Diffusion from Convection

M. Medenjak, J. De Nardis, T. Yoshimura, arXiv:1911.01995

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Introduction

- Microscopic origins of transport coefficients

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- Obtaining conductivity without an intricate integrability structure, e.g. form factor expansion

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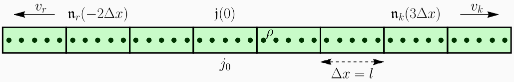
- Obtaining conductivity without an intricate integrability structure, e.g. form factor expansion
- Possible applications beyond integrable systems
- More formal/rigorous treatment

B. Doyon, Diffusion and superdiffusion from hydrodynamic projection, arXiv:1912.01551.

Hydrodynamic assumption



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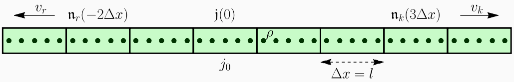

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- Divide the infinite lattice into fluid cells

$$\mathfrak{o}(x) = \sum_{i=x-\ell/2+1}^{x+\ell/2} o_i$$

for (quasi)localized o_i

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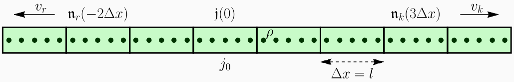

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  for (quasi)localized o_i
- Assuming the local equilibration
  State locally corresponds to a GGE

$$\rho(\underline{\beta}(x)) = \exp(\beta^r(x)\mathfrak{q}_r(x))$$

  $\mathfrak{q}_r(x)$ are hydrodynamic densities associated with quasilocal conservation laws $Q_r = \sum_{x;\Delta x} \mathfrak{q}_r(x)$.

Correlations on superlattice

- Connected correlation functions are defined as

$$\langle \mathfrak{o}(x)\mathfrak{q}_1(x_1)\cdots\mathfrak{q}_n(x_n)\rangle^c \equiv \frac{\partial^N \langle \mathfrak{o}(x)\rangle}{\partial_{\beta^1(x_1)}\cdots\partial_{\beta^N(x_N)}},$$

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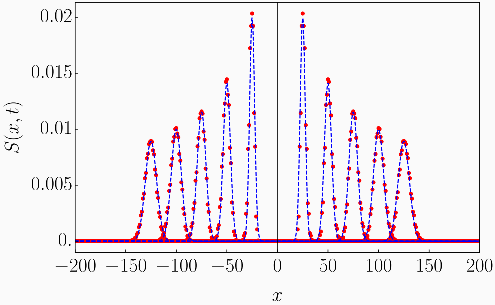
- and are orthogonal

$$\langle \mathfrak{o}(x)\mathfrak{q}_1(x_1)\cdots\mathfrak{q}_n(x_n)\rangle^c = \ell(\langle \mathfrak{o}(x)\mathfrak{q}_1(x_1)\cdots\mathfrak{q}_n(x_n)\rangle^c_n \\ \times\delta_{xx_1}\delta_{x_1x_2}\cdots\delta_{x_{N-1}x_N} + \mathcal{O}(\ell^{-1})),$$

  where we introduced
  $\langle \mathfrak{o}(x)\mathfrak{q}_1(x_1)\cdots\mathfrak{q}_n(x_n)\rangle^c_n = \langle o_x\mathfrak{q}_1(x_1)\cdots\mathfrak{q}_n(x_n)\rangle^c$

Normal modes

- Normal modes are excitations on top of homogeneous background $\beta^i(x) = \beta^i$ which propagate with well defined velocities, and can spread (super)diffusively.



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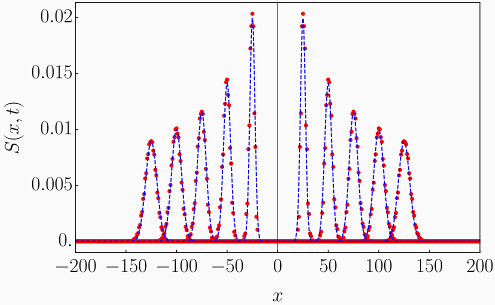

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- Drude matrix D_{lk} Encodes the velocities of normal modes

$$D_{lk} = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} \mathrm{d}t' \sum_x \langle j_l(0,t') j_k(x,0)\rangle^c$$

Normal modes

- Normal modes are a linear combination of conservation laws $\mathfrak{q}_i = (R^{-1})_i^{\ j} \mathfrak{n}_j$, obtained by diagonalizing the Drude matrix, and satisfy the continuity equation on top of the density matrix $\rho(\underline{\beta})$

$$\partial_t \hat{n}_i^{\rho}(k,t) + \mathrm{i}\omega_i(k)\hat{n}_i^{\rho}(k,t) = 0,$$

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- Onsager matrix $\mathfrak{L}_{lk}$ carries the information of how the normal modes spread

$$\mathfrak{L}_{lk} = \lim_{t\to\infty}\int_{-t}^{t} \mathrm{d}t' \sum_x (\langle j_l(0,t') j_k(x,0)\rangle^c - D_{lk}).$$

