Outline

- Digital CMOS Design
- Arithmetic Operators
- Floating Point Arithmetic Operators
  - Square root
  - division

Division - Direct Method

In *IEEE* floating point standard a real number is represented as :

$$(-1)^S \times M \times 2^E$$

In 32-bit representation :

normal	sub-normal
$S \in \{0,1\}$	
$M \in [1,2[$	*Or* $M \in [0,2[$
$E \in \{-126, \cdots, 127\}$	*if* $E = -127$

Division - Direct Method

Let U and V ($V \neq 0$) be two real numbers

We seek to calculate the real number $\frac{U}{V}$

$$U = (-1)^{S_U} \times M_U \times 2^{E_U}$$
$$V = (-1)^{S_V} \times M_V \times 2^{E_V}$$

then $$\frac{U}{V} = (-1)^{S_U \oplus S_V} \times \frac{M_U}{M_V} \times 2^{E_U - E_V}$$

Division - Direct Method

If U or V are sub-normal numbers, the calculation of $\frac{U}{V}$ may lead to a lost of precision

Therefore if $M_U = 0$ $\frac{U}{V} = 0$

and if M_U or $M_V \in]0,1[$

E_U or E_V are decreased and M_U or M_V are $\times 2$ until they can fit within $[1,2[$

Division - Direct Method

Since $M_U \in [1,2[$ and $M_V \in [1,2[$ $\quad \frac{M_U}{M_V} \in]0.5,2[$

Therefore if $M_U < M_V$

E_U is decreased and M_U is $\times 2$ so $\frac{M_U}{M_V} \in [1,2[$

Division - Direct Method

Then, the problem can be stated as :

Given two positive real numbers $A \in [1,4[$ and $B \in [1,2[$ such as $A \geq B$ we seek to calculate $X_{Th} = \frac{A}{B}$

$X_{Th} \in [1,2[$

Let X_n be an approximation of X_{Th} coded on n+1 bits

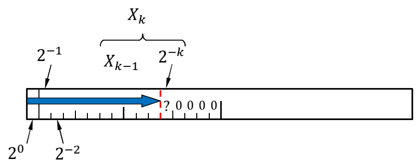
$$X_n = \sum_{j=0}^{n} x_{-j} \times 2^{-j} \quad \text{such as} \quad X_n \leq \frac{A}{B} < (X_n + 2^{-n})$$

We propose to calculate X_n digit-by-digit

Division - Direct Method

Let $X_{k-1} = \sum_{j=0}^{k-1} x_{-j} \times 2^{-j}$ such as

$$X_{k-1} \leq \frac{A}{B} < \left(X_{k-1} + 2^{-(k-1)}\right)$$



Division - Direct Method

Let $X_k = \sum_{j=0}^{n} x_{-j} \times 2^{-j}$ such as $X_k \leq \frac{A}{B} < \left(X_k + 2^{-k}\right)$

At each iteration X_k is obtained from X_{k-1}

$$X_k = X_{k-1} + x_{-k} 2^{-k} \qquad x_{-k} \in \{0,1\}$$

2^{-k} is denoted W_k

Division - Direct Method

$$X_k \le \frac{A}{B} < (X_k + 2^{-k})$$

$$BX_k \le A < B(X_k + 2^{-k})$$

$$0 \le A - BX_k < 2^{-k}B \qquad \text{Let} \quad \Delta_k = A - BX_k$$

$$0 \le \Delta_k < 2^{-k}B \qquad \text{yet} \quad B < 2$$

then $0 \le \Delta_k < 2^{-k} \times 2$

- $0 \le \Delta_0 < 2$
- At each iteration the upper bound of Δ_k is divided by 2

