- *B. Bertini, MF, L. Piroli, and P. Calabrese*
  Entanglement evolution and generalised hydrodynamics: noninteracting systems
  J. Phys. A ***51****, 39LT01 (2018)*

- *V. Alba, B. Bertini, and MF*
  Entanglement evolution and generalised hydrodynamics: interacting integrable systems
  Scipost ***7****, 005 (2019)*

Entanglement evolution and generalised hydrodynamics

Maurizio Fagotti

Plan

- ✦ Ingredients
- ✦ Entanglement evolution of inhomogeneous states within **GHD**

Ingredients

Bipartite entanglement

complete orthogonal bases of the corresponding spaces

Schmidt decomposition

$$\left|\Psi\right\rangle = \sum_{n=1}^{M} \sqrt{p_n} \left|\Psi_n^A\right\rangle \otimes \left|\Psi_n^B\right\rangle$$

pure state

no classical (thermal) correlations

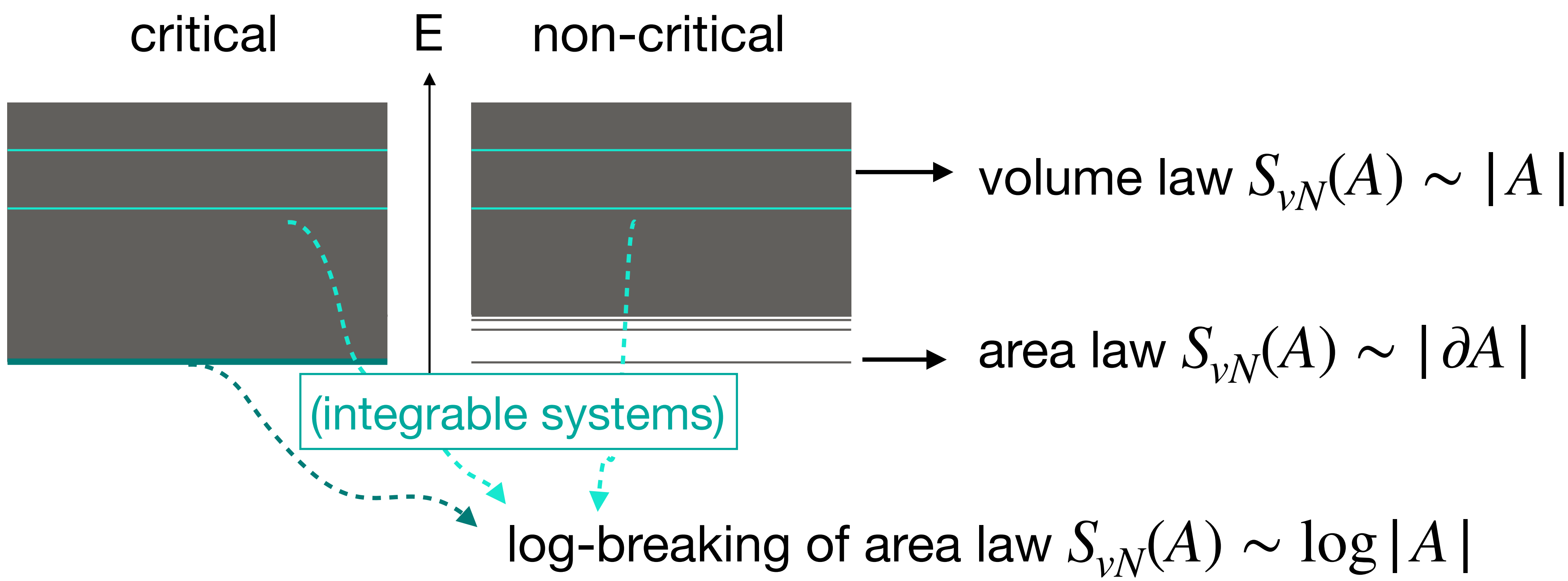
quantum correlations *between A and B*

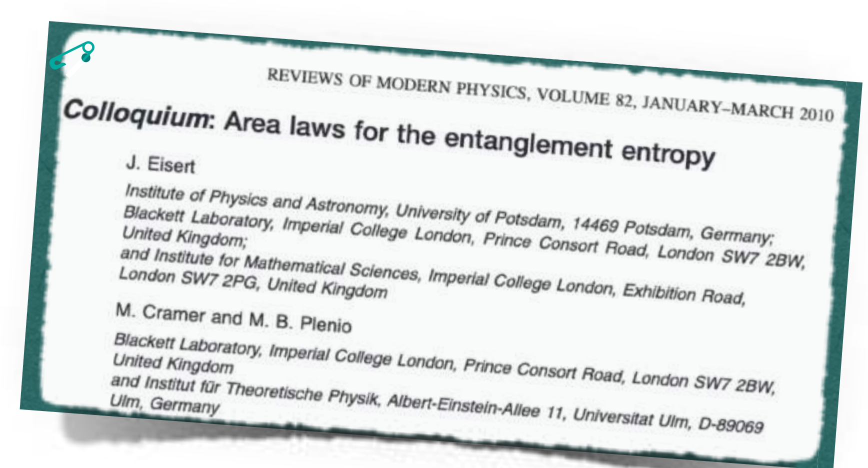
entanglement entropy $S_{vN} = -\sum_{n} p_n \log p_n$

Ingredients

Bipartite entanglement

in stationary states




REVIEWS OF MODERN PHYSICS, VOLUME 82, JANUARY–MARCH 2010

***Colloquium*: Area laws for the entanglement entropy**

J. Eisert

Institute of Physics and Astronomy, University of Potsdam, 14469 Potsdam, Germany; Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2BW, United Kingdom; and Institute for Mathematical Sciences, Imperial College London, Exhibition Road, London SW7 2PG, United Kingdom

M. Cramer and M. B. Plenio

Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2BW, United Kingdom and Institut für Theoretische Physik, Albert-Einstein-Allee 11, Universität Ulm, D-89069 Ulm, Germany

Ingredients

Quench dynamics

coined by J. Cardy

QUANTUM QUENCH $g_0 \to g$

$$\hat{H}(g_0)|\Psi_0> \; = E_{GS}|\Psi_0>$$
$$\hat{H} \; = \hat{H}(g)$$

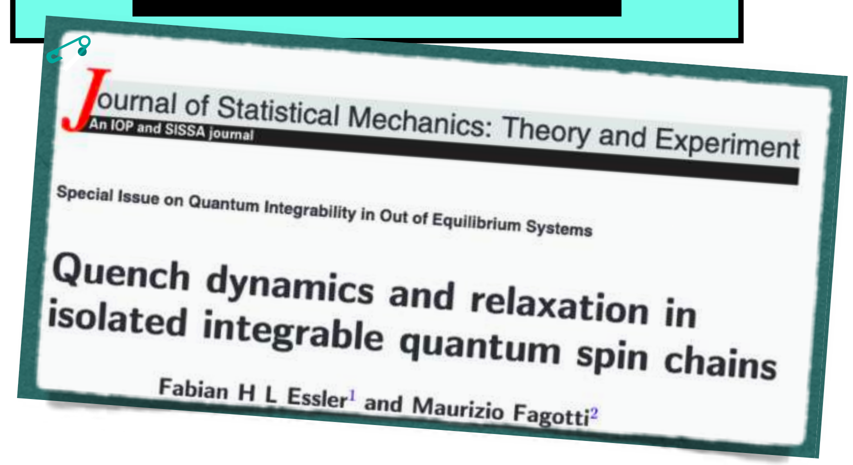
a many-body system time evolves unitarily

$$|\Psi_t> \; = e^{-i\hat{H}t}|\Psi_0> \qquad (\hat{\rho} = |\Psi><\Psi|)$$
$$\hat{\rho}_t = e^{-i\hat{H}t}\hat{\rho}_0 e^{i\hat{H}t}$$

thermodynamic limit

↓

Local Relaxation

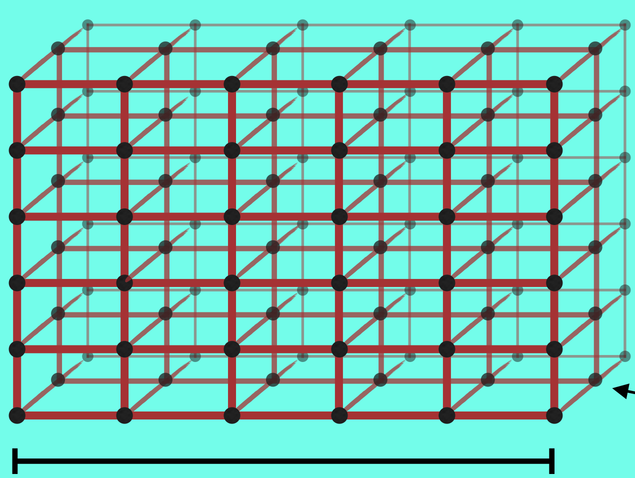

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Special Issue on Quantum Integrability in Out of Equilibrium Systems

