Functional Rate Theory: Index distribution for diluted random matrices

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Outline

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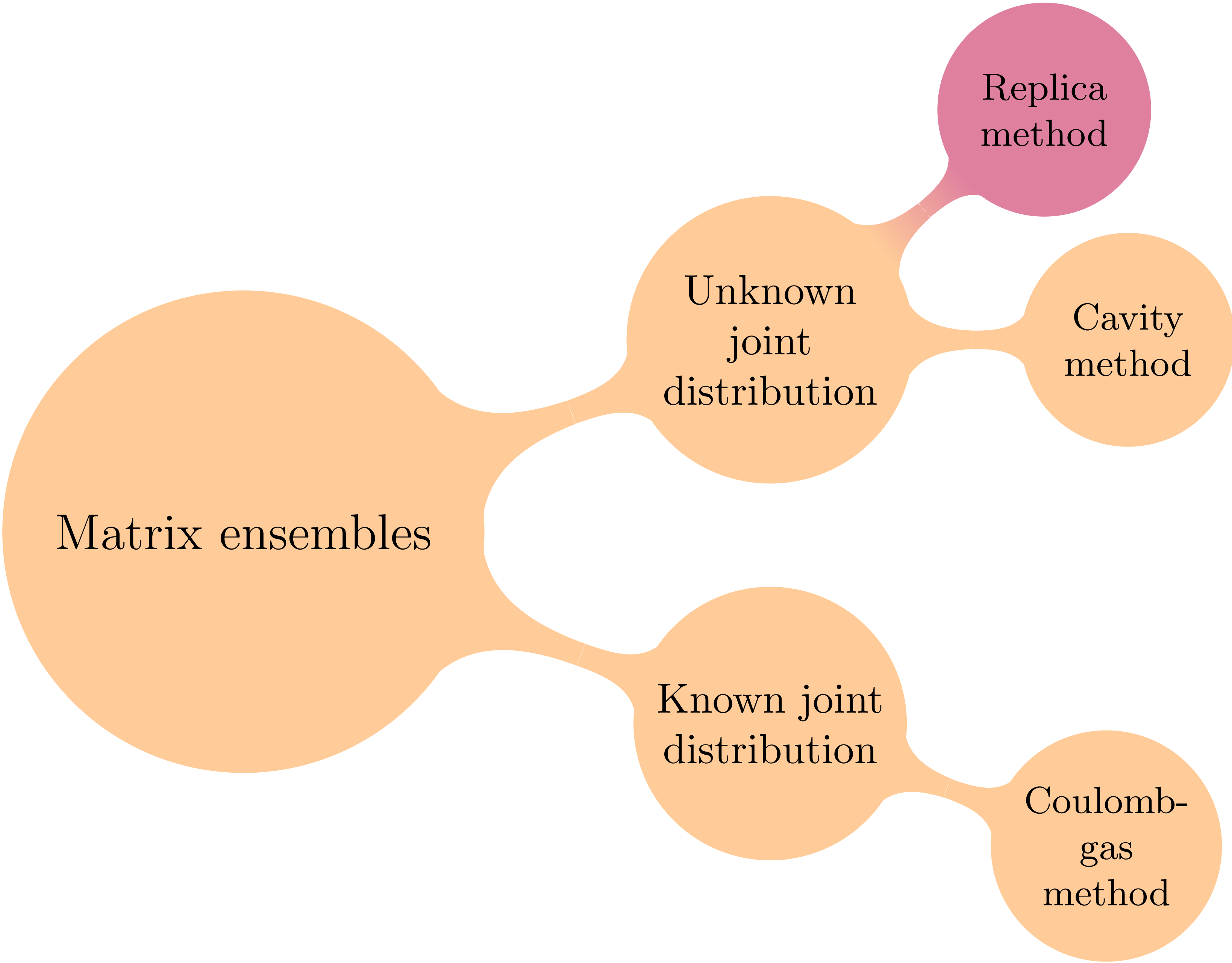
Applications of RMT

- Stability of complex systems
- Electron localization
- Quantum chaos
- The physics of glasses

Directions of RMT



Poissonian random graphs

Consider a $N \times N$ Poissonian random matrix C with average connectivity c, *i.e.* the adjacency matrix of the graph, that is constructed by choosing an edge with probability $\frac{c}{N}$.

$$Prob[C_{ij} = c_o] = \frac{c}{N}\delta_{c_o,1} + \left(1 - \frac{c}{N}\right)\delta_{c_o,0}$$

Physical quantity of interest

- Empirical spectral density:

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\lambda - \lambda_i\right),$$

where $\lambda_1, \ldots, \lambda_N$ are eigenvalues of matrix C.

