Many-body Systems and Beyond, ICTP, Trieste

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Collaborators

Will discuss collaborative work with Christoph Karrasch (TU Braunschweig) and Joel E. Moore (UC Berkeley), pictured:

Figure: left to right: C. Karrasch, J. E. Moore

Introduction

Superdiffusion of energy in one-dimensional metals

Kardar-Parisi-Zhang physics in isotropic magnets

What is normal transport?

- First, what is normal transport? Three generic scenarios in **classical** many-body systems:
  1. Free particles, different velocities lead to dispersion: ballistic transport.
  2. Interacting particles in the hydrodynamic regime: ballistic transport with diffusive corrections.
  3. Microscopic Brownian motion: diffusive transport.
- Broadly speaking, we expect the same pattern in **quantum** many-body systems.
- Today's talk: how one-dimensional quantum systems can confound these expectations

Classical intuition for anomalous transport

Anomalous transport is well-studied in classical systems. There are phenomena associated with anomalous diffusion in arbitary dimensions:

- **Fractional diffusion** e.g. the fractional heat equation,

$$\partial_t u = -(-\nabla^2)^{s/2} u, \quad s \neq 2$$

- **Nonlinear diffusion** e.g. the porous medium equation,

$$\partial_t u = \nabla^2 u^m, \quad m \neq 1$$

as well as fluctuation-dominated phenomena that are enhanced in low dimensions:

- **Nonlinear fluctuating hydrodynamics** e.g. the stochastic Burgers equation in $d = 1$:

$$\partial_t u + \partial_x (u^2 - D\partial_x u + \zeta) = 0$$

Each exhibits space-time scaling $|x| \sim t^\alpha$ with exponent distinct from "normal" possibilities, $\alpha = \frac{1}{2}, 1$.

New examples of anomalous transport in 1D quantum systems

- Subdiffusion: Approaching the MBL transition from the ergodic side, charge transport appears to be subdiffusive, with a spreading exponent $x \sim t^{\alpha}$ and $0 < \alpha < 1/2$ (e.g. recent review article *Gopalakrishnan, Parameswaran, '19*)
- Superdiffusion:
  - One-dimensional metals at low temperature can support nonlinear diffusion of heat, with $x \sim t^{\alpha}$ and $2/3 < \alpha < 1$ (*VBB, Karrasch, Moore, '19*).
  - Isotropic quantum magnets exhibit "integrability protected" superdiffusion in the KPZ universality class, with $\alpha = 2/3$ (exponent first observed in *Žnidarič, '11*, with many subsequent studies).
  - More exotic possibilities, e.g. $\alpha = 3/4$ in easy-plane XXZ, originating from infinite number of quasiparticle flavours (*Agrawal, Gopalakrishnan, Vasseur, Ware, '19*)
- Today's talk will focus on two of the superdiffusive examples: **nonlinear diffusion** in metals and **KPZ** in spin chains.

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Anomalous transport in one-dimensional metals

- Will argue that interacting, one-dimensional metals that exhibit generic, thermalizing behaviour in all other respects (level statistics, charge transport, . . .) can exhibit **superdiffusion of heat at low temperatures**.
- This type of superdiffusion persists at long times in non-integrable systems and should be observable in time-resolved experiments on quantum wires involving laser irradiation of a small region (c.f. *Hensel, Dynes, '77*)
- Unexpected violation of Fourier's law in a well-studied class of physical systems.