Hydrodynamic expansion

Hydrodynamic expansion

- Hydrodynamic expansion is a conjecture that on the level of correlation functions, for the calculation of integrated quantities the currents can be expanded in terms of conserved densities as

$$\delta \mathfrak{j}^{\rho}_{k}(x,t) = (\partial_{\mathfrak{q}_i(y)}\langle \delta \mathfrak{j}_k(x,t)\rangle)\delta q^{\rho}_{i}(y) + \\ + \frac{1}{2}(\partial_{\mathfrak{q}_j(z)}\partial_{\mathfrak{q}_i(y)}\langle \delta \mathfrak{j}_k(x,t)\rangle)\delta q^{\rho}_{i}(y)\delta q^{\rho}_{j}(z) + \mathrm{h.o.t.} + \mathcal{R},$$

where $\delta q^{\rho}_{i}(y) = q_i - \langle q_i \rangle$. The subtracted average is taken in the homogeneous state ρ, while the homogeneous limit is taken only after taking the derivatives. $\mathcal{R}$ can involve non-convective modes, such as quadratically extensive quantities. It can also be viewed as an expansion of the density matrix in terms of conserved quantities

$$\rho(x,t) = \partial_{\mathfrak{q}_i(x)}(\rho)\delta q^{\rho}_{i}(x) + \partial_{\mathfrak{q}_i(y)}\partial_{\mathfrak{q}_j(y)}(\rho)\delta q^{\rho}_{i}(x)\delta q^{\rho}_{j}(y) + \ldots$$

Drude weight

- On the super-lattice Drude weight reads

$$D_{kl} = \lim_{t\to\infty} \frac{1}{2t\ell} \sum_{x;\Delta x} \int_{-t}^{t} \mathrm{d}s \langle \mathfrak{j}_k(0,s)\mathfrak{j}_l(x,0)\rangle^c$$

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- Only the first order in the Hydrodynamic expansion contributes, yielding

$$D_{kl} = (\partial_{\mathfrak{q}_i(0)}\langle \mathfrak{j}_k(0)\rangle)\langle Q_i j_{l,0}\rangle = BC^{-1}B,$$

with $B_{ik} = \langle \mathfrak{q}_i(0)\mathfrak{j}_k(0)\rangle_n^c$ and $C_{ij} = \langle \mathfrak{q}_i(0)\mathfrak{q}_j(0)\rangle_n^c$. In normal modes basis $\mathcal{N}_i = R_i^{\,j} Q_j$

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- Takeaway message 1: In order to get the Drude weights diagonalize the susceptibility matrix $\langle Q_l Q_r\rangle^c$ of charges Q_l, and project the currents j_k.

Onsager matrix

- Onsager matrix can be expressed in terms of sub-Euler current $\mathfrak{j}_k^-(x,t) = \mathfrak{j}_k(x,t) - (\partial_{\mathfrak{q}_i(y)}\langle \mathfrak{j}_k(x)\rangle)\mathfrak{q}_i(y,t)$

$$\mathfrak{L}_{kl} = \ell^{-1} \sum_{x;\Delta x}^{\infty} \int \mathrm{d}t \langle \mathfrak{j}_k^-(x,t)\mathfrak{j}_l^-(0,0)\rangle^c.$$

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- Only the second order of the expansion contributes

$$\mathfrak{L}_{kl}^c = 2(R^{-1}\tilde{G}^2 R^{-\mathrm{T}})_{kl},$$

with G-matrix from NLFHD $\tilde{G}_{ij}^2 = \frac{G_{ii'j'}G_j{}^{i'j'}}{|v_{i'} - v_{j'}|}$,

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- In terms of the normal modes it simply reads

$$\mathfrak{L}_{kl}^c = \frac{\langle j_k^- \mathcal{N}_i \mathcal{N}_j\rangle^c \langle \mathcal{N}^i \mathcal{N}^j j_l^-\rangle^c}{2|v_i - v_j|}.$$

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- Takeaway message 2: Diagonalize the Drude weight of charges D_{lk} and project the currents j_k onto the products of normal modes (eigenvectors) and divide by the difference of velocities (eigenvalues)

Manifestations of the result

Integrable systems

- Lower bounds

T. Prosen, PRE 89, 012142 (2014)

M. Medenjak, C. Karrasch, and T. Prosen, Physical Review Letters 119, 080602 (2017)

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- We can show that for integrable systems $\mathfrak{L}^c_{ij} \equiv \mathfrak{L}_{ij}$ by calculating the G matrix.
- Explanation of the "Magic" formula relates the curvature of the self-Drude weight $D^{\mathrm{self}}_{s,s} = \int \mathrm{d}t \langle j_s(0,t) j_s(0,0)\rangle^c$ for the spin current to the spin diffusion constant $\mathfrak{D}_{s,s} = \mathfrak{L}_{s,s}/C_{s,s} = \partial^2_\nu D^{\mathrm{self}}_{s,s}(\nu)$.

Relation to the Nonlinear fluctuating hydrodynamics and superdiffusion

- Kardar-Parisi-Zhang equation

Kardar, Mehran; Parisi, Giorgio; Zhang, Yi-Cheng, Physical Review Letters. 56 (9): 889–892

$$\partial_t \phi = D\partial_x^2 \phi + \partial_x(\phi^2 + \xi).$$

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- Nonlinear fluctuating hydrodynamics (H. Spohn): If we have more then one conservation law, and no symmetry restrictions we will generically get superdiffusion. The expectation value of the current is a function of charges and we expand it to the second order

  H. Spohn, Journal of Statistical Physics 154, 1191 (2014).

  $$\partial_x \langle q \rangle + \partial_t (G \langle q \rangle^2 + D\partial_x \langle q \rangle + \xi) = 0$$

  $G = \partial^2_{\langle q \rangle} \langle j \rangle$. If the diagonal element is present the transport is superdiffusive. From our result the divergence of diffusion occurs whenever there are two normal modes with non-vanishing matrix element $\langle \mathcal{N}^i \mathcal{N}^j j_l^- \rangle^c \neq 0$, with degenerate velocities $v_i = v_j$.

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- Convective contributions to diffusion in higher dimensions

G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. 87, 081601 (2001).