- Number statistics: fraction of eigenvalues between x and y:

$$I_N(x, y) = \int_x^y d\lambda \rho(\lambda)$$

- **Index distribution**: fraction of eigenvalues less than x:

$$I_N(x) = \int_{-\infty}^{x} d\lambda \rho(\lambda)$$

Index distribution

Let $I_N(x)$ be the fraction of eigenvalues up to the point x. In general, the technique can be used to study the asymptotic behaviour of the following probability distribution:

$$P[\kappa(x)] = \text{Prob}\left[I_N(x) = \kappa(x), \forall x \in \mathbb{R}\right].$$

Particularly, as a preliminary check we calculate

- $\langle I(x) \rangle$
- $\langle I(x_1) I(x_2) \rangle - \langle I(x_1) \rangle \langle I(x_2) \rangle$.

Cumulant generating functional

In fact,

$$\text{Prob}[\kappa(x)] \asymp \exp\left[N\left(\int dx \mu^*(x)\kappa(x) - \mathcal{F}\left[\mu^*(x)\right]\right)\right]$$

where $\mu^*(x)$ extrimizes $\left(\int dx \mu(x)\kappa(x) - \mathcal{F}[\mu(x)]\right)$.
The task is reduced to studying the *cumulant generating functional*

$$\mathcal{F}[\mu(x)] = -\frac{1}{N}\log\left\langle \exp\left[-N\int dx \mu(x) I_N(x)\right]\right\rangle.$$

Outline

1. Introduction
2. Replica Method
3. Population dynamics algorithm
4. Results and conclusions

Creation of copies: very beginning

$$\exp\left[-N\int dx\mu(x)I_N(x)\right]=$$

- $\Theta(-y)=\lim_{\eta\to 0^+}\frac{1}{2\pi i}[\log(y+i\eta)-\log(y-i\eta)]$
- $I_N(x)=-\frac{1}{\pi i}\lim_{\eta\to 0^+}\log\left[\frac{\mathcal{Z}(x_\eta)}{\mathcal{Z}(x^\star_\eta)}\right]$, where $\mathcal{Z}\left(x^\eta\right)$ denotes $\left(\det\left(C-(x+i\eta)\mathbb{I}\right)\right)^{\frac{1}{2}}$
- $\int dx\mu(x)I_N(x)=\lim_{L\to\infty}\Delta x\sum_{\ell=1}^{L}\mu\left(x_\ell\right)I_N\left(x_\ell\right)$

$$=-\lim_{L\to\infty}\lim_{\eta\to 0^+}\lim_{\{n_\ell^\pm\}\to\left\{\pm\frac{\Delta x\mu(x_\ell)}{\pi i}\right\}}\prod_{\ell=1}^{L}\left[\mathcal{Z}\left(x_\ell^\eta\right)\right]^{n_\ell^+}\left[\mathcal{Z}^\star\left(x_\ell^\eta\right)\right]^{n_\ell^-}$$

Creation of copies: replica trick

$$\left\langle \prod_{\ell=1}^{L} \left[\mathcal{Z}\left(x_\ell^\eta\right)\right]^{n_\ell^+} \left[\mathcal{Z}^\star\left(x_\ell^\eta\right)\right]^{n_\ell^-} \right\rangle,$$

- When $n_\ell^\pm$ are **positive integers**
- Originally, $n_\ell^\pm$ are **imaginary.**
- Firstly, we will derive formula for positive integers, afterwards, taking analytical continuation (replica trick).

Saddle-point equations

Field theoretic description: exact coarse graining.

$$\left\langle \prod_{\ell=1}^{L} \left[\mathcal{Z}\left(x_\ell^\eta\right)\right]^{n_\ell^+} \left[\mathcal{Z}^\star\left(x_\ell^\eta\right)\right]^{n_\ell^-} \right\rangle_{\mathcal{E}} = \int D[\{P,\hat{P}\}] e^{-NS[\{P,\hat{P}\}]}$$

P and $\hat{P}$ are fields on $\prod_{\ell=1}^{L} \mathbb{R}^{n_\ell^+} \mathbb{R}^{n_\ell^-}$.

$$\int dx e^{NS(x)} \asymp e^{N max\{S(x)\}}$$

Self-consistent equations $\{P_{\min}, \hat{P}_{\min}\} = F(\{P_{\min}, \hat{P}_{\min}\})$.

Replica symmetric trick

Exchangeable if $\mu_{X_1,\ldots,X_n} = \mu_{X_{\pi(1)},\ldots,X_{\pi(n)}}$, for every permutation $\pi : \{1,\ldots,n\} \to \{1,\ldots,n\}$.