Quench dynamics and relaxation in isolated integrable quantum spin chains

Fabian H L Essler[1] and Maurizio Fagotti[2]

typical examples —
- *spin lattice systems*
- *quantum field theories*



L

$$\mathfrak{D} \sim \#^{L^d}$$

finite-dimensional local Hilbert space

Ingredients

GHD

Integrable systems

root densities *additional fields*

$$\hat{\rho}_t = \hat{\rho}[\rho_{x,t}(\lambda), \ldots]$$

rapidities

(used to parametrise stationary states in integrable systems)

$\hat{\rho}[\rho(\lambda), 0, \ldots, 0]$ *is stationary*

density matrix

$$\hat{\rho}_t = e^{-i\hat{H}t}\hat{\rho}_0 e^{i\hat{H}t}$$

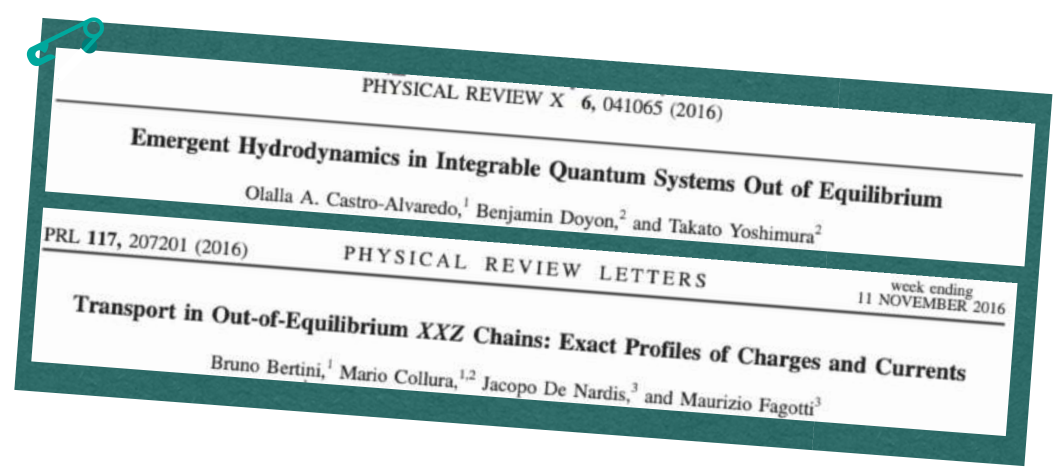
$$\partial_t \rho_{x,t}(\lambda) = F_x[\rho_{y,t}(\mu), \text{ nothing else?}](\lambda)$$

$$\approx -\partial_x\Big(v[\rho_{x,t}(\mu)](\lambda)\,\rho_{x,t}(\lambda)\Big) + O(\partial_x^2)$$

1st order GHD

classical equation: no trace of $\hbar$!

clever parametrisation depending
- *on the class of initial states*
- *on the Hamiltonian*


PHYSICAL REVIEW X 6, 041065 (2016)

Emergent Hydrodynamics in Integrable Quantum Systems Out of Equilibrium

Olalla A. Castro-Alvaredo,[1] Benjamin Doyon,[2] and Takato Yoshimura[2]

PRL 117, 207201 (2016) PHYSICAL REVIEW LETTERS week ending 11 NOVEMBER 2016

Transport in Out-of-Equilibrium XXZ Chains: Exact Profiles of Charges and Currents

Bruno Bertini,[1] Mario Collura,[1,2] Jacopo De Nardis,[3] and Maurizio Fagotti[3]

Ingredients

GHD

Integrable systems

Exactly solvable example: NONINTERACTING SPIN CHAINS

auxiliary field describing the "off-diagonal" elements of the density matrix

Wick's theorem at any time →

$$\hat{\rho}_t = \hat{\rho}[\rho_{x,t}(p), \psi_{x,t}(p)]$$

$$\hat{H} = \hat{H}[\varepsilon_x(p)]$$

excitation energy

EXACT PARAMETRISATION

$$i\hbar\partial_t \rho_{x,t}(p) = \varepsilon_x(p) \star \rho_{x,t}(p) - \rho_{x,t}(p) \star \varepsilon_x(p)$$

$$i\hbar\partial_t \psi_{x,t}(p) = \varepsilon_x(p) \star \psi_{x,t}(p) + \psi_{x,t}(p) \star \varepsilon_x(-p)$$

phase-space formulation of quantum mechanics in *noninteracting spin chains*

Moyal star product

$$a_x(p) \star b_x(p) = a_x(p) e^{i\hbar \frac{\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overrightarrow{\partial}_x \overleftarrow{\partial}_p}{2}} b_x(p)$$


Sci Post

SciPost Phys. 8, 048 (2020)

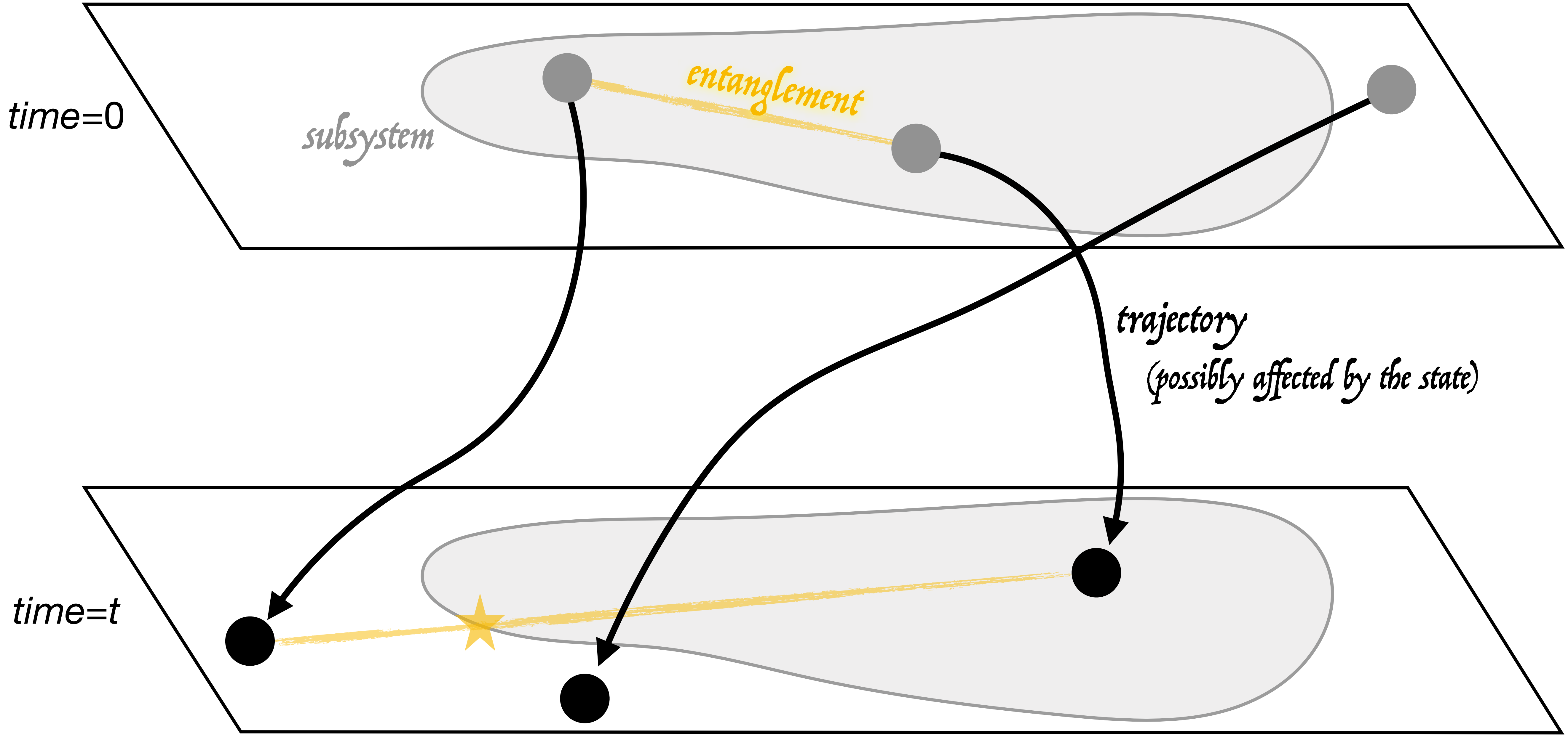
Locally quasi-stationary states in noninteracting spin chains