One-dimensional metals as perturbed Luttinger liquids

- Ideal 1D metals are described by the free, Luttinger liquid theory, yielding divergent linear response transport coefficients (ballistic transport).
- Unperturbed Hamiltonian maps to free bosons,

$$H_0 = \frac{u}{2} \int_0^L dx \, \left(\Pi^2 + (\partial_x \phi)^2 \right) .$$

- Realistic 1D metals have finite transport coefficients, due to interactions or disorder (diffusive transport).
- Typical 1D interactions generate density-wave instabilities. In the metallic phase, these show up as irrelevant vertex operators:

$$\delta H \sim \cos \alpha\phi, \quad \alpha^2 > 8\pi.$$

Lattice realization of perturbed Luttinger liquid

- A microscopic realization of an interacting Luttinger liquid is spin-1/2 XXZ in an integrability-breaking staggered field:

$$H = \sum_{i=1}^{N} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z + (-1)^i h S_i^z. \quad (1)$$

- Low-energy field theory has the form

$$H = \frac{u}{2} \int_0^L dx \, \left(\Pi^2 + (\partial_x \phi)^2\right) + ch \int_0^L dx \cos\left(2\sqrt{\pi K}\phi\right) + \dots$$

  i.e. staggered field is the most relevant perturbation.
- A previous work (*Huang, Karrasch, Moore, '13*) numerically verified **non-integrability** for $h > 0$:
  1. For $h > 0$, level statistics flow from Poisson to Wigner-Dyson.
  2. Charge transport is diffusive, with linear-response conductivity matching the analytical result (*Luther, Peschel, '74, Sirker, Pereira, Affleck, '11*)

$$\sigma_c(T) \sim T^{3-2K}. \quad (2)$$

Anomalous diffusion model for heat transport

- What about heat transport? Minimal assumption is power-law divergence, e.g.

$$\kappa(T) \sim T^{\lambda(K)}, \quad T \to 0 \tag{3}$$

  for some $\lambda(K) < 0$, e.g. WF scaling, $\lambda(K) = 4 - 2K$.
- Unfortunately, accessing $\kappa(T)$ directly is beyond present numerical and analytical techniques, and WF need not hold.
- Even so, power-law ansatz has non-trivial, testable consequence for transport: energy density ρ_E should satisfy **fast diffusion equation**

$$\partial_t \rho_E = D \partial_x^2 \left(\rho_E^m\right), \quad m = \frac{1+\lambda}{2} \tag{4}$$

- Fundamental solutions are superdiffusive "Barenblatt-Pattle profiles", with anomalous scaling

$$x \sim t^{\alpha}, \quad \alpha = \frac{2}{\lambda + 3} \tag{5}$$

Testing the theoretical model

- To test this, we simulated XXZ in staggered field, with localized thermal wavepacket initial condition (*VBB, Karrasch, Moore, '19*)

$$\beta(x) = \beta - (\beta - \beta_M) e^{-(x/L)^2} \tag{6}$$

and low bulk temperature $\beta \gg 1$.
- Can probe space-time scaling by looking at **logarithmic derivatives of moments**:

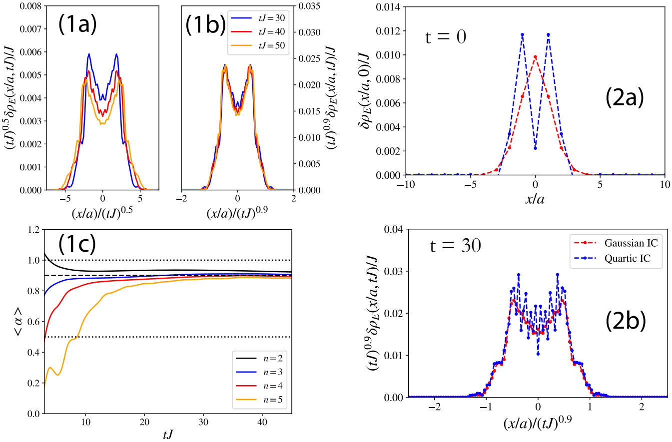
$$\frac{1}{n} \frac{d \log \langle |x|^n \rangle (t)}{d \log t} \to \alpha, \quad t \to \infty. \tag{7}$$

- Non-trivial prediction of our model: these should converge to the same, superdiffusive exponent, $2/3 < \alpha < 1$.
- At any non-zero bulk temperature, expect eventual crossover to diffusive behaviour on a timescale

$$\tau_D(T) \sim T^{\lambda - 1}. \tag{8}$$

Numerical results for anomalous diffusion of heat

- Clear numerical evidence for superdiffusive, rather than diffusive, spreading of wavepacket ($\beta J = 12$):



- Long-time shape of profiles inconsistent with simple nonlinear diffusion model - kinetic description? proximate integrability?

