Optimization of full-RSB spherical spin glasses

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מכון ויצמן למדע
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⋆ **Pure *p*-spin:** for $p \geq 2$,

$$H_{N,p}(\mathbf{x}) = \sqrt{N} \sum_{i_1,\dots,i_p=1}^{N} J_{i_1,\dots,i_p} x_{i_1} x_{i_2} \cdots x_{i_p},$$

where $J_{i_1,\dots,i_p} \sim \mathrm{N}(0,1)$ i.i.d., $\mathbf{x} = (x_1, \dots, x_N)$.

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$\star$ **Mixed *p*-spin:** fix some coefficients $\gamma_p \geq 0$, $p \geq 2$,

$$H_N(\mathbf{x}) = \sum_{p=2}^{\infty} \gamma_p H_{N,p}(\mathbf{x}).$$

Optimization

Given all the coefficients $J_{i_1,\ldots,i_p}$, find a ground-state configuration,

$$\frac{1}{N} H_N(\mathbf{x}_\star) \approx E_\star := \lim_{N\to\infty} \frac{1}{N} \max_{\mathbf{x}\in\mathbb{S}^N} H_N(\mathbf{x}),$$

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Theorem (S. '18)

If at zero temperature the support of the **Parisi measure** is $[0,1]$ and $p_{\max} < \infty$, then there is an algorithm that finds $\mathbf{x}_\star$ with

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Define the mixture polynomial

$$\nu(t) = \sum_{p=2}^{\infty} \gamma_p^2 t^p.$$

Support $= [0,1] \iff \nu''(t)^{-\frac{1}{2}}$ **is concave** on $(0,1]$.

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Theorem (Montanari '18, El Alaoui-Montanari-Sellke '20)

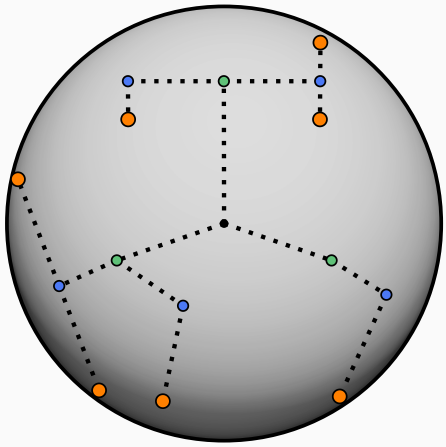
Similar result for **Ising spins**, i.e. optimization over $\boldsymbol{\sigma} \in \{\pm 1\}^N$, using massage passing algorithms.

The ultrametric tree at zero temp.

W.p.$\to 1$, there exists a tree with vertex set

$$T \subset \mathbb{B}^N = \{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\| \leq 1\}$$

with root $= 0$, leaves $\in \mathbb{S}^N$ such that:



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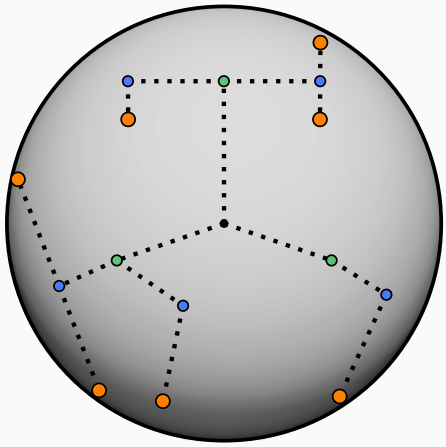

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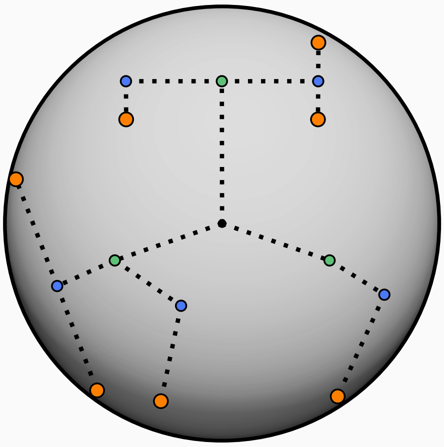

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$$\frac{1}{N} H_N(\mathbf{x}) = \frac{1}{N} \max_{\|\mathbf{y}\|=\|\mathbf{x}\|} H_N(\mathbf{y}) + o(1).$$

Remark: $H_N(\mathbf{x})$ is defined on $\mathbb{B}^N$ as on the sphere $\mathbb{S}^N$.

