Project 11: Anomalous diffusion in random dynamical systems
Auto-correlation functions of random maps

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Background: Stochastic Langevin Equation

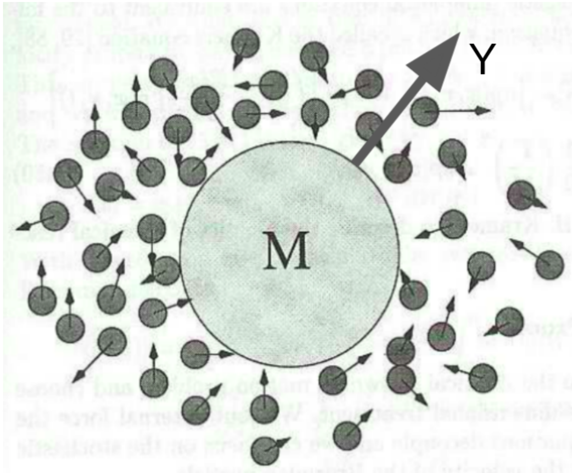

Figure: Particle of mass M undergoing Brownian motion. [L Sjögren]

Brownian Motion can be modeled by the Langevin Equation:

$$M\dot{Y} = -\gamma Y + \eta(t)$$

Y: velocity, γ: damping constant, $\eta(t)$: delta-correlated Gaussian white noise i.e:

$$E[\eta(t)] = 0$$
$$\langle \eta(t)\eta(t') \rangle = \sigma^2 \delta(t - t')$$

σ^2: variance of Gaussian white noise $\delta(t)$: Dirac delta function.

Deterministic Langevin Equation

Replace the stochastic term $\eta(t)$ with chaotic dynamics generated by a deterministic map B [C. Beck, 1996] :

$$\eta(t) = \tau^{1/2} \sum_{n=1}^{\infty} (x_n - \langle x \rangle) \delta(t - n\tau)$$

$$x_{n+1} = B(x_n) = 2x_n \ (mod 1)$$

$\tau > 0$: time difference between subsequent impulses of kick force, strength given by map B, known as the Bernoulli shift map.
Integrate the original equation to get $Y = e^{-\gamma(t-n\tau)} y_n$:

$$y_{n+1} = \lambda y_n + \tau^{1/2}(x_{n+1} - \langle x \rangle) \qquad \lambda = e^{-\gamma\tau}$$

$$x_{n+1} = B(x_n) \qquad x_n \in [0, 1]$$

We will consider a simplified case with $\gamma\tau \to 0$ and $\tau \to 1$.

Langevin Equation driven by a random dynamical system

In the simplified form:

$$y_{n+1} = y_n + (x_{n+1} - \langle x \rangle)$$

$$x_{n+1} = B(x_n)$$

For this project, the deterministic map B is replaced by:

$$x_{n+1} = T(x_n) = \begin{cases} 2x_n \ (mod 1) & p \in [0,1] \\ \frac{1}{2}x_n & 1-p \end{cases}$$

$x_n \in [0,1]$ where p is the probability.
T: random dynamical system. [S. Pelikan, 1984]
$p = 1 \implies$ reproduces the deterministic map B, positive Lyapunov exponent
$p = 0 \implies$ negative Lyapunov exponent
$p \to \frac{1}{2} \implies$ (intermittency) transition point with zero Lyapunov exponent

The invariant density of the Pelikan Map

Analytic result ($\frac{1}{2} < p \leq 1$)

The explicit form of the invariant density $\rho_p(x)$ was derived to be:

$$a_j = \frac{2p-1}{3p-2}\left[1 - \left(\frac{2(1-p)}{p}\right)^{(j+1)}\right]$$

in each interval $I_j = [\frac{1}{2^{j+1}}, \frac{1}{2^j}]$ $j = 0, 1, 2...$ [S. Pelikan, 1984]