Theorem (De Finetti's)

$$P(\{X_a = x_a\}) = \int d\mu(\Delta) \prod_{a=1}^{N} \Pi_\Delta(x_a)$$

$$P\left[\{\Phi_{\beta_i}, \Psi_{\alpha_i}\}\right] = \int \prod_{i=1}^{L} d\Delta_i W[\{\Delta_i\}] \prod_{\beta_i=1}^{n_i^-} \frac{\exp\left(-\iota \frac{\Phi_{\beta_i}^2}{2\Delta_i^*}\right)}{\sqrt{-2\pi\iota\Delta_i^*}} \prod_{\alpha_i=1}^{n_i^+} \frac{\exp\left(\iota \frac{\Psi_{\alpha_i}^2}{2\Delta_i}\right)}{\sqrt{2\pi\iota\Delta_i}}$$

Cumulant generating functional

$$W\left[\Delta(x)\right] = \frac{1}{\mathcal{N}} \sum_{k=0}^{\infty} \frac{e^{-c} c^k}{k!} \int \prod_{s=1}^{k} \mathcal{D}\Delta_s(x) W\left[\{\Delta_s(x)\}\right]$$

$$\delta\left(\Delta(x) - \frac{1}{\sum_{s=1}^{k} \Delta_s(x) - x - \iota\eta}\right)$$

$$exp\left(\frac{1}{2\pi\iota} \int dx \mu(x) \log\left(-\frac{\Delta(x)}{\Delta^*(x)}\right)\right)$$

$$\mathcal{N} = \sum_{k=0}^{\infty} \frac{e^{-c} c^k}{k!} \int \left[\prod_{s=1}^{k} \mathcal{D}\Delta_s(x) W\left[\{\Delta_s(x)\}\right]\right] e^{\frac{1}{2\pi\iota} \int dx \mu(x) \log\left(-\frac{\sum_{k}^{k} = \Delta_s^*(x) - x_\eta^*}{\sum_{s=1}^{k} \Delta_s(x) - x\eta}\right)}$$

$$\mathcal{F}[\mu(x)] = \frac{c}{2} \int \mathcal{D}\Delta \mathcal{D}\Delta' W[\Delta(x)] W\left[\Delta'(x)\right] e^{-\int \frac{dx}{2\pi i} \mu(x) \log \frac{1+\Delta(x)\Delta'(x)}{1+\Delta^*(x)\Delta'^*(x)}}$$

$$- \frac{c}{2} + \log\left(\mathcal{N}\right)$$

Expression for mean and covariance

$$\langle I_N(x_1)\rangle = \frac{\delta\mathcal{F}[\mu]}{\delta\mu(x_1)} = -\frac{c}{4\pi\iota}\int d\Delta d\Delta' W(\Delta)W(\Delta')\log\left(\frac{1+\Delta\Delta'}{1+\Delta^*\Delta'^*}\right) - \frac{1}{\pi\iota}\int d\Delta W(\Delta)\log\left(-\frac{\Delta}{\Delta^*}\right)$$

$$\begin{aligned}\langle I_N(x_1)I_N(x_2)\rangle_c &= \frac{\delta^2\mathcal{F}[\mu]}{\delta\mu(x_1)\delta\mu(x_2)} \\ &= -\frac{c}{8\pi^2}\int d\Delta_1 d\Delta'_1 d\Delta_2 d\Delta'_2 W(\Delta_1,\Delta_2)W(\Delta'_1,\Delta'_2) \\ &\qquad \log\left(\frac{1+\Delta_1\Delta'_1}{1+\Delta_1^*\Delta_1'^*}\right)\log\left(\frac{1+\Delta_2\Delta'_2}{1+\Delta_2^*\Delta_2'^*}\right) \\ &\quad + \frac{1}{4\pi^2}\int d\Delta_1 d\Delta_2 W(\Delta_1,\Delta_2)\log\left(-\frac{\Delta_1}{\Delta_1^*}\right)\log\left(-\frac{\Delta_2}{\Delta_2^*}\right) \\ &\quad - \frac{1}{4\pi^2}\int d\Delta_1 d\Delta_2 W(\Delta_1)W(\Delta_2)\log\left(-\frac{\Delta_1}{\Delta_1^*}\right)\log\left(-\frac{\Delta_2}{\Delta_2^*}\right)\end{aligned}$$

Outline

Population dynamics algorithm.