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Numerical observations

- A large body of work has confirmed "integrability protected" KPZ physics in one-dimensional magnets with isotropic symmetry.
- Diagnostic is the long-time behaviour of the spin autocorrelation function

$$\langle \mathbf{S}(x,t)\cdot\mathbf{S}(0,0)\rangle_\beta \sim t^{-\alpha} f(x/t^\alpha), \quad t\to\infty. \qquad (9)$$

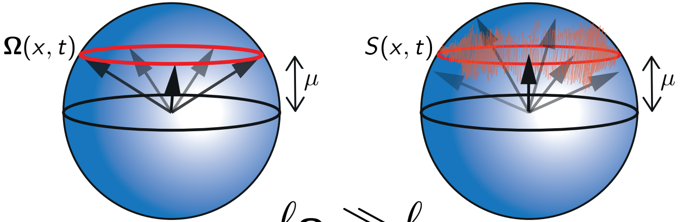
- Numerics: $\alpha = 2/3$ for integrable models and $\alpha = 1/2$ for non-integrable models (with divergent log correction and crossover from $\alpha = 2/3$, next talk by Jacopo de Nardis).
- Some recent numerical studies:
  Classical: *Das, Kulkarni, Spohn, Dhar, '19, Krajnik, Prosen, '19, de Nardis, Medenjak, Karrasch, Ilievski, '20, Krajnik, Ilievski, Prosen, '20*, **Quantum:** *Ljubotina, Žnidarič, Prosen, '19, de Nardis, Medenjak, Karrasch, Ilievski, '19, Dupont, Moore, '19, Weiner, Schmitteckert, Bera, Evers, '19, Fava, Ware, Gopalakrishnan, Vasseur, Parameswaran, '20*

State of theory

- Until recently, the closest approach to a theoretical explanation for the dynamical exponent $\alpha = 2/3$ was a self-consistent derivation (*Gopalakrishnan, Vasseur, '19*) for the spin-$1/2$ Heisenberg model, based on generalized hydrodynamics (*Bertini, Collura, de Nardis, Fagotti, '16, Castro-Alvaredo, Doyon, Yoshimura, '16*)
- However, three fundamental questions had not been addressed:
  1. Why does the same phenomenon occur for both quantum and classical systems?
  2. Why are both integrability and isotropic symmetry necessary for this phenomenon to be stable at long times? (*Dupont, Moore, '19, Krajnik, Prosen, '19*)
  3. Why is a collapse to universal, Kardar-Parisi-Zhang scaling functions observed numerically? (*Ljubotina, Žnidarič, Prosen, '19, Das, Kulkarni, Spohn, Dhar, '19*)
- These questions reflected a basic lack of understanding of the physical mechanism underlying this phenomenon

Kardar-Parisi-Zhang universality from soft gauge modes

- Recently, we pointed out that certain local equilibrium states support "soft modes" of the magnetization, that are missed by standard hydrodynamic approaches (*VBB, '19*).
- These modes are nonlinear and separated in scale from short-wavelength hydrodynamics, yielding a channel for superdiffusive spin transport.
- Provides a physical mechanism for the emergence of KPZ physics in isotropic spin chains, similar to 1D Bose gases (*Arzamasovs, Bovo, Gangardt, '14*)



$$\ell_{\mathbf{\Omega}} \gg \ell$$

Example: spin-1/2 Heisenberg

- Consider the spin-1/2 Heisenberg Hamiltonian

$$H = -J \sum_{i=1}^{N} \mathbf{S}_i \cdot \mathbf{S}_{i+1} - 2\mathbf{h} \cdot \sum_{i=1}^{N} \mathbf{S}_i. \tag{10}$$