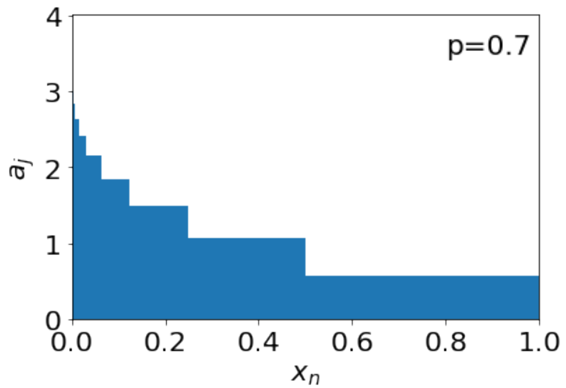

Figure: The invariant density of the Pelikan map at p=0.7

The transition in the invariant density

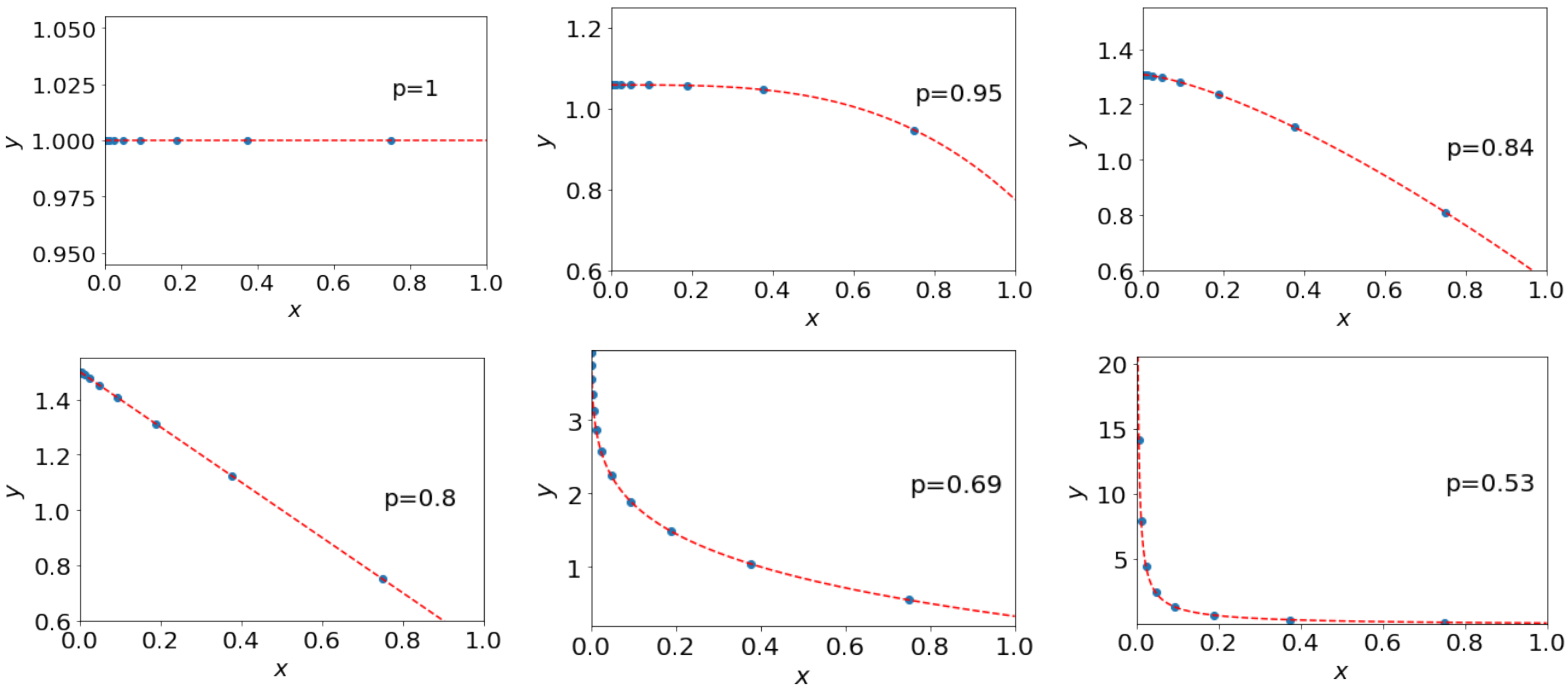

Figure: The invariant density as a curve derived using midpoint interpolation.

$$y(p,x) = A(p)(1 - B(p)x^{-1+C(p)})$$

The invariant density changes from a uniform to an unbounded function.
[Jin Yan, LML Summer School 2019]