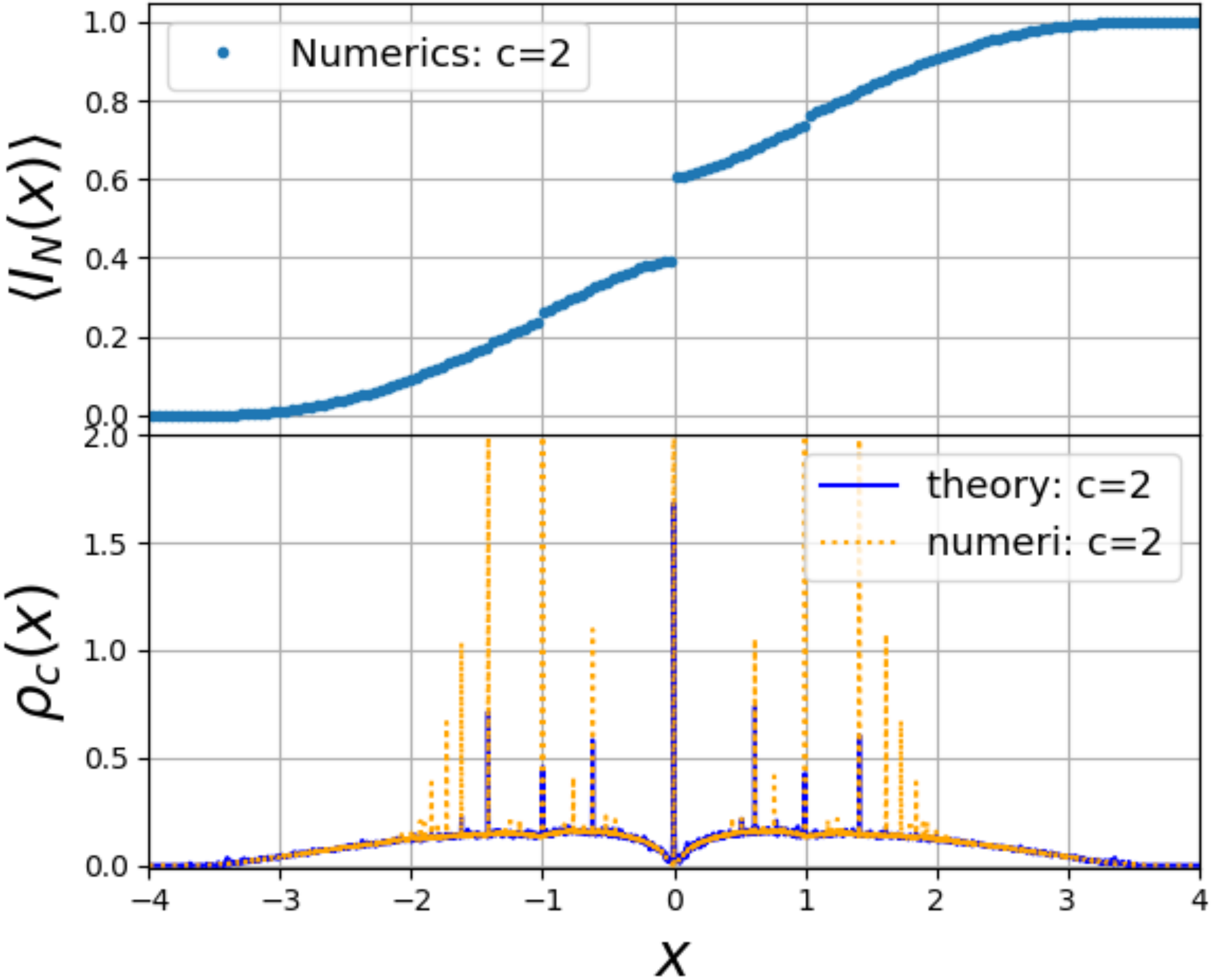
$$W\left[\{\Delta_i\}\right] = \sum_{k=0}^{\infty} \frac{e^{-c}c^k}{k!\mathcal{N}} \int \prod_{i=1}^{L} \prod_{s=1}^{k} d\Delta_{si} W\left[\{\Delta_{si}\}\right]$$
$$\prod_{i=1}^{L} \delta\left(\Delta_i - \frac{1}{\sum_{s=1}^{k} \Delta_{si} + x - \iota\eta}\right)$$

To solve these *self consistent equation* we use the population dynamics algorithm. Steps of the algorithm are

