London Mathematical Laboratory
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Project 7: Learning Random Dynamical Systems from Data

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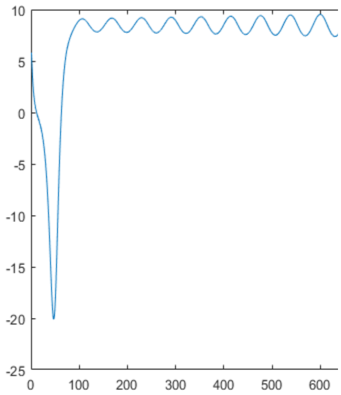
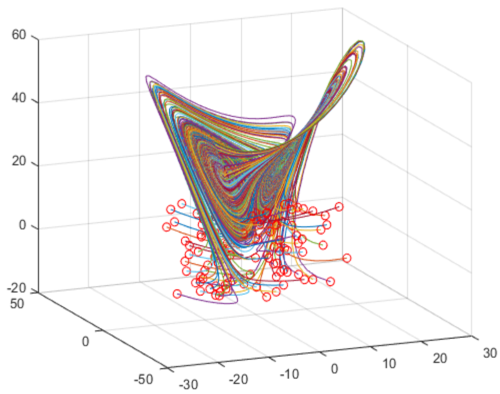
Motivation

Q1 : Given a partial observation of a system, can we determine/recover/learn the underlying dynamical system?





Q2 : Learn dynamical properties of trajectories Vs Averaged quantities?
Q3 : Why Learn about trajectories? Modelling in low-dimensional space with noise Vs Modelling in High Dimensional space

Direct Construction of Random Dynamical Systems Using Additive Noise

- What if we add some additive noise to a discrete time mapping? (climate science, laser dynamics, etc.)

$$\begin{aligned} x_{n+1} &= a(y_n - x_n), \\ y_{n+1} &= x_n(b - z_n) - y_n, \\ z_{n+1} &= x_n y_n - c z_n. \end{aligned} \qquad \text{(Lorenz System)}$$

- What kind of random phenomena can we observe and describe, in particular in multiple dimensions?

$$\begin{aligned} x_{n+1} &= a(y_n - x_n) + \sigma\omega_n^x, \\ y_{n+1} &= x_n(b - z_n) - y_n + \sigma\omega_n^y, \\ z_{n+1} &= x_n y_n - c z_n + \sigma\omega_n^z. \end{aligned} \qquad \text{(Random Lorenz System)}$$

where $\omega_n^x, \omega_n^y, \omega_n^z \epsilon [0, \varepsilon]$ and $\sigma > 0$

Taken's Embedding

Taken's theorem provides the conditions under which a smooth attractor can be reconstructed from the observations made with a generic function.