Simulations using arbitrary precision computation

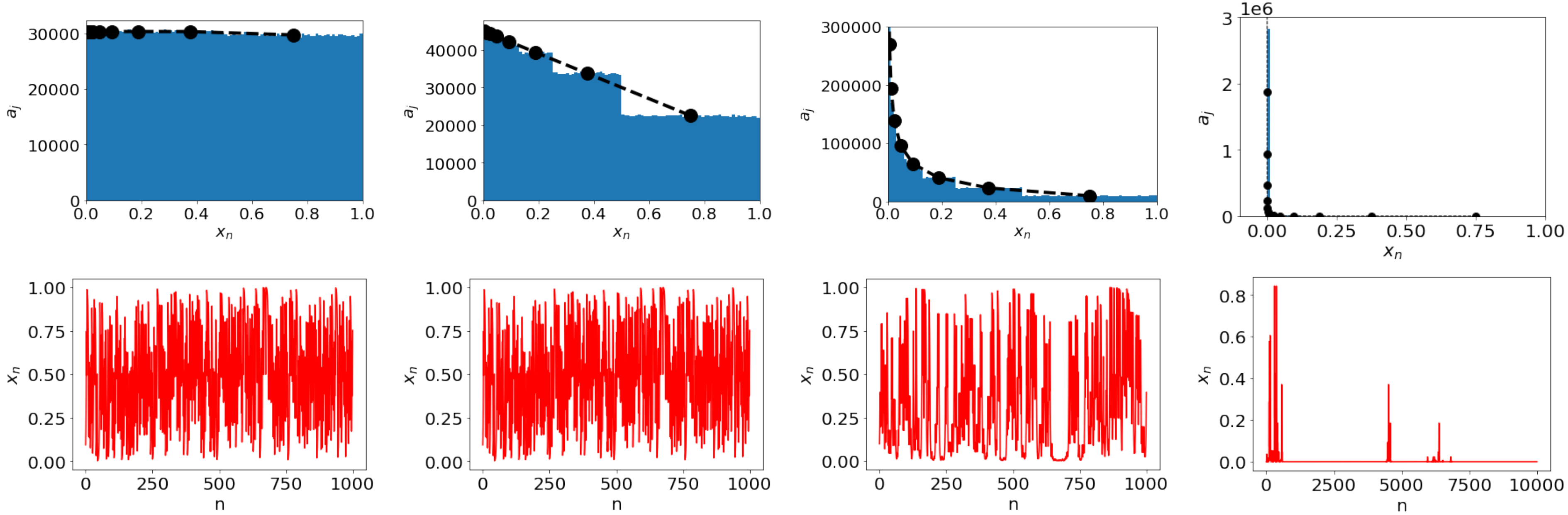

Figure: Simulations (left to right) at p=0.99, 0.8, 0.6 and 0.501 with an ensemble of 10^3 initial conditions and corresponding time series plots (bottom panel). They were computed using the GNU MPFR library, with precision up to 10^{10^7} digits.

$$x_{n+1} = T(x_n) = \begin{cases} 2x_n \ (mod 1) & p \in [0,1] \\ \frac{1}{2}x_n & 1-p \end{cases}$$

Auto-correlation functions

The velocity auto-correlation function captures the decay of memory with time. For the Langevin equation, it is related to the position auto-correlation function of the previously defined random dynamical system:

$$\langle (y_k - y_{k-1})(y_1 - y_0) \rangle = \langle x_k x_0 \rangle - \langle x \rangle^2$$

A semi-Markovian analytical approximation for $\langle x_k x_0 \rangle$ in terms of p has been derived. [Jin Yan, 2021]

Goal

To compare the auto-correlation functions obtained from theory with numerical results computed using infinite precision.

To compare across different p, we compute the *normalized* auto-correlation function:

$$CF(p) = \frac{\langle x_k x_0 \rangle - \langle x \rangle^2}{\langle x^2 \rangle - \langle x \rangle^2}$$