  Recall that Bethe's ansatz builds eigenstates out of spin-waves on the "pseudovacuum", $|\Omega\rangle = |\uparrow\uparrow \ldots \uparrow\rangle$.
- In the absence of an applied magnetic field, $\mathbf{h} = 0$, Bethe's solution is $SU(2)$ symmetric - direction of the pseudovacuum, $\boldsymbol{\Omega} \in S^2$, is arbitrary.
- An applied magnetic field $\mathbf{h} \neq 0$ breaks this symmetry; only pseudovacuum directions $\boldsymbol{\Omega} \parallel \mathbf{h}$ are allowed. TBA predicts

$$\langle \mathbf{S} \rangle / \ell = \boldsymbol{\Omega} \left[\frac{1}{2} - \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dk \, n \rho_k^n \right]. \tag{11}$$

- In local equilibrium states, formation of a local magnetization on a fluid cell spontaneously breaks $SU(2)$ symmetry - **pseudovacuum becomes a dynamical degree of freedom**.

Coarse-grained pseudovacuum dynamics

- Effective, long-wavelength vacuum dynamics $\mathbf{\Omega}$ is described by the Landau-Lifshitz equation:

$$\partial_t \mathbf{\Omega} = \lambda \mathbf{\Omega} \times \partial_x^2 \mathbf{\Omega} \tag{12}$$

- Curvatures in the Frenet-Serret frame of a fictitious space curve with $\hat{\mathbf{t}}(s) \equiv \mathbf{\Omega}(x)$ yield $SU(2)$ invariant hydrodynamic modes (c.f. *Lakshmanan, Ruijgrok, Thompson, '76*)
- Nonlinear fluctuating hydrodynamics (*Van Beijeren, '11, Spohn, '13*) of the pseudovacuum is **stochastic Burgers equation** for the torsion:

$$\partial_t \tau + \partial_x (\lambda \tau^2 - D \partial_x \tau + \sigma \zeta_\tau) = 0, \tag{13}$$

  (by fluct.-diss. $\langle \tau \tau \rangle_\mu = \sigma^2 / 2D$).
- Follows that "height function" η, defined by $\tau = \partial_x \eta$, satisfies the KPZ equation, and that the correlation functions of the torsional mode have superdiffusive scaling form $C(x,t) = \mathbb{E}[\tau(x,t)\tau(0,0)] = f_{KPZ}(x/(\Gamma t)^{3/2})/(\Gamma t)^{3/2}$, where $\Gamma = 2\sqrt{2}\lambda$ (*Spohn, '13*).

A hydrodynamic explanation for integrability protection

- Short answer: integrable models support distinct "gauge" and "quasiparticle" excitations of spin. No such distinction exists in non-integrable models.
- For the "Landau-Lifshitz" states that we considered earlier, the only difference in the short-wavelength hydrodynamics is the nature of the scalar bath.
- **Non-integrable models**: two scalar conserved modes $\{S, E\}$ per fluid cell. Total variance scales as $\sigma_\ell^2 \sim \ell^{-1}$. As $\ell \to \infty$, hydrodynamics of S becomes deterministic[1] and recouples to slow dynamics of $\mathbf{\Omega}$
- **Integrable models**: extensively many scalar modes $\{S, E, Q_3, \ldots, Q_n\}_{n\sim\ell}$, whose variance scales as $\sigma_\ell^2 \sim \ell^0$. As $\ell \to \infty$, S continues to fluctuate as part of a bath.

[1] in fact, decay renders nonlinearity in $\mathbf{\Omega}$ *marginally* irrelevant, see next talk!