Maurizio Fagotti*

Semiclassical picture

$$\partial_t \rho_{x,t}(\lambda) + \partial_x v_{x,t}(\lambda)\rho_{x,t}(\lambda) = O(\partial_x^2)$$

interpreted as → **density of quasi-localised (semiclassical) particles**



the semiclassical particles time evolve classically: entanglement is simply transported

Semiclassical picture

$$\partial_t \rho_{x,t}(\lambda) + \partial_x v_{x,t}(\lambda)\rho_{x,t}(\lambda) = O(\partial_x^2)$$

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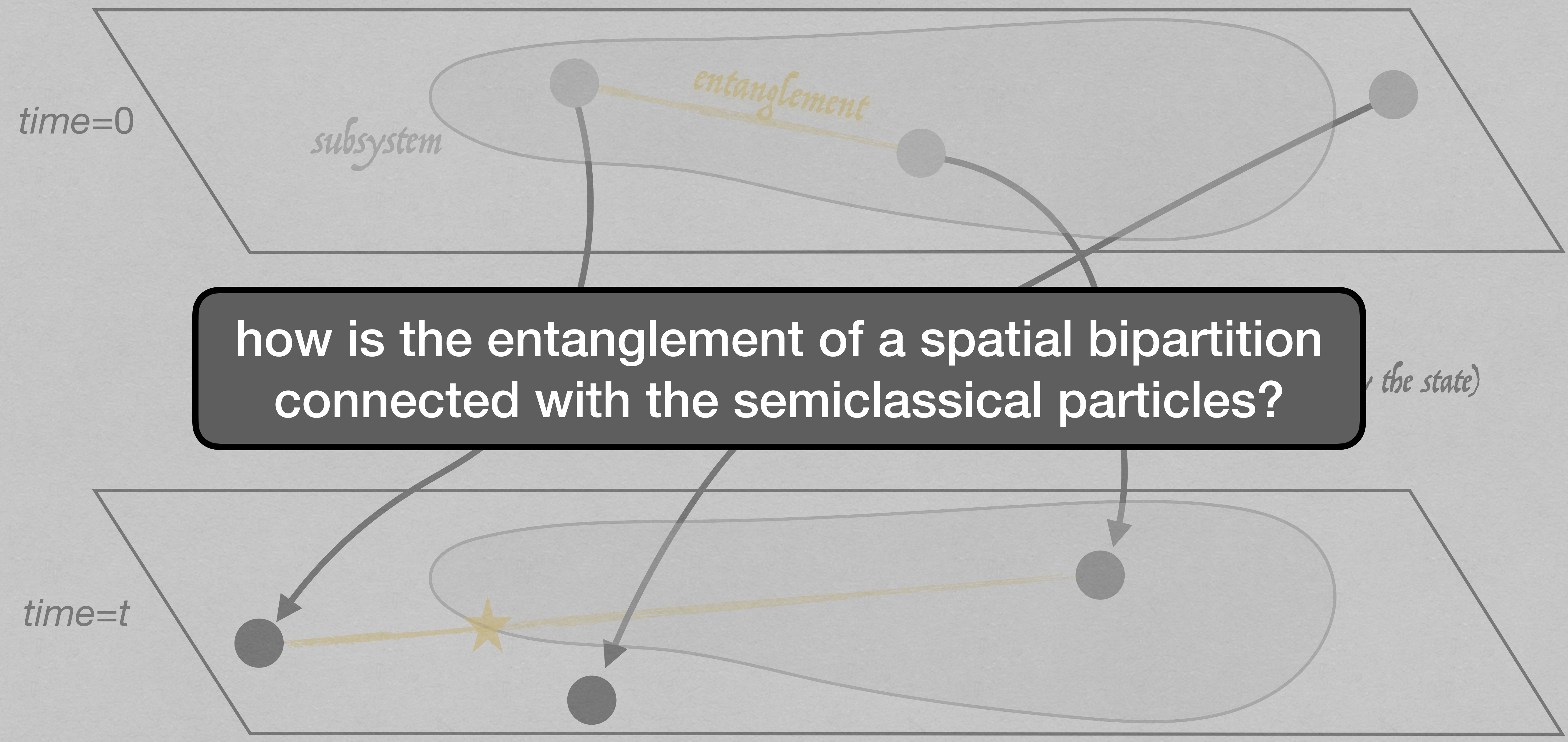
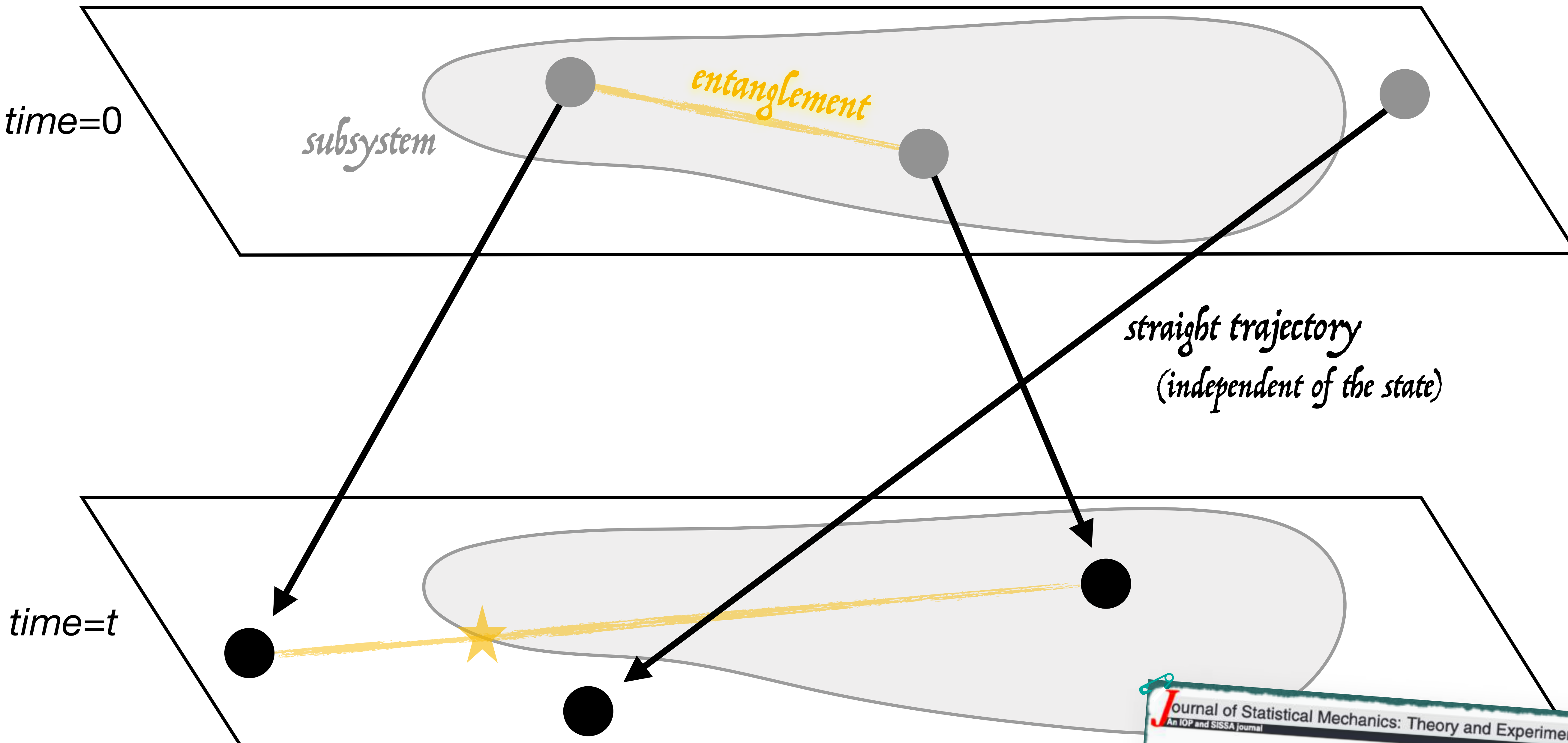


