The Evolution of Income or Wealth Distribution with Higher Order Autoregressive Processes

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Introduction

How does intergenerational mobility effect inequality? We shall investigate using autoregressive process models.
Could a 'snapshot' inequality estimate mobility between generations?

From shirtsleeves to shirtsleeves in three generations vs.
Shirtsleeves stay as shirtsleeves over many generations.

Model

Let X_t be the log of income/wealth and ϵ_t be the cultural and genetic endowment of generation t. Assume the following model with microeconomic foundations [Solon, 2018] [1]:

$$X_t = \alpha + \beta X_{t-1} + \epsilon_t, \quad \text{where } \epsilon_t = \theta\epsilon_{t-1} + \eta_t \tag{1}$$

α trend in average incomes, $\eta_t \sim \mathcal{N}(0, \sigma^2)$ random endowment noise. Note endowments AR(1) and if $\theta = 0$ then X_t is AR(1).

[1] adapted from classical model from Becker and Tomes 1979

Model can be Expressed as AR(2)

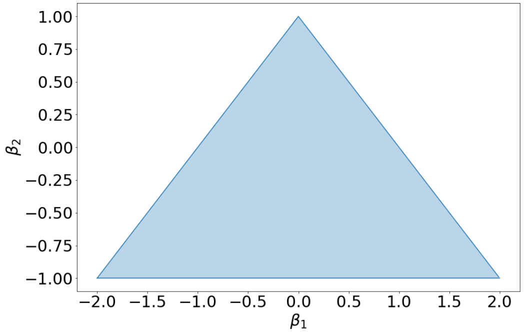
$$
\begin{aligned}
X_t &= \alpha + \beta X_{t-1} + \theta \epsilon_{t-1} + \eta_t \\
&= \alpha + \beta X_{t-1} + \theta(X_{t-1} - (\alpha + \beta X_{t-2})) + \eta_t \\
&= \alpha(1 - \theta) + (\beta + \theta) X_{t-1} - \beta\theta X_{t-2} + \eta_t
\end{aligned}
$$

We would expect $\beta > 0$ and $\theta > 0$ which then implies $-\beta\theta < 0$ - a negative relation between children's and grandparents income?!

General Results for AR(2)

$$X_t = \alpha + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \eta_t, \quad \eta_t \sim \mathcal{N}(0, \sigma^2)$$

Stationary when mean and variance independent of time.



$$\text{Stat. condns.:} \quad \begin{aligned} \beta_2 - \beta_1 &< 1 \text{ for } \quad -2 < \beta_1 \leq 0 \text{ and} \\ \beta_2 + \beta_1 &< 1 \text{ for } \quad 0 \leq \beta_1 < 2 \end{aligned}$$

$$\mathbb{E}[X_t] := \mu = \frac{\alpha}{1 - \beta_1 - \beta_2}, \quad \text{var}(X_t) := \gamma_0 = \frac{(1 - \beta_2)\sigma^2}{(1 + \beta_2)((1 - \beta_2)^2 - \beta_1^2)},$$

$$\rho_1 := \frac{\text{cov}(X_{t-1}, X_t)}{\text{var}(X_t)} = \frac{\beta_1}{1 - \beta_2}$$

$$X_t \sim \mathcal{N}(\mu, \gamma_0) \Rightarrow \quad \text{income lognormal.}$$

Intergenerational Income Elasticity (IGE) and Variance of Logs (VL)

For an AR(1) process $X_t = \alpha + \beta X_{t-1} + \epsilon_t$ where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ the OLS estimator for β is

$$\hat{\beta} = \frac{\text{cov}(X_{t-1}, X_t)}{\text{var} X_t}$$

Assuming Solon model:

$$\hat{\beta} = \frac{\beta + \theta}{1 + \beta\theta} \quad \text{(IGE)}$$

$$\gamma_0 = \frac{(1 + \beta\theta)\sigma^2}{(1 - \beta\theta)(1 - \theta^2)(1 - \beta^2)} \quad \text{(VL)}$$

Note **symmetry** and that VL is an **inequality measure**.