Division - Direct Method

$$\Delta_k = A - BX_k$$

$$\Delta_k = A - B(X_{k-1} + x_{-k}2^{-k})$$

$$\Delta_k = A - BX_{k-1} - x_{-k}2^{-k}B$$

$$\Delta_k = \Delta_{k-1} - x_{-k}2^{-k}B$$

$$2^k\Delta_k = 2^k\Delta_{k-1} - x_{-k}B \qquad \text{Let} \quad D_k = 2^k\Delta_k$$

$$D_k = 2D_{k-1} - x_{-k}B \qquad x_{-k} = \begin{cases} 0 \\ 1 \end{cases} \text{ such as } 0 \le D_k$$

Division - Direct Method

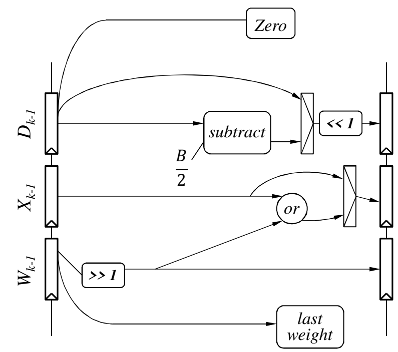
Iteration scheme :

$$\begin{cases} D_k = 2\left(D_{k-1} - x_{-k}\dfrac{B}{2}\right) \\ X_k = X_{k-1} + x_{-k}2^{-k} \end{cases} \qquad x_{-k} = \begin{cases} 0 \\ 1 \end{cases} \text{ such as } 0 \le D_k$$

Division - Direct Method

Implementation

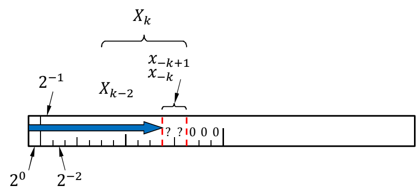
$$D_k = 2\left(D_{k-1} - x_{-k}\frac{B}{2}\right)$$
$$X_k = X_{k-1} + x_{-k}2^{-1}W_{k-1}$$
$$W_k = 2^{-k} = 2^{-1}W_{k-1}$$



Division - Direct Method - Improvement

radix 4

$$X_k = X_{k-2} + (2x_{-k+1} + x_{-k})2^{-k}$$



Division - Direct Method - Improvement

radix 4

$$\Delta_k = A - BX_k$$

$$\Delta_k = A - B\left(X_{k-2} + \left(x_{-k+1}2^{-k+1} + x_{-k}2^{-k}\right)\right)$$

$$\Delta_k = \Delta_{k-2} - B\left(x_{-k+1}2^{-k+1} + x_{-k}2^{-k}\right)$$

$$2^k\Delta_k = 2^k\Delta_{k-2} - B(2x_{-k+1} + x_{-k})$$

Let $D_k = 2^k\Delta_k$

$$D_k = 4\left(D_{k-2} - \frac{B}{4}(2x_{-k+1} + x_{-k})\right)$$

Division - Direct Method - Improvement

radix 4

Iteration scheme :

$$\begin{cases} D_k = 4\left(D_{k-2} - \frac{B}{4}(2x_{-k+1} + x_{-k})\right) \\ X_k = X_{k-2} + (2x_{-k+1} + x_{-k})2^{-k} \end{cases}$$

$$x_{-k+1}, x_{-k} = \begin{cases} 00 \\ 01 \\ 10 \\ 11 \end{cases} \text{ such as } 0 \le D_k$$

Division - Direct Method - Improvement

radix 4

$$x_{-k+1} = 0,\ x_{-k} = 0 \quad \begin{cases} D_k = 4\,D_{k-2} \\ X_k = X_{k-2} \end{cases}$$

$$x_{-k+1} = 0,\ x_{-k} = 1 \quad \begin{cases} D_k = 4\left(D_{k-2} - \frac{B}{4}\right) \\ X_k = X_{k-2} + 1 \times 2^{-k} \end{cases}$$

$$x_{-k+1} = 1,\ x_{-k} = 0 \quad \begin{cases} D_k = 4\left(D_{k-2} - 2\frac{B}{4}\right) \\ X_k = X_{k-2} + 2 \times 2^{-k} \end{cases}$$

$$x_{-k+1} = 1,\ x_{-k} = 1 \quad \begin{cases} D_k = 4\left(D_{k-2} - 3\frac{B}{4}\right) \\ X_k = X_{k-2} + 3 \times 2^{-k} \end{cases}$$