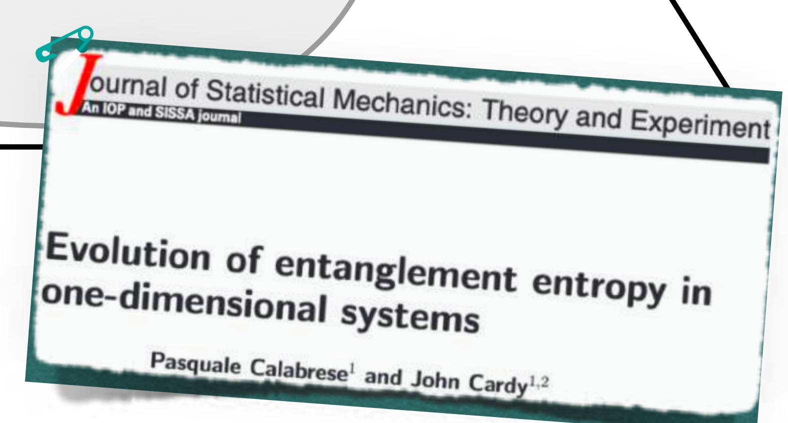
how is the entanglement of a spatial bipartition connected with the semiclassical particles?

the semiclassical particles time evolve classically: entanglement is simply transported

Semiclassical picture noninteracting systems

$$\partial_t \rho_{x,t}(\lambda) + v(\lambda)\partial_x \rho_{x,t}(\lambda) = \hbar^2 \frac{v''(\lambda)}{24}\partial_x^3 \rho_{x,t}(\lambda) + \dots$$




Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

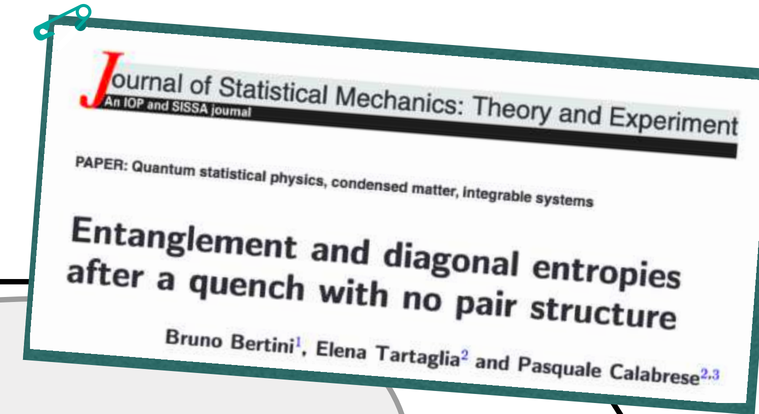
Evolution of entanglement entropy in one-dimensional systems

Pasquale Calabrese[1] and John Cardy[1,2]

Semiclassical picture

noninteracting systems

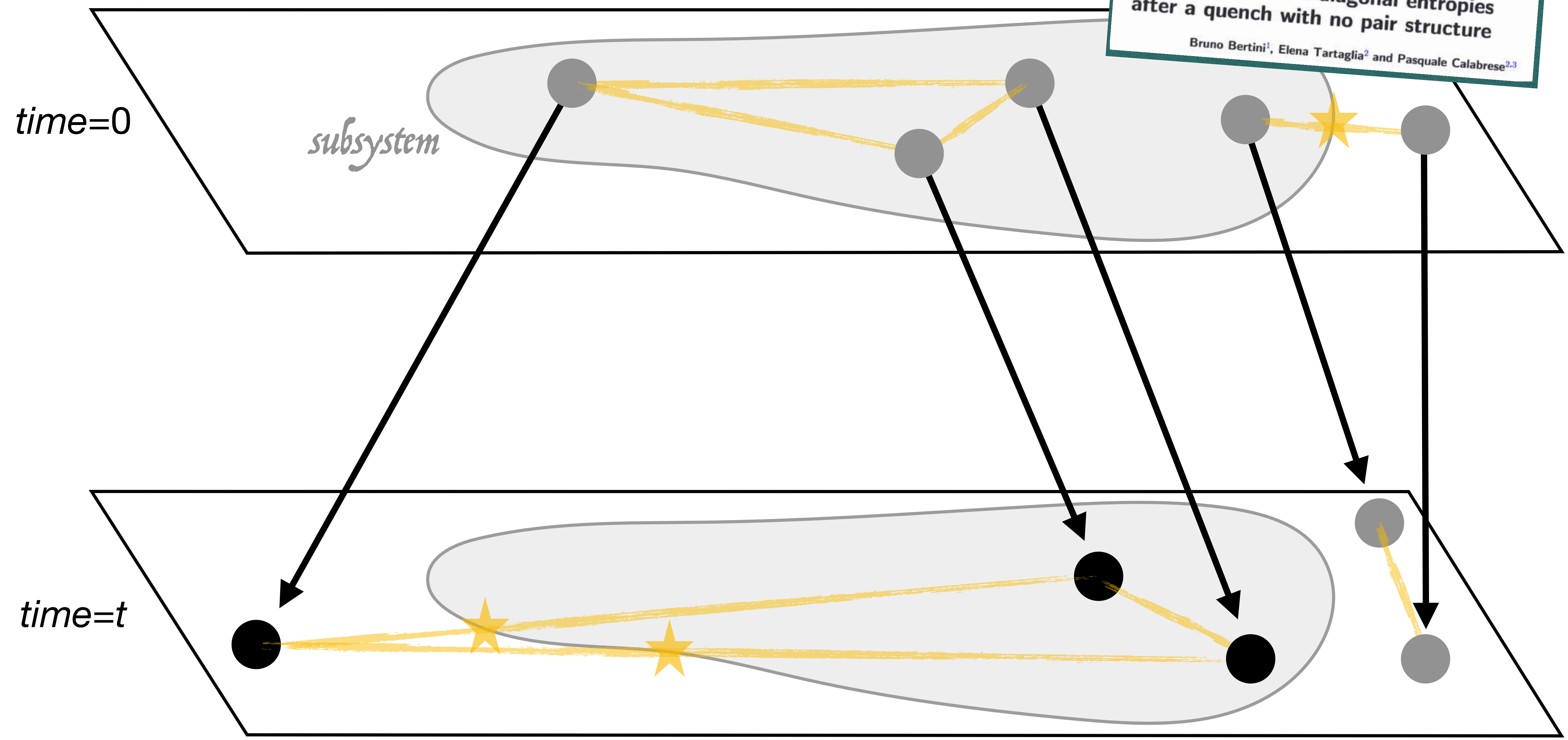
$$\partial_t \rho_{x,t}(\lambda) + v(\lambda)\partial_x \rho_{x,t}(\lambda) = \hbar^2 \frac{v''(\lambda)}{24}\partial_x^3 \rho_{x,t}(\lambda) + \dots$$


Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

PAPER: Quantum statistical physics, condensed matter, integrable systems

Entanglement and diagonal entropies after a quench with no pair structure

Bruno Bertini[1], Elena Tartaglia[2] and Pasquale Calabrese[2,3]