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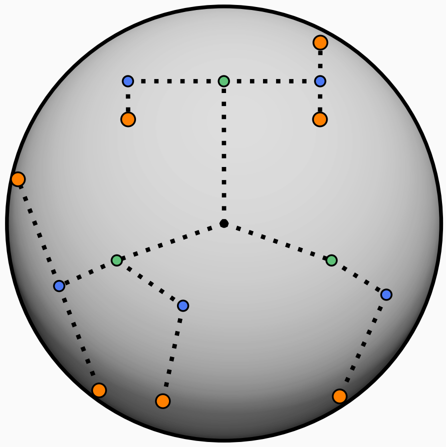

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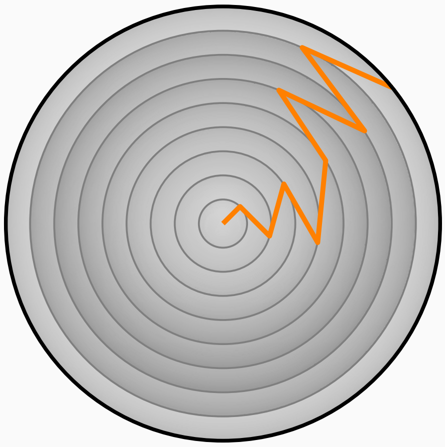
3. $\#$ levels depends on the size of the support of the Parisi measure. If support $= [0, 1]$: $\#$ levels$\to \infty$ as $N \to \infty$, vertices at all radii.

One path from the tree

What do we have for a single path from the tree?

$$\mathbf{x}_q : [0,1] \to \mathbb{R}^N, \quad \|\mathbf{x}_q\| = \sqrt{q}$$

(linearly interpolated between nodes of the tree...)



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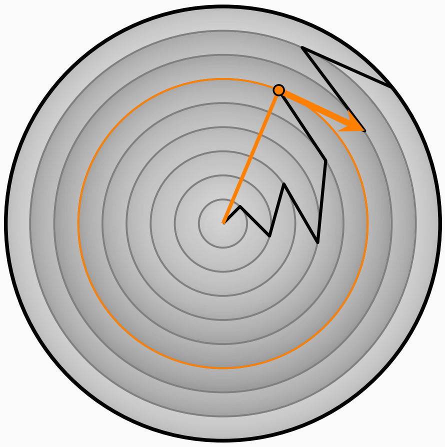

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2. The energy is consistently maximal,

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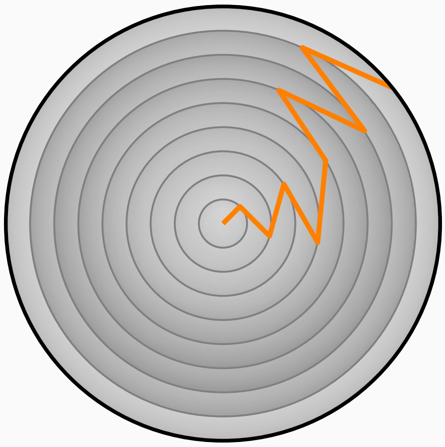
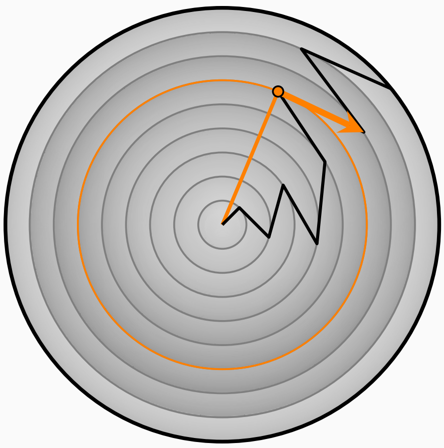


Constructing a good path

To 'imitate' the path on the tree, we set

$$\mathbf{x}_i = \sqrt{\frac{1}{k}} \sum_{j=0}^{i-1} \mathbf{v}_j, \quad i = 0, 1, \ldots, k,$$

where k is large and $\mathbf{v}_i$, $\|\mathbf{v}_i\| = 1$ are directions with $\mathbf{v}_i \perp \mathbf{x}_i$.