Solving for β, θ given IGE and VL

To solve numerically we fix $\hat{\beta}$ and estimate γ_0 as stationary AR(1) variance:

$$\hat{\gamma}_0 = \frac{\sigma^2}{1 - \hat{\beta}^2}.$$

From data (X_t, X_{t-1}) could find $\hat{\beta}$, $\hat{\gamma}_0$ exactly. Want to solve for β and θ from

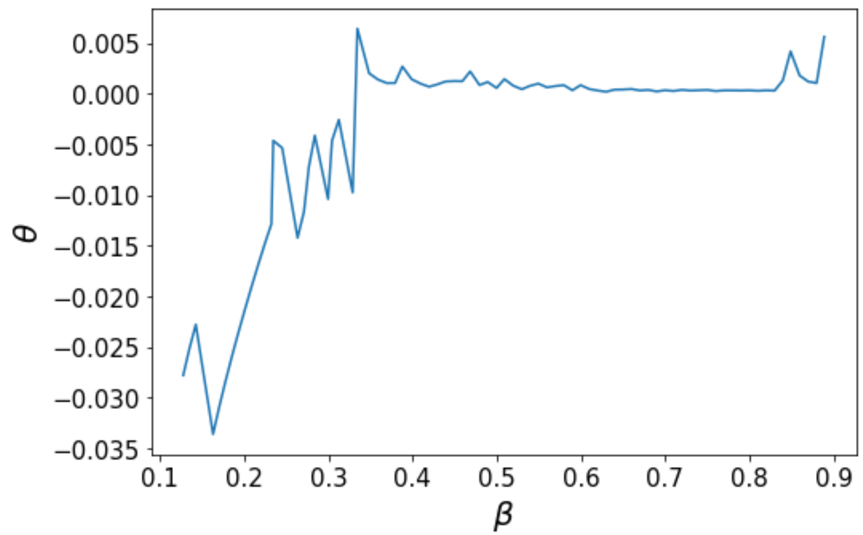
$$\hat{\beta} = \frac{\beta + \theta}{1 + \beta\theta}, \quad \hat{\gamma}_0 = \frac{(1 + \beta\theta)\sigma^2}{(1 - \beta\theta)(1 - \theta^2)(1 - \beta^2)}.$$

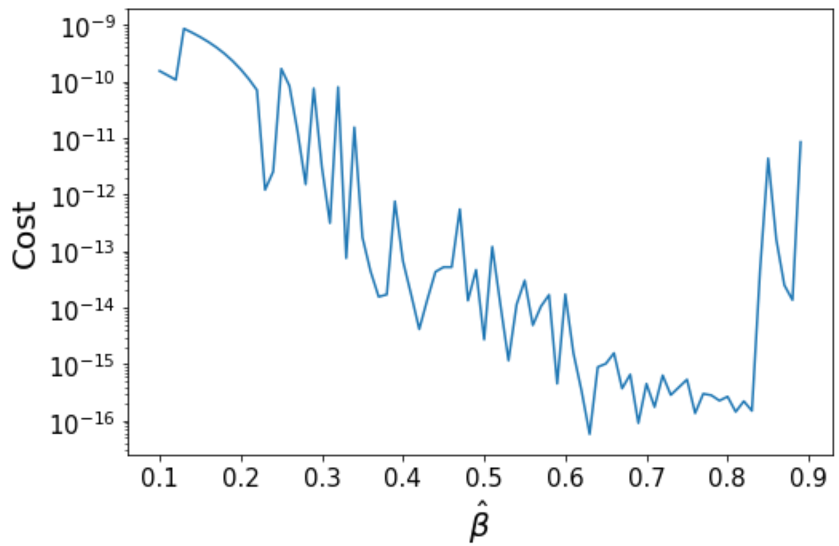
There are four (complicated) analytical solutions (β, θ) from a quartic.

Solving Numerically for β, θ given IGE and VL

Minimise cost function ($\sigma^2 = 1$):

$$\underset{0<\beta<1, |\theta|<1}{\operatorname{argmin}} \left(\left(\frac{\beta + \theta}{1 + \beta\theta} - \hat{\beta} \right)^2 + \left(\frac{(1 + \beta\theta)\sigma^2}{(1 - \beta\theta)(1 - \theta^2)(1 - \beta^2)} - \hat{\gamma}_0 \right)^2 \right)$$