- **low-entanglement assumption:**
  only finite sets of particles are entangled with each others

Semiclassical picture — noninteracting systems

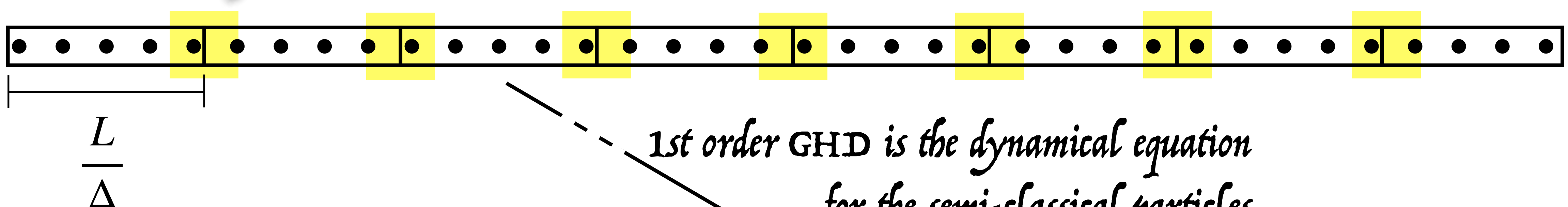
spatial bipartition → ? **particle bipartition**

$\hat\rho(0) = \otimes_{\mathcal{S}} \hat\rho_{\mathcal{S}}$ $\mathcal{S} = \{k\}$ **represents the sets of entangled particles**

$$\sim \otimes_{j=1}^{\frac{L}{\Delta}} \otimes_{\mathcal{S}} \hat\rho_{\mathcal{S}}^{(x_j)}$$

some quantum correlations are lost

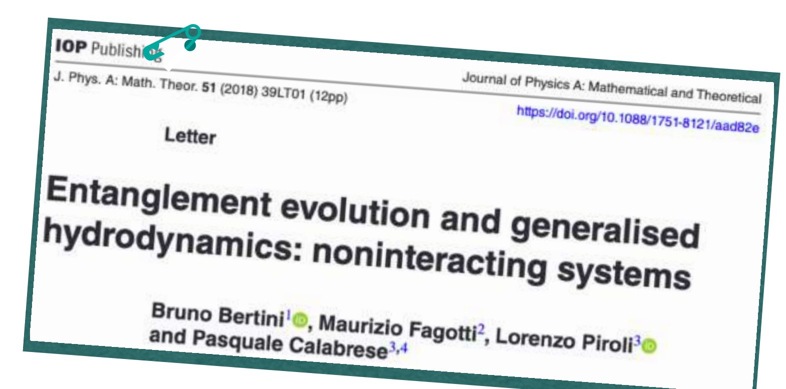
but they are not expected to contribute at the leading order



1st order GHD is the dynamical equation for the semi-classical particles

$$\partial_t \rho_{x,t}(\lambda) + \partial_x v_{x,t}(\lambda)\rho_{x,t}(\lambda) = 0$$

$$\mathrm{tr}_B[\hat\rho] \sim \mathrm{tr}_{\text{particles}\in \mathrm{B}}\left[e^{-i\hat H t} \otimes_{j=1}^{\frac{L}{\Delta}} \otimes_{\mathcal{S}} \hat\rho_{\mathcal{S}}^{(x_j)} e^{i\hat H t}\right]$$


IOP Publishing
J. Phys. A: Math. Theor. 51 (2018) 39LT01 (12pp)
Journal of Physics A: Mathematical and Theoretical
https://doi.org/10.1088/1751-8121/aad82e

Letter

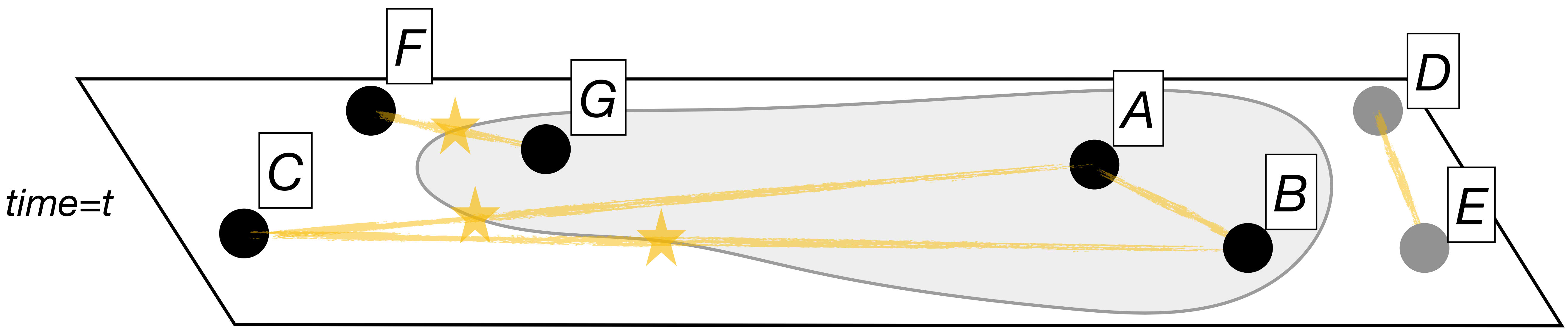
Entanglement evolution and generalised hydrodynamics: noninteracting systems

Bruno Bertini[1], Maurizio Fagotti[2], Lorenzo Piroli[3] and Pasquale Calabrese[3,4]

Semiclassical picture

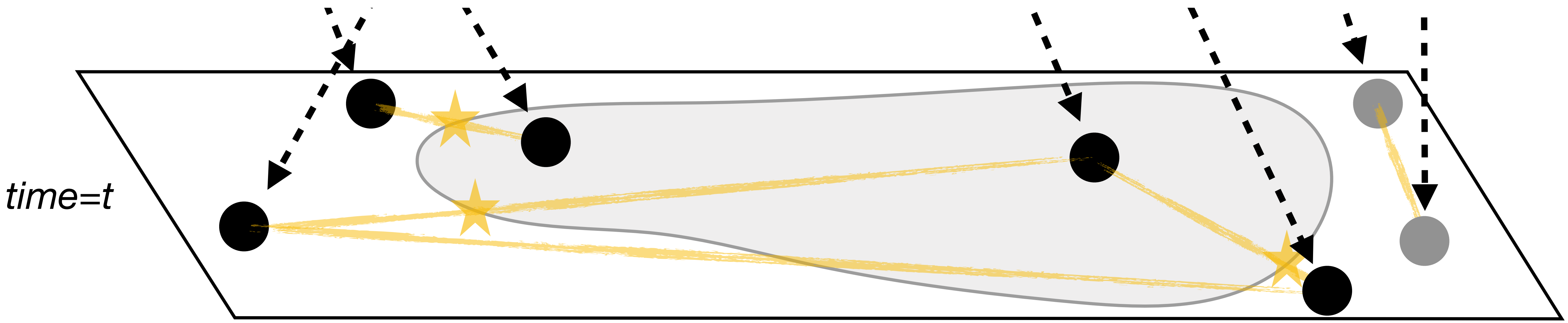
noninteracting systems

$$\hat{\rho}_A \sim \mathrm{tr}_C\left[\hat{\rho}_{ABC}\right]\mathrm{tr}_F\left[\hat{\rho}_{FG}\right]$$