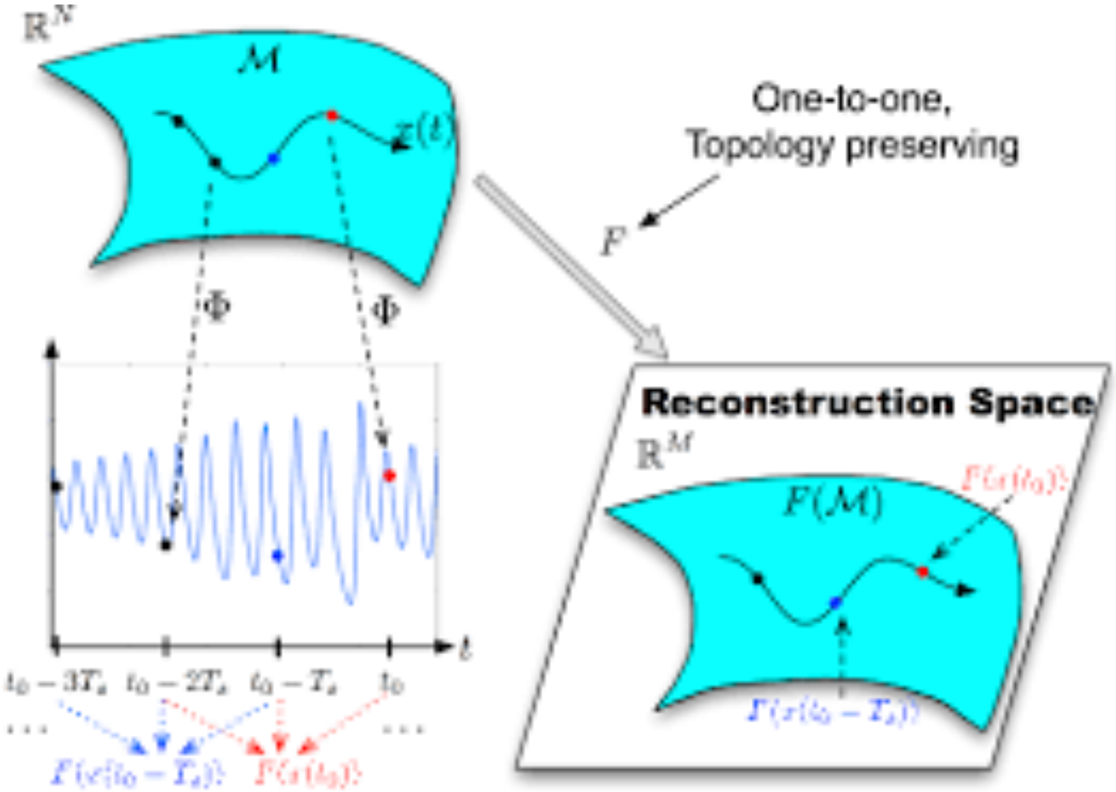
- Assume we have a map $x_{n+1} = f(x_n)$ and $x \epsilon \mathrm{R^d}$ for discrete time $t \epsilon \mathrm{Z}^+$.
- Assume we have an observation $(y_1, y_2, y_3, ..., y_n)$ time series.

The delay embedding theorem states that

Theorem

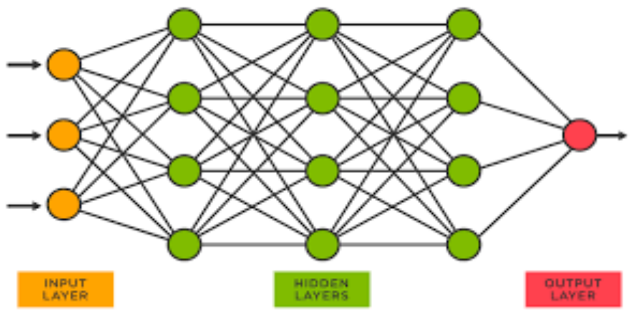
The delay embedding theorem states that if the dynamics of x is finite dimensional e.g. $d-$ dimensional attractor, then the dynamics of x is "topologically conjugate" to the evolution of y in so called delay co-ordinates, $y_n^{'} = (y_n, y_{n-1}, ..., y_{n-D}) = (y(n), y(n-\tau), y(n-(D-1)\tau)$ i.e on its attractor f is conjugated to the dynamics of $F : \mathbb{R}^D \to \mathbb{R}^D$ on its attractor where $y_{n+1}^{'} = F(y_n^{'})$ for at least $D > 2d$

Taken's Embedding

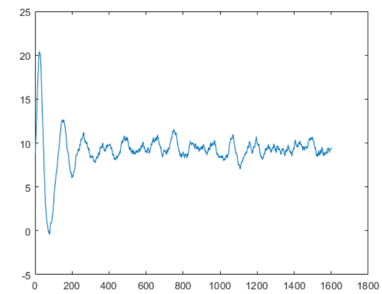


Learning using Neural Network

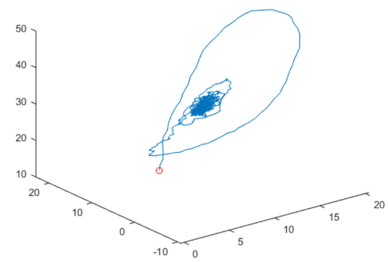
- Goal of NN : Learn an update rule/function which advances state space from x_k to x_{k+1} , i.e. find a function f such that $x_{k+1} = f(x_k)$
- Accurately determining the solution requires a **non-linear transfer function** since the underlying system is non-linear
- Input is the matrix of the partial observation at x_k
- Output is the matrix of the partial observation at x_{k+1}
- We have used a **3-layer network** with **10 nodes in each layer** with three different **activation functions**