Exponential to power law decay

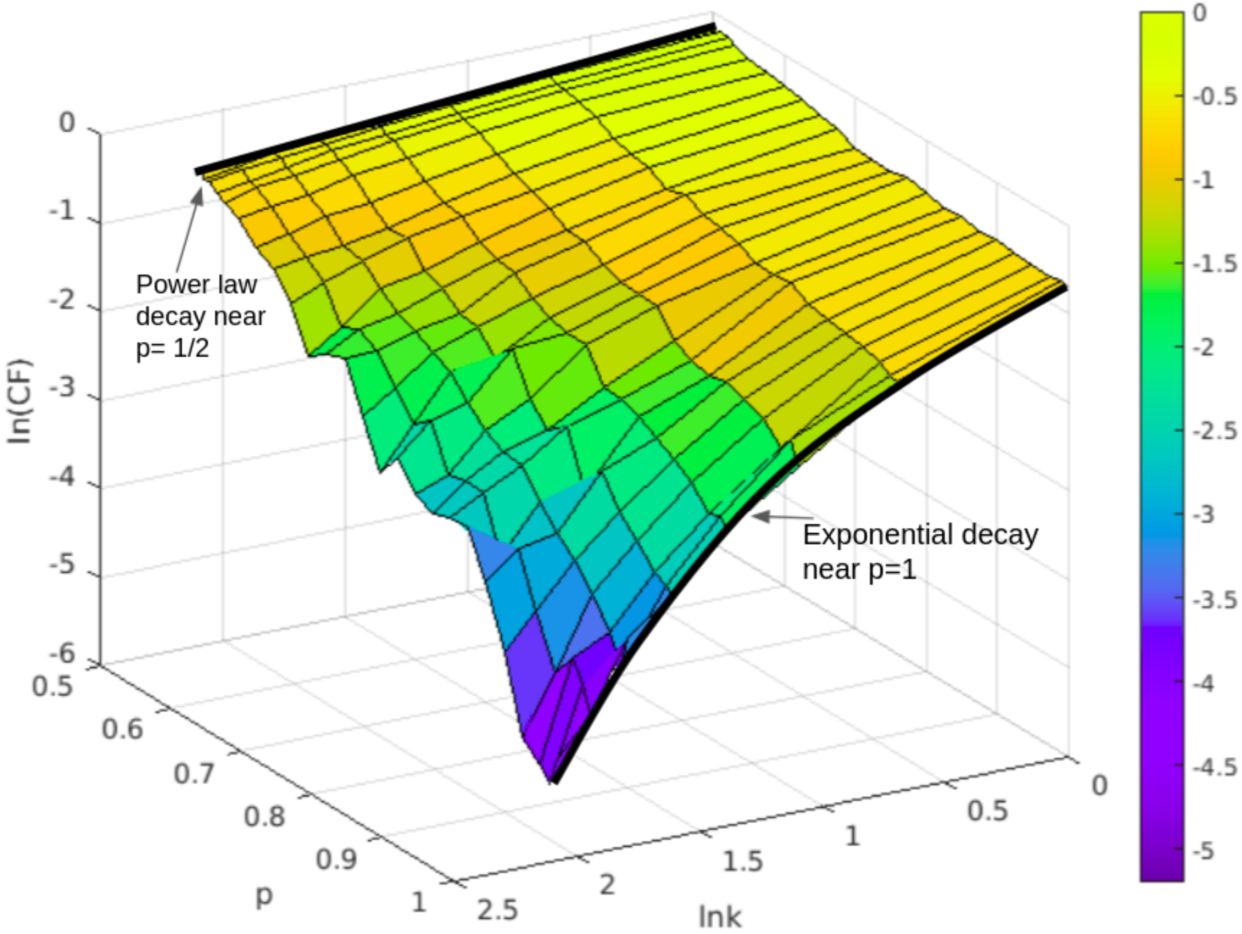

Figure: The log-log plot of the normalized auto-correlation function for 25 values of $p \in (\frac{1}{2}, 1)$, which decays exponentially at $p = 1$, by monotonically changing to a power law decay as $p \to \frac{1}{2}$