Semiclassical picture

noninteracting systems

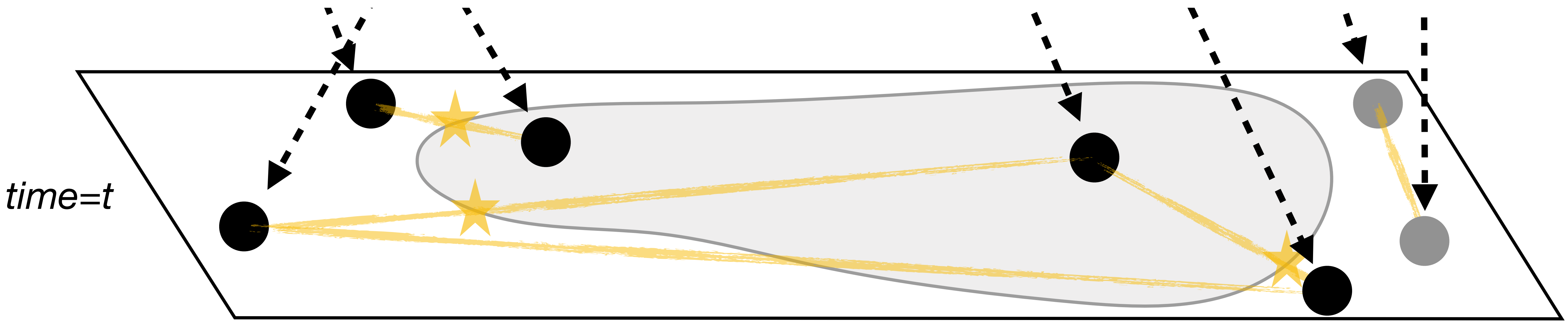


at **late times**, only **one** of the particles in an entangled set remains in the subsystem

Semiclassical picture **integrable** systems

local relaxation:

at late times the state becomes locally equivalent to a stationary state



at **late times**, only **one** of the particles in an entangled set remains in the subsystem

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 10, NUMBER 7 JULY 1969

Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interaction

C. N. YANG
Institute for Theoretical Physics, State University of New York, Stony Brook, New York

AND

C. P. YANG*
Ohio State University, Columbus, Ohio

(Received 10 October 1968)

The equilibrium thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction is shown to be derivable from the solution of a simple integral equation. The excitation spectrum at any temperature T is also found.

I. INTRODUCTION

The ground-state energy of a system of N bosons with repulsive delta-function interaction in one dimension with periodic boundary condition was calculated by Lieb and Liniger.[1] The Hamiltonian for the system is

$$H = -\sum_1^N \frac{\partial^2}{\partial x_i^2} + 2c\sum_{i>j}\delta(x_i - x_j), \quad c > 0, \quad (1)$$

and the periodic box has length L. Using Bethe's hypothesis[2] they showed that the k's in the hypothesis satisfy

$$(-1)^{N-1}\exp(-ikL) = \exp\left[i\sum_{k'}\theta(k' - k)\right], \quad (2)$$

where

$$\theta(k) = -2\tan^{-1}(k/c), \quad -\pi < \theta < \pi. \quad (3)$$

Now, for any set of real I's, $I_1, I_2, \cdots, I_N$, Eq. (4) has a unique real solution for the k's, $k_1, k_2, \cdots, k_N$. The proof of this statement (similar to but simpler than the proof of a corresponding statement[3] for the Heisenberg–Ising problem) follows. Let

$$\theta_1(k) = \int_0^k \theta(k)\,dk.$$

Define

$$B(k_1, \cdots, k_N) = \tfrac{1}{2}L\sum_1^N k_j^2 - 2\pi\sum_1^N I_j k_j - \tfrac{1}{2}\sum_{j,S}\theta_1(k_j - k_S). \quad (6)$$

Equation (4) is the condition for the extrema of B. Now the second-derivative matrix B_2 of B is positive-definite. [The first sum in (6) contributes a positive-definite part to B_2. The second sum contributes

By a continuity argument with respect to c^{-1} we obtain the following:

Theorem: For any set of I's satisfying (5), no two of which are identical, there is a unique set of real k's satisfying (4), with no two k's being identical. With this set of k's, one eigenfunction of H, of Bethe's form, can be constructed. The totality of such eigenfunctions form a complete set for the boson system.

The numbers I are quantum numbers for the problem.

III. ENERGY AND ENTROPY FOR A SYSTEM WITH $N = \infty$

We now consider the problem for $N = \infty$ and $L = \infty$ at a fixed density $D = N/L$. For the ground state, the quantum numbers I/L form[1] a uniform lattice between $-D/2$ and $D/2$. The k's then form[1] a nonuniform distribution between a maximum k and a minimum k. For an excited state, (5) shows that the quantum numbers I/L are still on the same lattice, but not all lattice sites are taken, and the limits $-D/2$ and $D/2$ are no longer respected. We shall call the omitted lattice sites J_j/L. We would want to define corresponding "omitted k values" to be called holes. This can be easily done: Given the I's, Eq. (4) defines the set of k's as proved in the last section. Now,

$$Lh(p) \equiv pL - \sum_{k'} \theta(p - k') \qquad (8)$$

is a continuous monotonic function of p. At $p = \pm\infty$, it is equal to $\pm\infty$. Those values of p where $Lh(p) = 2\pi I$ are k's. Those values of p where $Lh(p) = 2\pi J$ will be defined as holes.

For a large system, there is thus a density distribution of holes as well as one of k's:

$$L\rho(k)\, dk = \text{No. of } k\text{'s in } dk,$$

The energy per particle for the state is

$$E/N = D^{-1}\int_{-\infty}^{\infty} \rho(k)k^2\, dk, \qquad (12)$$

where

$$D = N/L = \int_{-\infty}^{\infty} \rho(k)\, dk. \qquad (13)$$

The entropy of the "state" is not zero since the existence of the omitted quantum numbers J_j allows many wavefunctions of approximately the same energy to be described by the same ρ and ρ_h. In fact, for given ρ and ρ_h, the total number of k's and holes in dk is $L(\rho + \rho_h)\, dk$, of which $L\rho\, dk$ are k's and $L\rho_h\, dk$ are holes. Thus the number of possible choices of states in dk consistent with given ρ and ρ_h is

$$\frac{[L(\rho + \rho_h)\, dk]!}{[L\rho\, dk]!\,[L\rho_h\, dk]!}.$$

The logarithm of this gives the contribution to the entropy from dk. Thus, the total entropy is, putting the Boltzman constant equal to 1,

$$S = \sum \{(L\rho\, dk + L\rho_h\, dk) \ln(\rho + \rho_h) - L\rho\, dk \ln \rho - L\rho_h\, dk \ln \rho_h\}$$

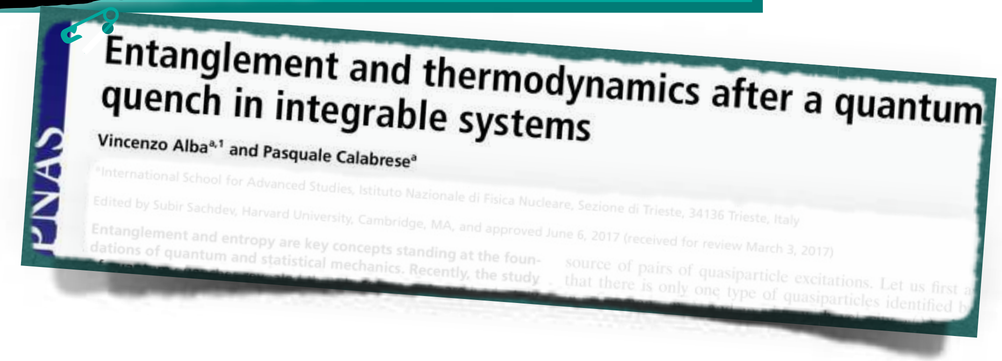
or

$$S/N = D^{-1}\int_{-\infty}^{\infty} [(\rho + \rho_h)\ln(\rho + \rho_h) - \rho \ln \rho - \rho_h \ln \rho_h]\, dk. \qquad (14)$$