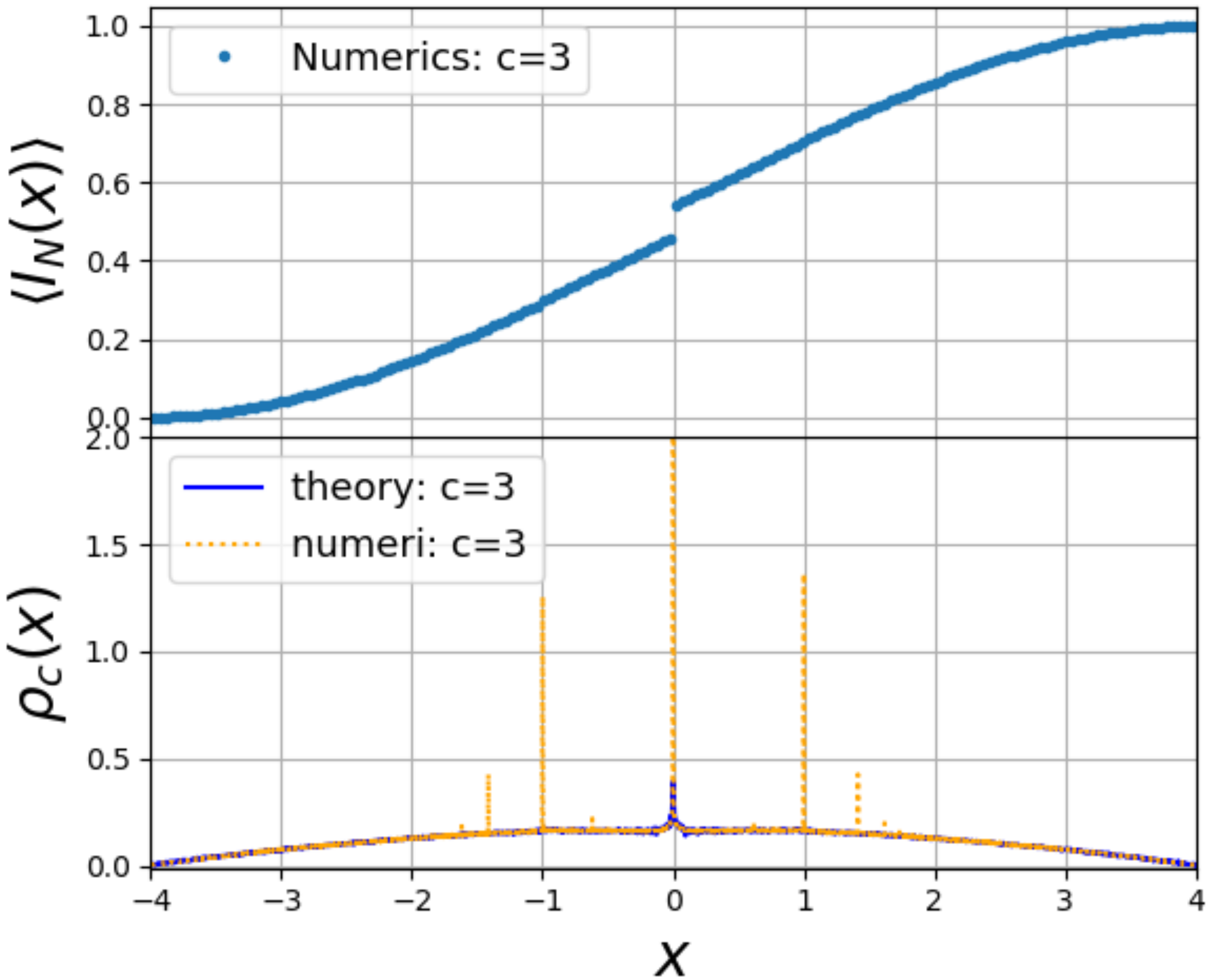
- Consider a population $(\Delta_1, \Delta_2 .... \Delta_{NS})$.
- Uniformly choose k members from the population $(\Delta_{i_1}, \Delta_{i_2}, \Delta_{i_3} ...... \Delta_{i_k})$
- Uniformly chosen member $\Delta_i = \frac{1}{\sum_{j=1}^{k} \Delta_{i_j} + x - \iota\eta}$
- Repeat till the steady state is reached.
- Now we have a population which is sampled from the distribution function W.