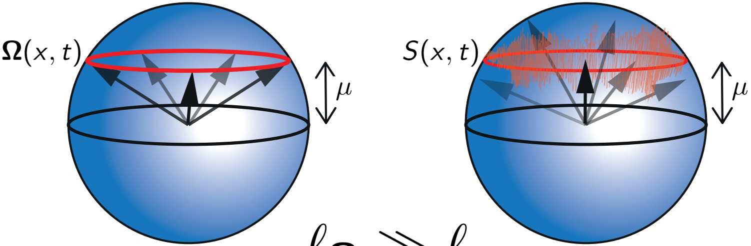
Summary and outlook

- We identified the nonlinear modes giving rise to KPZ physics in isotropic quantum and classical magnets.
- The "soft gauge mode" effective theory has since been:
  - tested in a wide range of classical and quantum spin models (*De Nardis, Medenjak, Karrasch, Ilievski, '20*)
  - justified microscopically within GHD (*De Nardis, Gopalakrishnan, Ilievski, Vasseur, '20*)
  - succesfully applied to the $SU(2) \times SU(2)$ symmetric 1D Hubbard model (*Fava, Ware, Gopalakrishnan, Vasseur, Parameswaran, '20*)
- An exciting goal: extend to superuniversality, i.e. observation of same physics for higher internal symmetry groups $G = SU(3), SO(5)$ etc. (*Dupont, Moore, '19, Krajnik, Ilievski, Prosen, '20*)

Thank you for listening!

Relevant papers:

- **Superdiffusive transport of energy in one-dimensional metals**: *VBB, Christoph Karrasch, Joel E. Moore*, PNAS 1916213117 (2020)
- **Kardar-Parisi-Zhang universality from soft gauge modes**: *VBB*, PRB Rapid Communication, 101, 041411 (2020)



$$\ell_{\boldsymbol{\Omega}} \gg \ell$$

A primer on nonlinear diffusion

- The **nonlinear diffusion equation** is given by

$$\partial_t u = D\nabla^2 u^m \tag{14}$$

i.e. effective diffusion constant is nonlinear,

$$D_{eff}[u] = mDu^{m-1}.$$

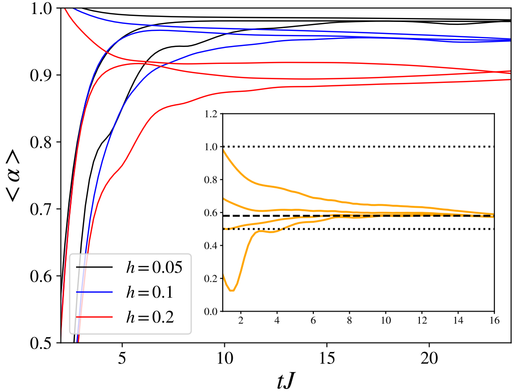
- For $m = 1$, recover normal diffusion. For $m > 1$, this is the "porous medium equation". For $m < 1$, this is the "fast diffusion equation" (for a review: *Vázquez, '06*).
- Fundamental solutions for $m \neq 1$ are non-Gaussian. Instead, nonlinearity yields **Barenblatt-Pattle** profiles, characterized by anomalous space-time scaling.
- e.g. in $d = 1$, these have the form

$$u_{B.P.}(x,t) = t^{-\alpha}\max[(C - k(x/t^{\alpha})^2)^{-\frac{1}{m-1}}, 0] \tag{15}$$

with space-time scaling exponent $\alpha = 1/(m+1)$, $k = k(m)$ constant and C fixed by initial area.

Numerical results for anomalous diffusion of heat II

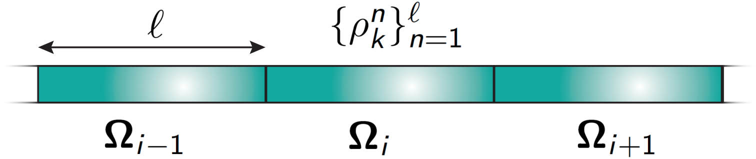
- Increasing strength of integrability-breaking field lowers superdiffusive exponent (main figure):



- Sanity checks : expansion into ground state yields superdiffusion, higher temperatures begin to recover normal diffusion (inset, at $\beta = 1$, $h = 0.49$).

