Square Root - Direct Method

In *IEEE* floating point standard a real number is represented as :

$$(-1)^S \times M \times 2^E$$

In 32-bit representation :

normal	sub-normal
$S \in \{0,1\}$	
$M \in [1,2[$	*Or* $M \in [0,2[$
$E \in \{-126,\cdots,127\}$	*if* $E = -127$

Square Root - Direct Method

Let R be a positive real number ($S = 0$) $R = (-1)^S \times M \times 2^E$

We seek to calculate the positive real number $\sqrt{R}$

$$\sqrt{R} = (-1)^S \times \sqrt{M} \times 2^{E/2}$$

$\frac{E}{2}$ is easy to calculate ... except when E is odd

Square Root - Direct Method

To help the calculation of $\sqrt{R}$

the representation of R must be changed into

$R = (-1)^S \times M' \times 2^{2E'}$ where $S = 0$

$E' = \text{Int}\left(E/2\right)$

$M' \in [0,4[$

Then $\sqrt{R} = (-1)^S \times \sqrt{M'} \times 2^{E'}$ with $\sqrt{M'} \in [0,2[$

Square Root - Direct Method

However, when $M' \in [0,1[$ the calculation of $\sqrt{M'}$ may lead to a lost of precision

Therefore if $M' = 0$ $\sqrt{M'} = 0$

and if $M' \in]0,1[$ E' is decreased and M' is $\times 2$ until it can fit within $[1,4[$

Square Root - Direct Method

Then, the problem can be stated as :

Given a positive real number $A \in [1, 4[$
we seek to calculate X_{Th} such as $X_{Th} = \sqrt{A}$

$X_{Th} \in [1, 2[$

Let X_n be an approximation of X_{Th} coded on n+1 bits

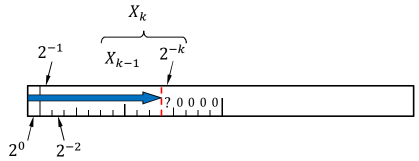
$$X_n = \sum_{j=0}^{n} x_{-j} \times 2^{-j} \quad \text{such as} \quad {X_n}^2 \leq A < (X_n + 2^{-n})^2$$

We propose to calculate X_n digit-by-digit

Square Root - Direct Method

Let $X_{k-1} = \sum_{j=0}^{k-1} x_{-j} \times 2^{-j}$ such as

$${X_{k-1}}^2 \leq A < \left(X_{k-1} + 2^{-(k-1)}\right)^2$$



Square Root - Direct Method

Let $X_k = \sum_{j=0}^{k} x_{-j} \times 2^{-j}$ such as ${X_k}^2 \leq A < \left(X_k + 2^{-k}\right)^2$

At each iteration X_k is obtained from X_{k-1}

$$X_k = X_{k-1} + x_{-k} 2^{-k} \qquad x_{-k} \in \{0, 1\}$$

2^{-k} is denoted W_k

Square Root - Direct Method

$${X_k}^2 \leq A < \left(X_k + 2^{-k}\right)^2$$

$$0 \leq A - {X_k}^2 < \left(X_k + 2^{-k}\right)^2 - {X_k}^2$$

Let $\Delta_k = A - {X_k}^2$

$$0 \leq \Delta_k < 2^{-k}\left(2X_k + 2^{-k}\right)^2$$

yet $X_k < 2$

and $X_k + 2^{-k} \leq 2$

then $0 \leq \Delta_k < 4 \times 2^{-k}$

- $0 \leq \Delta_0 < 4$
- At each iteration the upper bound of Δ_k is divided by 2

Square Root - Direct Method

$$\Delta_k = A - {X_k}^2$$

$$\Delta_k = A - \left(X_{k-1} + x_{-k}2^{-k}\right)^2$$

$$\Delta_k = A - \left({X_{k-1}}^2 + 2x_{-k}2^{-k}X_{k-1} + x_{-k}2^{-k}2^{-k}\right)$$

$$\Delta_k = \Delta_{k-1} - x_{-k}2^{-k}\left(2X_{k-1} + 2^{-k}\right)$$

$$2^k\Delta_k = 2^k\Delta_{k-1} - x_{-k}\left(2X_{k-1} + 2^{-k}\right) \qquad \text{Let } D_k = 2^k\Delta_k$$

$$D_k = 2D_{k-1} - x_{-k}\left(2X_{k-1} + 2^{-k}\right) \quad x_{-k} = \begin{cases} 0 \\ 1 \end{cases} \text{such as } 0 \le D_k$$