IV. THERMAL EQUILIBRIUM

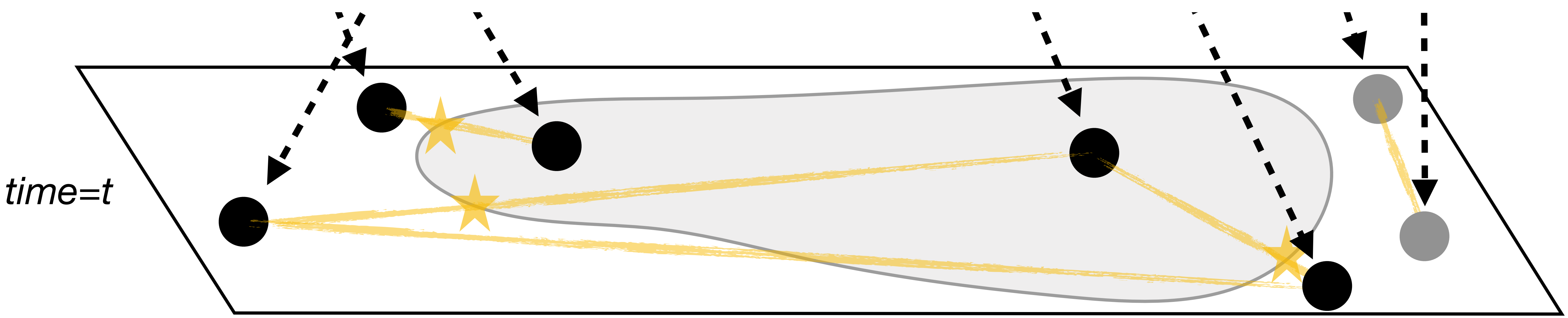
At temperature T, we should maximize the contribution to the partition function from the states described by ρ and ρ_h. In other words, given ρ, ρ_h is defined by (11). One then computes the contribution to the partition function

Semiclassical picture integrable systems



$$S_\lambda \sim -[\rho(\lambda)+\rho^h(\lambda)]\left[\log\frac{\rho(\lambda)}{\rho(\lambda)+\rho^h(\lambda)}+\log\frac{\rho^h(\lambda)}{\rho(\lambda)+\rho^h(\lambda)}\right]$$

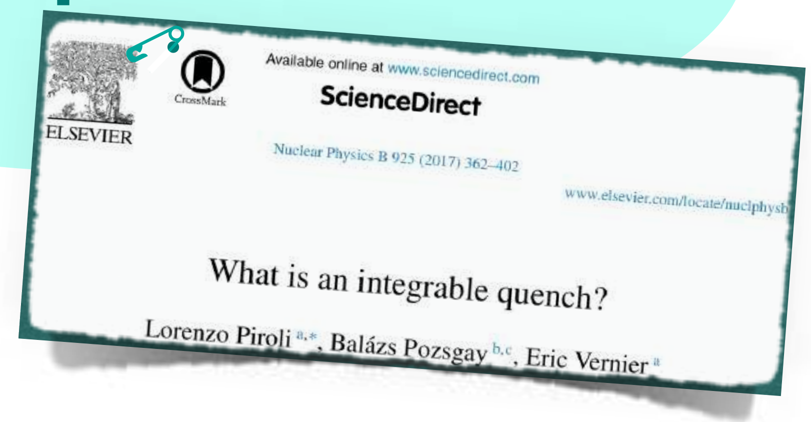
in particular $\lim_{t\to\infty} S_{vN}[A] = |A|\int \mathrm{d}\lambda S_\lambda$



Semiclassical picture

integrable systems

state with a pair structure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Nuclear Physics B 925 (2017) 362–402

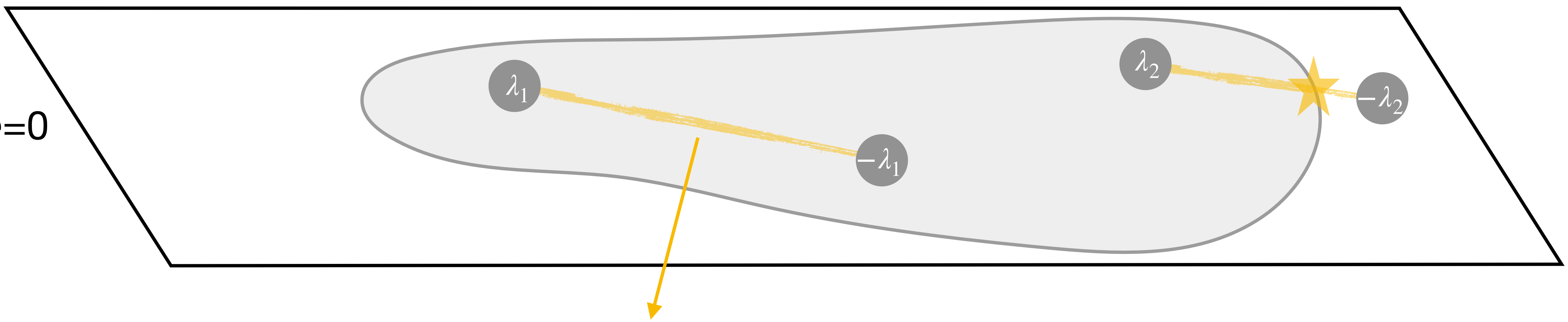
www.elsevier.com/locate/nuclphysb

What is an integrable quench?

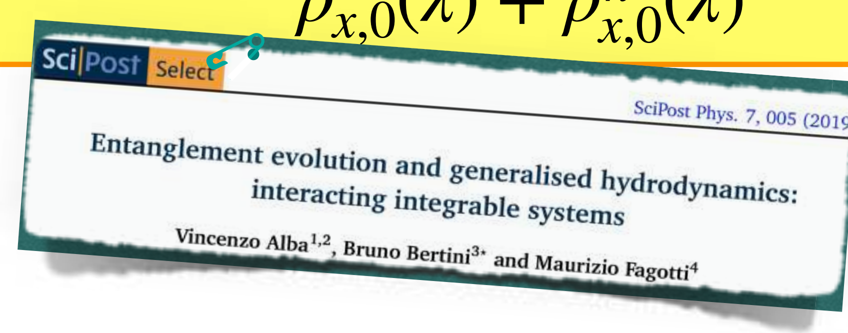
Lorenzo Piroli [a,*], Balázs Pozsgay [b,c], Eric Vernier [a]

$$\partial_t \rho_{x,t}(\lambda) + \partial_x v_{x,t}(\lambda)\rho_{x,t}(\lambda) = O(\partial_x^2)$$

time=0



$$S_\lambda \sim -[\rho_{x,0}(\lambda) + \rho^h_{x,0}(\lambda)][\log \frac{\rho_{x,0}(\lambda)}{\rho_{x,0}(\lambda) + \rho^h_{x,0}(\lambda)} + \log \frac{\rho^h_{x,0}(\lambda)}{\rho_{x,0}(\lambda) + \rho^h_{x,0}(\lambda)}]$$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SciPost Phys. 7, 005 (2019)

Entanglement evolution and generalised hydrodynamics: interacting integrable systems

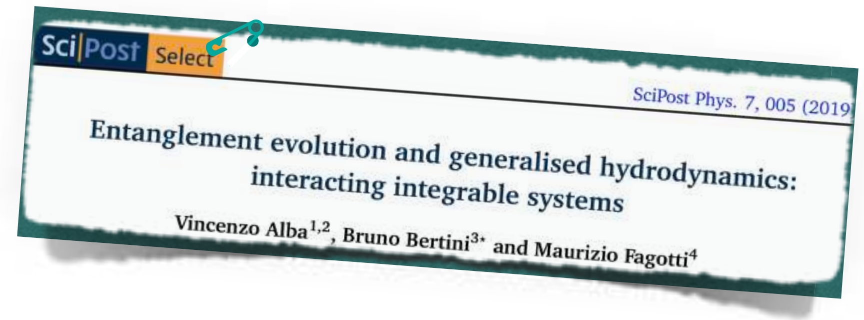
Vincenzo Alba [1,2], Bruno Bertini [3*] and Maurizio Fagotti [4]