Landau-Lifshitz states in isotropic integrable magnets

- Consider local equilibrium states with constant quasiparticle occupancies per fluid cell and a pseudovacuum $\boldsymbol{\Omega}(x,t) \in S^2$ varying on a scale $\ell_{\boldsymbol{\Omega}} \gg \ell$:



- In the limit $\ell/\ell_{\boldsymbol{\Omega}} \to 0$, the quasiparticle dynamics is decoupled from the gauge dynamics.
- Slow modulations of the pseudovacuum are mean-field states by definition, with effective dynamics ($\lambda = J/2$)

$$\partial_t \boldsymbol{\Omega} = \lambda \boldsymbol{\Omega} \times \partial_x^2 \boldsymbol{\Omega} + \mathcal{O}(\ell_{\boldsymbol{\Omega}}^{-4}). \tag{16}$$

- At zero temperature, recovers previous results (*Gamayun, Miao, Ilievski, '19, Misguich, Pavloff, Pasquier, '19*). At finite temperature, Eq. (16) is coupled to a thermal bath.

Coarse-grained Landau-Lifshitz dynamics I

- Effective field theory describing KPZ physics is **Landau-Lifshitz dynamics at finite temperature**. We therefore consider nonlinear fluctuating hydrodynamics of this equation. First we need the Euler hydrodynamics.
- Since integrability is broken microscopically, mean-field evolution is not integrable and there are two conserved modes. Exactly the same reasoning as discretized GPE (*Kulkarni, Huse, Spohn, '15*).
- Standard parameterizations of the sphere (e.g. spherical polar) are not gauge invariant. An elegant solution is to regard x as arc-length and $\mathbf{\Omega}$ as the tangent vector of a space curve (*Lakshmanan, Ruijgrok, Thompson, '76*).
- In terms of the curvature κ and torsion τ of this curve, the Landau-Lifshitz evolution becomes

$$\dot{\kappa} + \lambda(2\kappa'\tau + \kappa\tau') = 0, \quad \dot{\tau} + \lambda(\tau^2 - \kappa''/\kappa - \kappa^2/2)' = 0. \quad (17)$$

Coarse-grained Landau-Lifshitz dynamics II

- Discarding the dispersive term and changing variable from κ to energy density $\mathcal{E} = \kappa^2/2$ yields

$$\partial_t \mathcal{E} + \partial_x[\lambda(2\mathcal{E}\tau)] = 0, \tag{18}$$

$$\partial_t \tau + \partial_x[\lambda(\tau^2 - \mathcal{E})] = 0. \tag{19}$$

- Linearizing, we find imaginary velocities and violation of sum rules - **instability**. Suggests that two-mode hydrodynamics of Landau-Lifshitz is unphysical.
- Intuition: soft modes can't transport extensive energy. More precisely, total energy in a fluid cell of characteristic length $\ell_{\mathbf{\Omega}}$ is subextensive, $\mathcal{E} \sim 1/\ell_{\mathbf{\Omega}}$. Thus energy is not a true hydrodynamic variable and tends to zero as $\ell_{\mathbf{\Omega}} \to \infty$.
- This leaves a single Burgers equation for the torsion (i.e. magnetization density),

$$\partial_t \tau + \partial_x(\lambda\tau^2) = 0. \tag{20}$$

Coarse-grained Landau-Lifshitz dynamics III

- Mesosopic coupling to noise and dissipation yields stochastic Burgers equation

$$\partial_t \tau + \partial_x(\lambda \tau^2 - D\partial_x \tau + \sigma \zeta_\tau) = 0, \qquad (21)$$

  with coefficients constrained by fluctuation dissipation relation $\langle \tau\tau \rangle_\mu = 2\sigma D$.
- Follows that "height function" η, defined by $\tau = \partial_x \eta$, satisfies the KPZ equation, and that the correlation functions of the torsional mode have superdiffusive scaling form $C(x,t) = \mathbb{E}[\tau(x,t)\tau(0,0)] = f_{KPZ}(x/(\Gamma t)^{3/2})/(\Gamma t)^{3/2}$, where $\Gamma = 2\sqrt{2}\lambda$ (*Spohn, '16*).