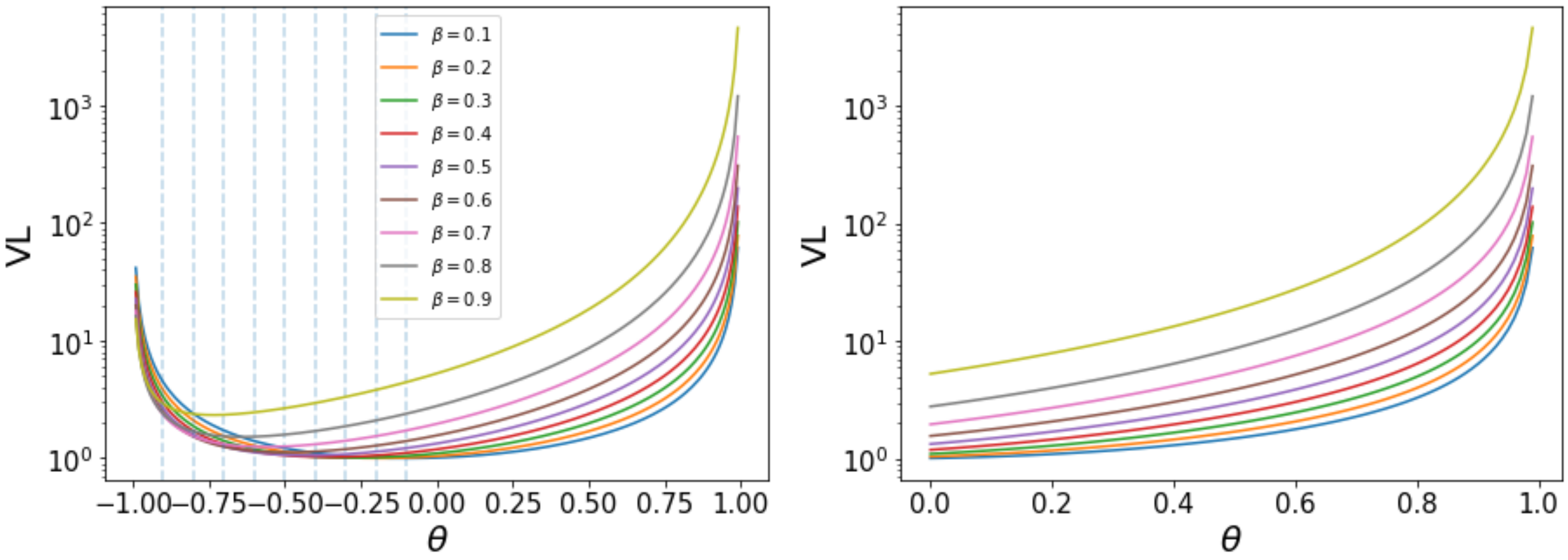


Analysis of VL

$$\gamma_0 = \frac{(1+\beta\theta)\sigma^2}{(1-\beta\theta)(1-\theta^2)(1-\beta^2)}$$

- Asymptotes: $\theta = \pm 1$, $\beta = \pm 1$ and $\beta\theta = 1$.
- Symmetry between β and θ.
- If $0 \leq \beta < 1$ and $0 \leq \theta < 1$ then highest VL (or inequality) for highest β and θ.
- If $0 \leq \beta < 1$ fixed and $0 \leq \theta < 1$ then $\theta = 0$ gives lowest VL (or highest equality).

Analysis of VL



Negative Grandparental Relationship

As before

$$X_t = \alpha + \beta X_{t-1} + \epsilon_t, \quad \text{where } \epsilon_t = \theta \epsilon_{t-1} + \eta_t, \quad \eta_t \sim \mathcal{N}(0, \sigma^2)$$

gives

$$X_t = \alpha(1 - \theta) + (\beta + \theta) X_{t-1} - \beta\theta X_{t-2} + \eta_t$$

How to reconcile this? [Solon, 2018]

- Subtle implication of higher grandparental income **conditional** on the amount of parental income.
- Perhaps it is incomplete - we could consider for example higher order processes.

Higher Order Endowments

Take endowments AR(2):

$$X_t = \alpha + \beta X_{t-1} + \epsilon_t, \quad \text{where } \epsilon_t = \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \eta_t, \quad \eta_t \sim \mathcal{N}(0, \sigma^2)$$

gives the following AR(3) for log income:

$$X_t = \alpha(1 - \theta_1 - \theta_2) + (\beta + \theta_1) X_{t-1} + (\theta_2 - \beta\theta_1) X_{t-2} - \theta_2 \beta X_{t-3} + \eta_t$$

Positive grandparent relation if $\theta_2 > \beta\theta_1$ but great grandparent coefficient now negative?! Can extend this to general case of endowments AR($p-1$) in which case have negative coefficient $-\theta_{p-1}\beta$ on X_{t-p}, the pth generation back.

Final Remarks

- We showed evidence from autoregressive model theory that lower mobility translates to higher inequality.
- Policy implications: process or outcome?
- Test theory against data! Especially test residuals/noise part of model.