Outline

Result: Mean, c=2

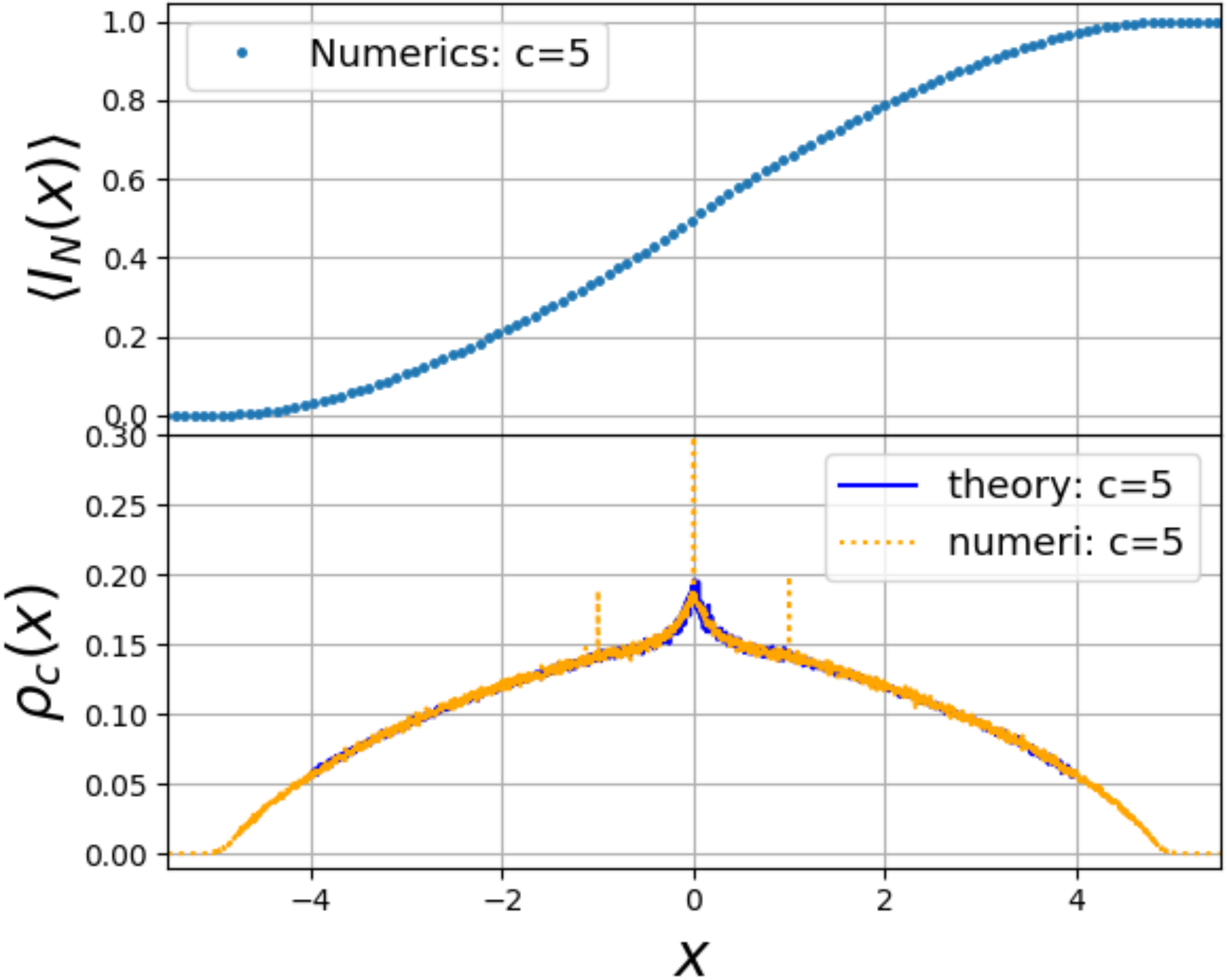


c=3



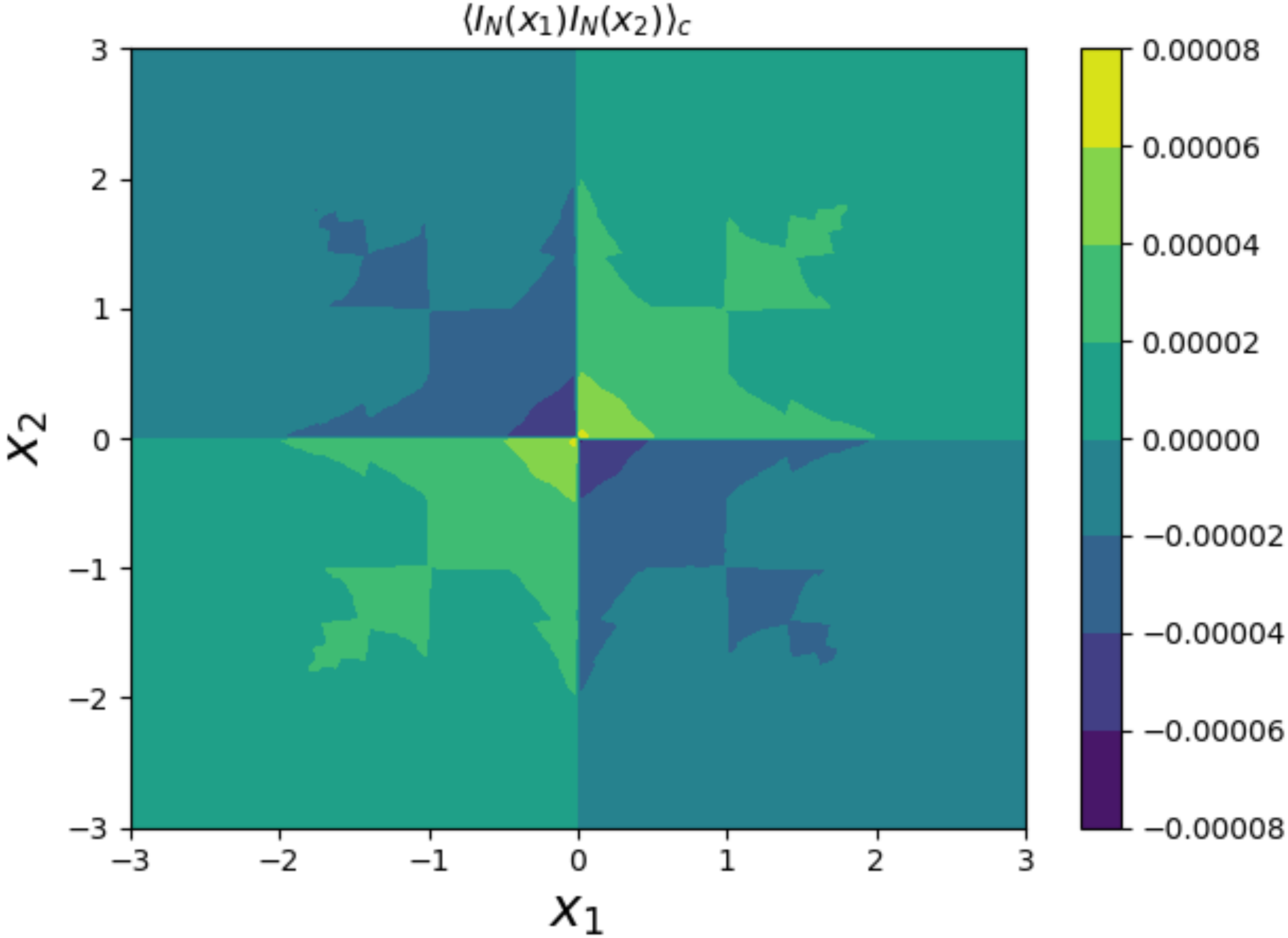
Numerics: c=3
theory: c=3
numeri: c=3
$\langle I_N(x) \rangle$
$\rho_c(x)$
x
1.0
0.8
0.6
0.4
0.2
0.0
2.0
1.5
1.0
0.5
0.0
−4
−3
−2
−1
0
1
2
3
4