Square Root - Direct Method

$$D_k = 2D_{k-1} - x_{-k}\left(2X_{k-1} + 2^{-k}\right) \quad x_{-k} = \begin{cases} 0 \\ 1 \end{cases} \text{such as } 0 \le D_k$$

Iteration scheme :

$$\begin{cases} D_k = 2\left(D_{k-1} - x_{-k}\left(X_{k-1} + 2^{-k-1}\right)\right) \\ X_k = X_{k-1} + x_{-k}2^{-k} \end{cases} \qquad x_{-k} = \begin{cases} 0 \\ 1 \end{cases} \text{such as } 0 \le D_k$$

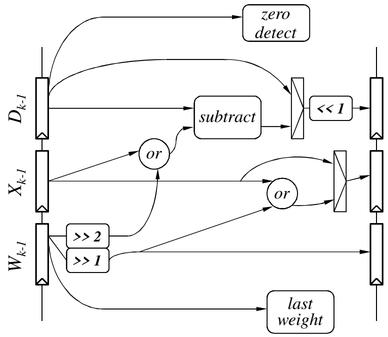
Square Root - Direct Method

Implementation

$$D_k = 2(D_{k-1} - x_{-k}(X_{k-1} + 2^{-2}W_{k-1}))$$

$$X_k = X_{k-1} + x_{-k}2^{-1}W_{k-1}$$

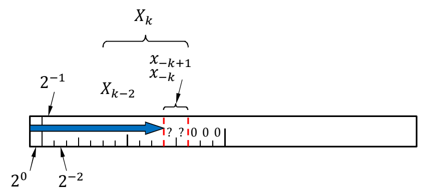
$$W_k = 2^{-k} = 2^{-1}W_{k-1}$$



Square Root - Direct Method - Improvement

radix 4

$$X_k = X_{k-2} + (2x_{-k+1} + x_{-k})2^{-k}$$



Square Root - Direct Method - Improvement

radix 4

$$x_{-k+1} = 0,\ \ x_{-k} = 0 \quad \begin{cases} D_k = 4\, D_{k-2} \\ X_k = X_{k-2} \end{cases}$$

$$x_{-k+1} = 0,\ \ x_{-k} = 1 \quad \begin{cases} D_k = 4\left(D_{k-2} - {}^1\!/_2\left(X_{k-2} + 1 \times 2^{-k-1}\right)\right) \\ X_k = X_{k-2} + 1 \times 2^{-k} \end{cases}$$

$$x_{-k+1} = 1,\ \ x_{-k} = 0 \quad \begin{cases} D_k = 4\left(D_{k-2} - \quad \left(X_{k-2} + 2 \times 2^{-k-1}\right)\right) \\ X_k = X_{k-2} + 2 \times 2^{-k} \end{cases}$$

$$x_{-k+1} = 1,\ \ x_{-k} = 1 \quad \begin{cases} D_k = 4\left(D_{k-2} - {}^3\!/_2\left(X_{k-2} + 3 \times 2^{-k-1}\right)\right) \\ X_k = X_{k-2} + 3 \times 2^{-k} \end{cases}$$

Square Root - Direct Method - Improvement

Implementation

$$\begin{cases} D_k = 4\, D_{k-2} \\ X_k = X_{k-2} \end{cases}$$

$$\begin{cases} D_k = 4\left(D_{k-2} - {}^1\!/_2\left(X_{k-2} + 1 \times 2^{-k-1}\right)\right) \\ X_k = X_{k-2} + 1 \times 2^{-k} \end{cases}$$

$$\begin{cases} D_k = 4\left(D_{k-2} - \quad \left(X_{k-2} + 2 \times 2^{-k-1}\right)\right) \\ X_k = X_{k-2} + 2 \times 2^{-k} \end{cases}$$

$$\begin{cases} D_k = 4\left(D_{k-2} - {}^3\!/_2\left(X_{k-2} + 3 \times 2^{-k-1}\right)\right) \\ X_k = X_{k-2} + 3 \times 2^{-k} \end{cases}$$