Comparison of theoretical and numerical results

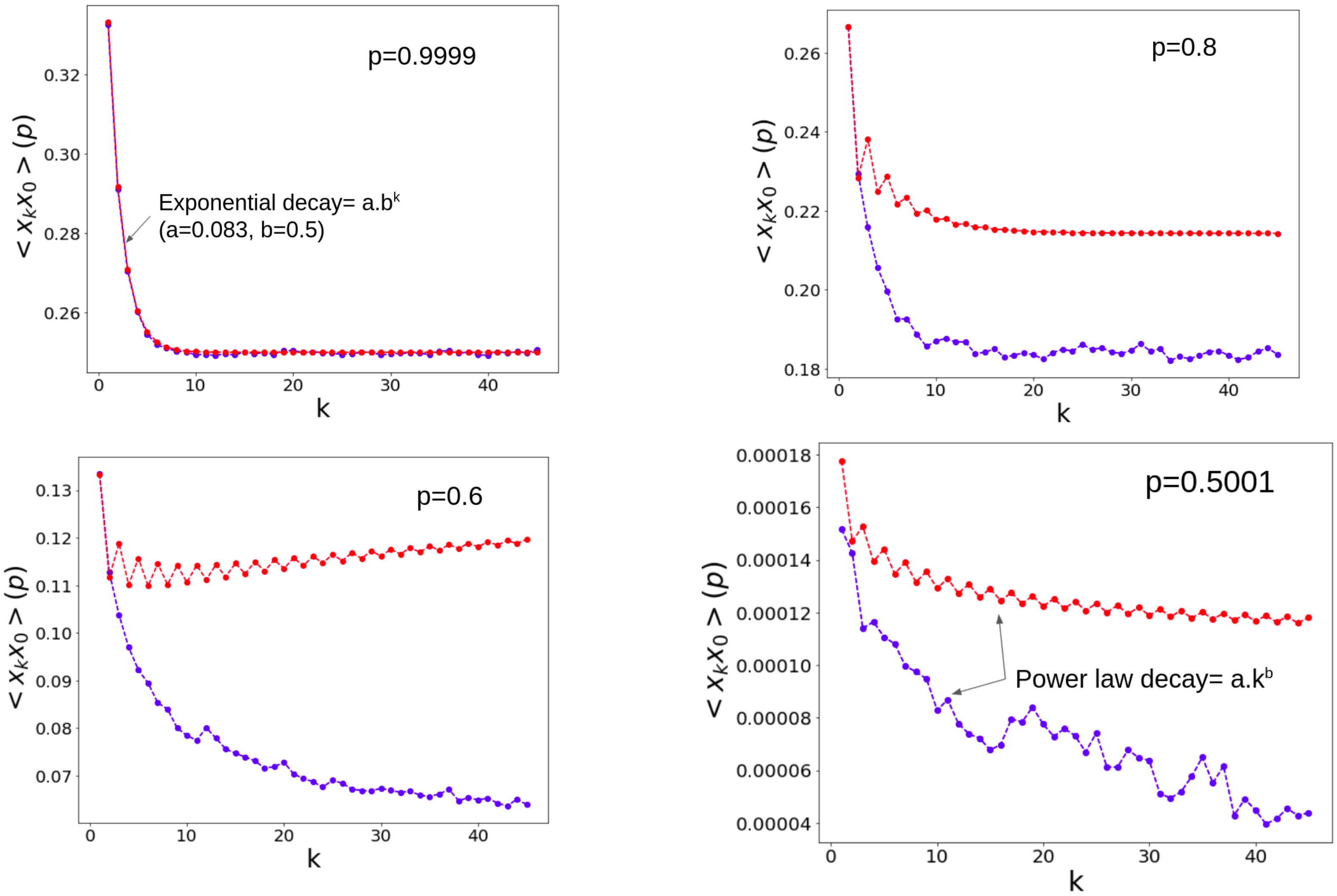

Figure: $\langle x_k x_0 \rangle$ from theory (red) and simulations (blue). At $p = 0.5001$, the power law exponent $b = -0.3584$ from simulations, and $b = -0.067$ from theory.

Summary

1.) The auto-correlation functions for the random dynamical system were numerically calculated using arbitrary precision computation.

2.) For $p \to 1$, the theoretical result shows good agreement with the numerical computation of the auto-correlation function.

3.) For $p \to \frac{1}{2}$, the auto-correlation function calculated from theory shows the expected power law decay, however the exponent is different from the one observed in simulations.

Outlook

Next Step

To compute the mean square displacement (MSD=$\langle(x_n - x_0)^2\rangle$) for Langevin dynamics driven by this random dynamical system.

Conjecture

The MSD exhibits a transition from linear to sub-linear growth in time t (MSD $\sim$t$^\alpha$: with $\alpha < 1$, showing subdiffusion) under variation of p.
[Y. Sato R. Klages, 2019]

References

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2.) Pelikan S. (1984), *Invariant Densities for Random Maps of the Interval*, AMS, Vol.281 (1984)

3.) Yan J. (2021), *Complex Behaviour in Coupled Oscillators, Coupled Map Lattices and Random Dynamical Systems* [Unpublished doctoral dissertation], Queen Mary University of London

4.) Sato, Y., Klages, R. (2019). Anomalous Diffusion in Random Dynamical Systems. Physical Review Letters, 122(17). doi:10.1103/physrevlett.122.174101