c=5

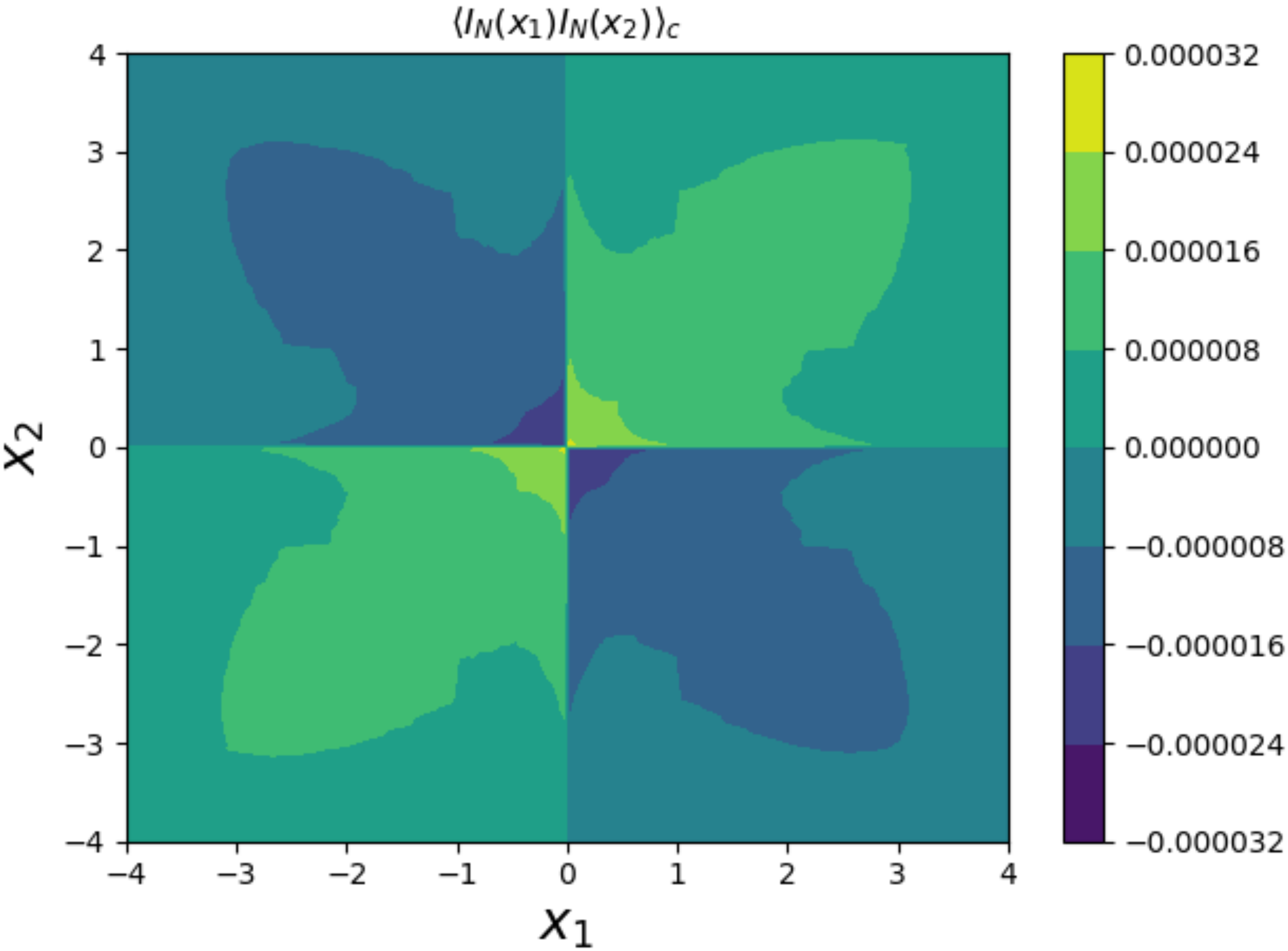


Numerics: c=5
theory: c=5
numeri: c=5
$\langle I_N(x) \rangle$
$\rho_c(x)$
x
1.0
0.8
0.6
0.4
0.2
0.0
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−4
−2
0
2
4