Semiclassical picture

integrable systems

state with a pair structure

$$\partial_t \rho_{x,t}(\lambda) + \partial_x v_{x,t}(\lambda)\rho_{x,t}(\lambda) = O(\partial_x^2)$$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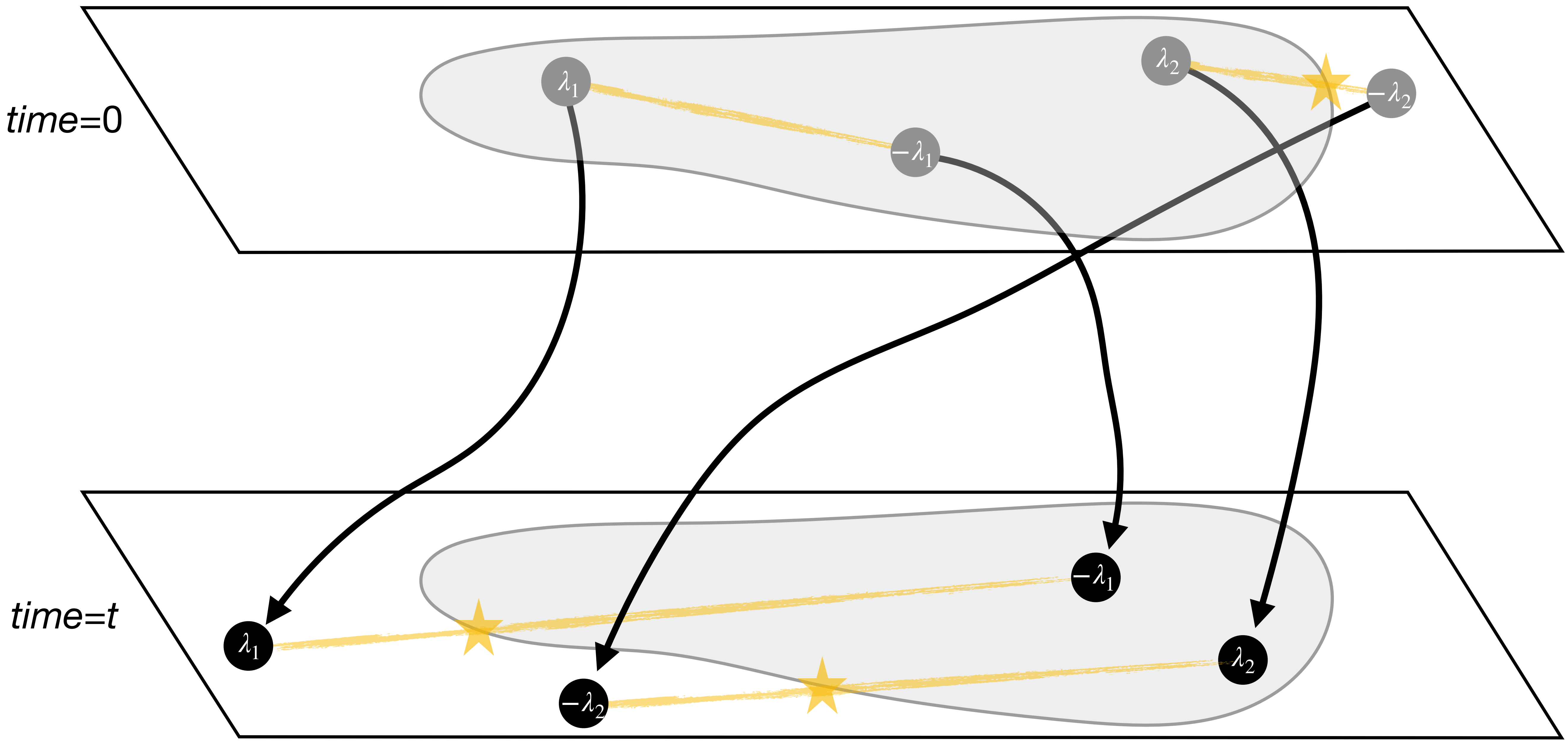
Entanglement evolution and generalised hydrodynamics: interacting integrable systems

Vincenzo Alba[1,2], Bruno Bertini[3*] and Maurizio Fagotti[4]

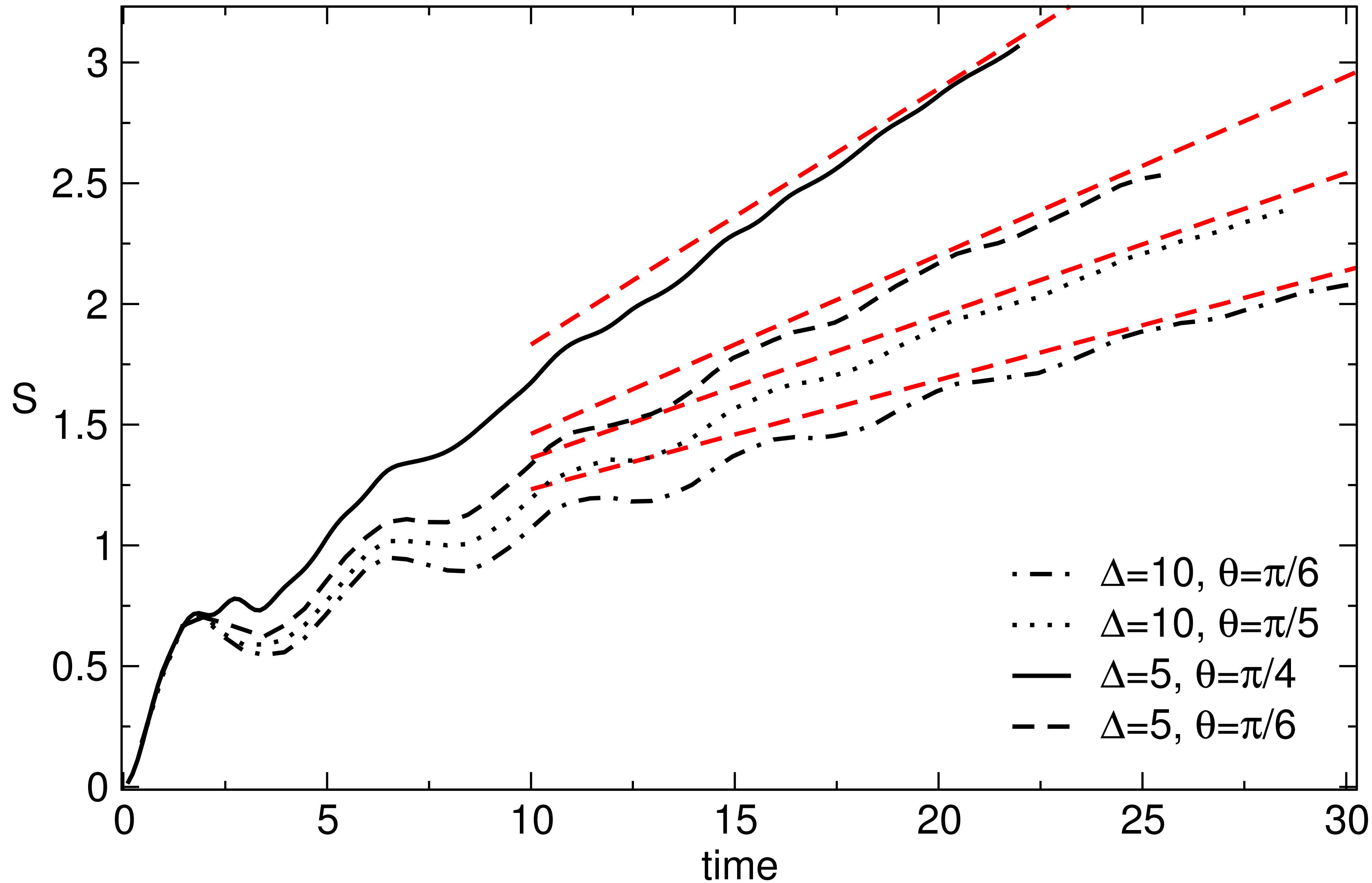


Half-chain entropy

$$H = \sum_{\ell} s^x_{\ell} s^x_{\ell+1} + s^y_{\ell} s^y_{\ell+1} + \Delta s^z_{\ell} s^z_{\ell+1}$$

$$|\Psi_0\rangle = |\Psi_L\rangle \otimes |\Psi_R\rangle$$

$$|\Psi_L\rangle = |\ldots \uparrow \downarrow \ldots\rangle \qquad |\Psi_R\rangle = |\ldots \nearrow \nearrow \ldots\rangle$$



Summary

- 1st order **GHD** supports the semiclassical picture for the time evolution of the entanglement entropy after quantum quenches

- Predictions can be obtained even in the presence of interactions

Summary

Open problem

In the presence of interactions, the semiclassical picture in terms of the density matrix of entangled particles has a fault:

no analytic expression for the time evolution of the Rényi entropies yet!

thank You for your attention!