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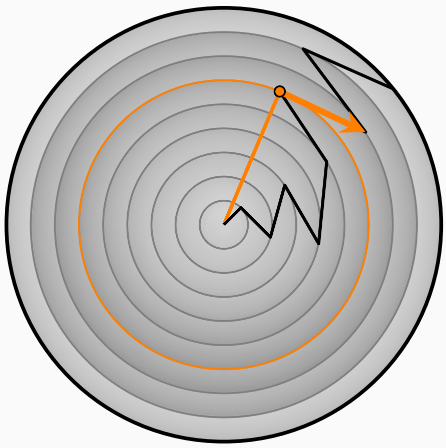

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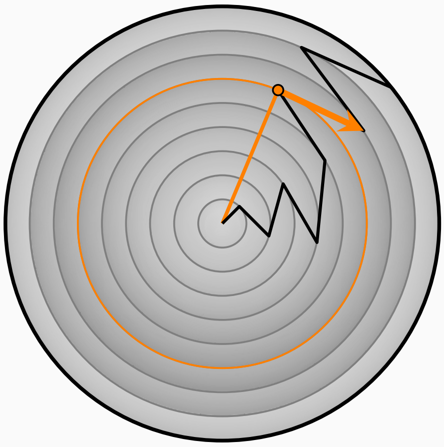

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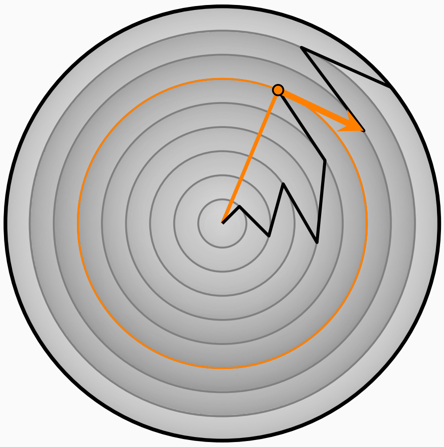
- Go in a direction of large e.v. of $\nabla^2_\perp H_N(\mathbf{x}_i)$.



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$$\frac{1}{N}\mathbf{v}^T \nabla^2 H_N(\mathbf{x}_i)\mathbf{v} \geq \max_{\mathbf{u}} \frac{1}{N}\mathbf{u}^T \nabla^2 H_N(\mathbf{x}_i)\mathbf{u} - \delta,$$

and update

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Energy on path: one can show that

$$\tfrac{1}{N} H_N(\mathbf{x}_i) \geq \max_{\|\mathbf{y}\|=\|\mathbf{x}_i\|} \tfrac{1}{N} H_N(\mathbf{y}) - \epsilon,$$

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where $\epsilon \to 0$ as $k \to \infty$ and $\delta \to 0$.

Complexity: Each step takes time $O(C(\delta)N^{\deg \nu})$.

Proof sketch

Proposition (S. '18)

W.h.p., denoting $q = \|\mathbf{x}\|^2$, uniformly in $\|\mathbf{x}\| \leq 1$:

$$\text{'edge' of } \tfrac{1}{N}\nabla_\perp^2 H_N(\mathbf{x}) \approx 2\nu''(q)^{\frac{1}{2}}.$$

For $\mathbf{x}_i$, $\mathbf{v}$ as before, by a Taylor expansion,

$$\frac{1}{N}H_N(\mathbf{x}_{i+1} + \tfrac{1}{\sqrt{k}}\mathbf{v}) \geq \frac{1}{N}H_N(\mathbf{x}_i) + \tfrac{1}{k}\nu''(q)^{\frac{1}{2}} - O(\tfrac{1}{k^{3/2}}).$$

If we concatenate many small intervals like above, we get a path

$$\frac{1}{N}H_N(\mathbf{x}_q) \gtrsim \int_0^q \nu''(q)^{\frac{1}{2}} \qquad (\|\mathbf{x}_q\| = \sqrt{q}).$$

From (e.g.) Parisi formula, assuming the condition on the support,

$$\lim_{N\to\infty} \frac{1}{N} \max_{\|\mathbf{x}\|=\sqrt{q}} H_N(\mathbf{x}) = \int_0^q \nu''(q)^{\frac{1}{2}}.$$

Thank You!