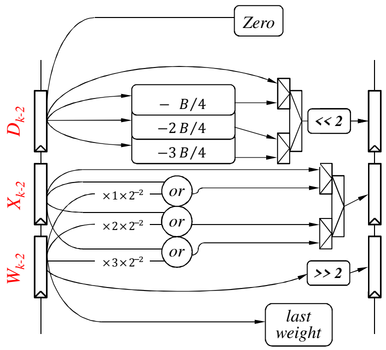
Division - Direct Method - Improvement

$x_{-k+1}=0,\ x_{-k}=0 \quad \begin{cases} D_k = 4\,D_{k-2} \\ X_k = X_{k-2} \end{cases}$

$x_{-k+1}=0,\ x_{-k}=1 \quad \begin{cases} D_k = 4\left(D_{k-2}-\frac{B}{4}\right) \\ X_k = X_{k-2}+1\times 2^{-k} \end{cases}$

$x_{-k+1}=1,\ x_{-k}=0 \quad \begin{cases} D_k = 4\left(D_{k-2}-2\frac{B}{4}\right) \\ X_k = X_{k-2}+2\times 2^{-k} \end{cases}$

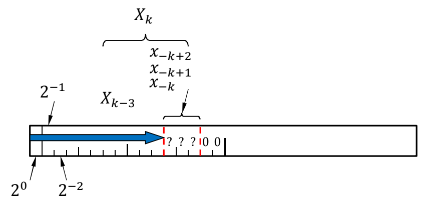
$x_{-k+1}=1,\ x_{-k}=1 \quad \begin{cases} D_k = 4\left(D_{k-2}-3\frac{B}{4}\right) \\ X_k = X_{k-2}+3\times 2^{-k} \end{cases}$



Division - Direct Method - Improvement

radix 8

$$X_k = X_{k-3} + (4x_{-k+2} + 2x_{-k+1} + x_{-k})2^{-k}$$



Division - Direct Method - Improvement

radix 8

$$\Delta_k = A - BX_k$$

$$\Delta_k = A - B\left(X_{k-3} + \left(x_{-k+2}2^{-k+2} + x_{-k+1}2^{-k+1} + x_{-k}2^{-k}\right)\right)$$

$$\Delta_k = \Delta_{k-3} - B\left(x_{-k+2}2^{-k+2} + x_{-k+1}2^{-k+1} + x_{-k}2^{-k}\right)$$

$$2^k\Delta_k = 2^k\Delta_{k-3} - B(4x_{-k+2} + 2x_{-k+1} + x_{-k})$$ Let $D_k = 2^k\Delta_k$

$$D_k = 8\left(D_{k-3} - \frac{B}{8}(4x_{-k+2} + 2x_{-k+1} + x_{-k})\right)$$

Division - Direct Method - Improvement

Iteration scheme :

radix 8

$$\begin{cases} D_k = 8\left(D_{k-3} - \frac{B}{9}(4x_{-k+2} + 2x_{-k+1} + x_{-k})\right) \\ X_k = X_{k-3} + (4x_{-k+2} + 2x_{-k+1} + x_{-k})2^{-k} \end{cases}$$

$$x_{-k+2}, x_{-k+1}, x_{-k} = \begin{cases} 000 \\ 001 \\ 010 \\ 011 \\ 000 \\ 001 \\ 010 \\ 011 \end{cases}$$ such as $0 \le D_k$

Division - Direct Method - Improvement

$$D_k = 8\left(D_{k-3} - \frac{B}{8}\left(4x_{-k+2} + 2x_{-k+1} + x_{-k}\right)\right)$$

$$X_k = 8\left(X_{k-3} + \left(4x_{-k+2} + 2x_{-k+1} + x_{-k}\right)\right)$$