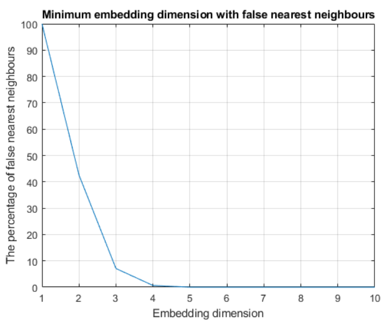
Partial Observation and Reconstructed Random Dynamical System



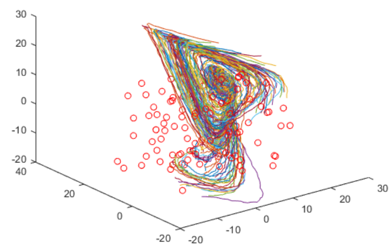
(a) Partial Observation



(b) 1 Embedded Trajectory



(c) The right embedding dimension for delay 10



(d) 100 Embedded Trajectories

Learning form the Taken's Embedding of the Random Lorenz

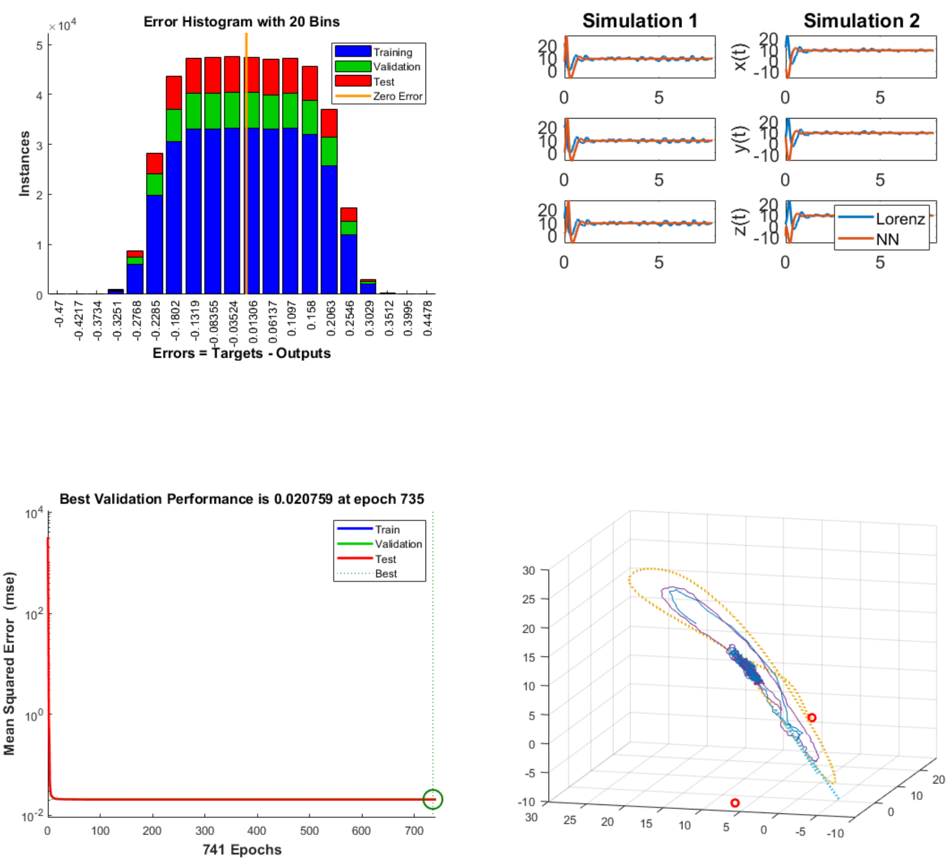


Figure: Learning Reports

References

Yuruzu Sato, Jeroen S. W. Lamb (2018)
Dynamic Characterization of stochastic bifurcations in a random logistic map
arXiv preprint

Floris Taken (1981)
Taken's Embedding Theorem
Wikipedia.

Steven L. Bruton (2019)
Data-Driven Science and Engineering: Machine Learning, Dynamical Systems, and Control
Cambridge University Press 12(3), 45 – 678.