Result: Correlations, c=2

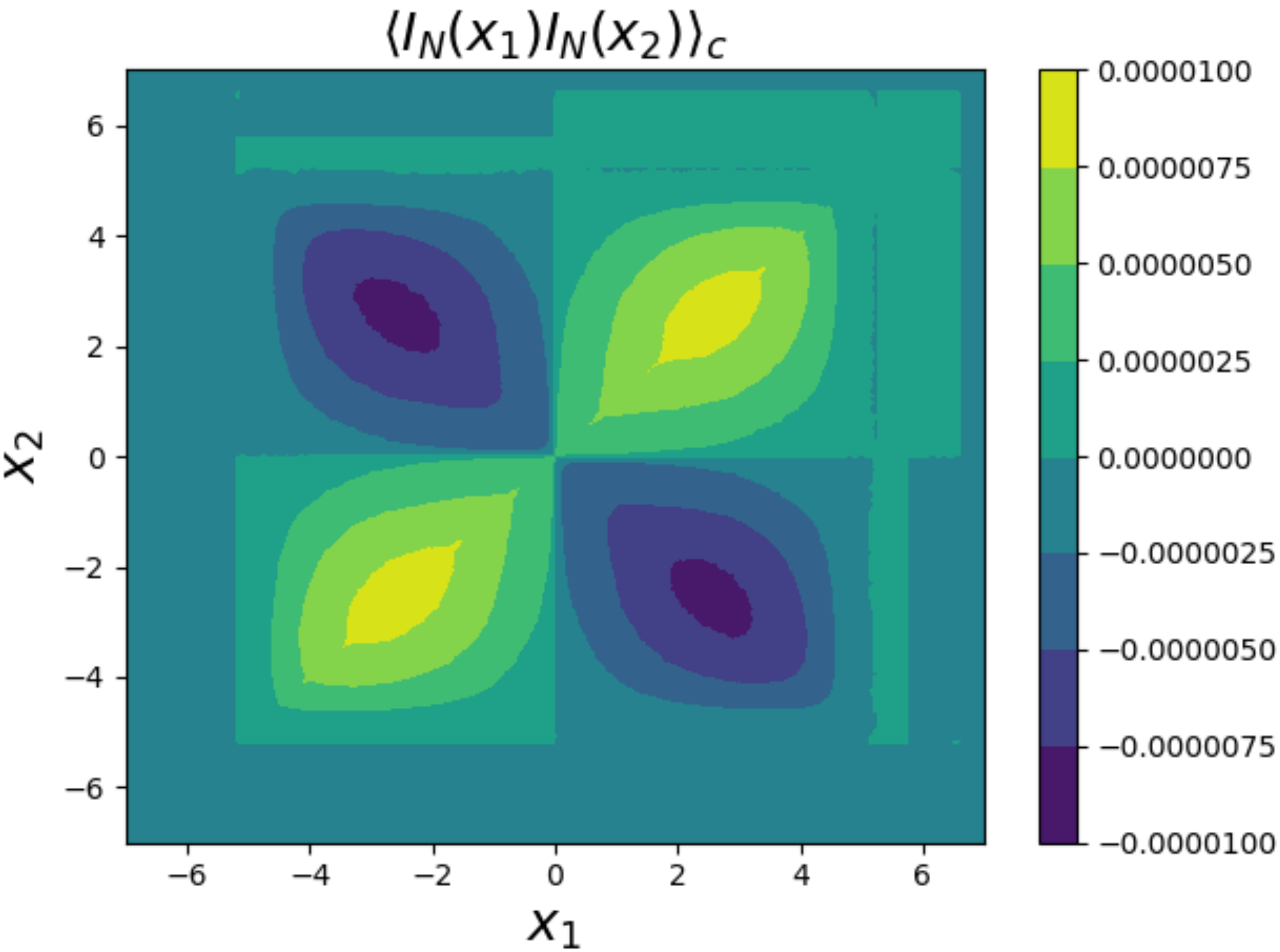


c=3



$\langle I_N(x_1)I_N(x_2)\rangle_c$
4
3
2
1
0
−1
−2
−3
−4
x_2
−4
−3
−2
−1
0
1
2
3
4
x_1
0.000032
0.000024
0.000016
0.000008
0.000000
−0.000008
−0.000016
−0.000024
−0.000032

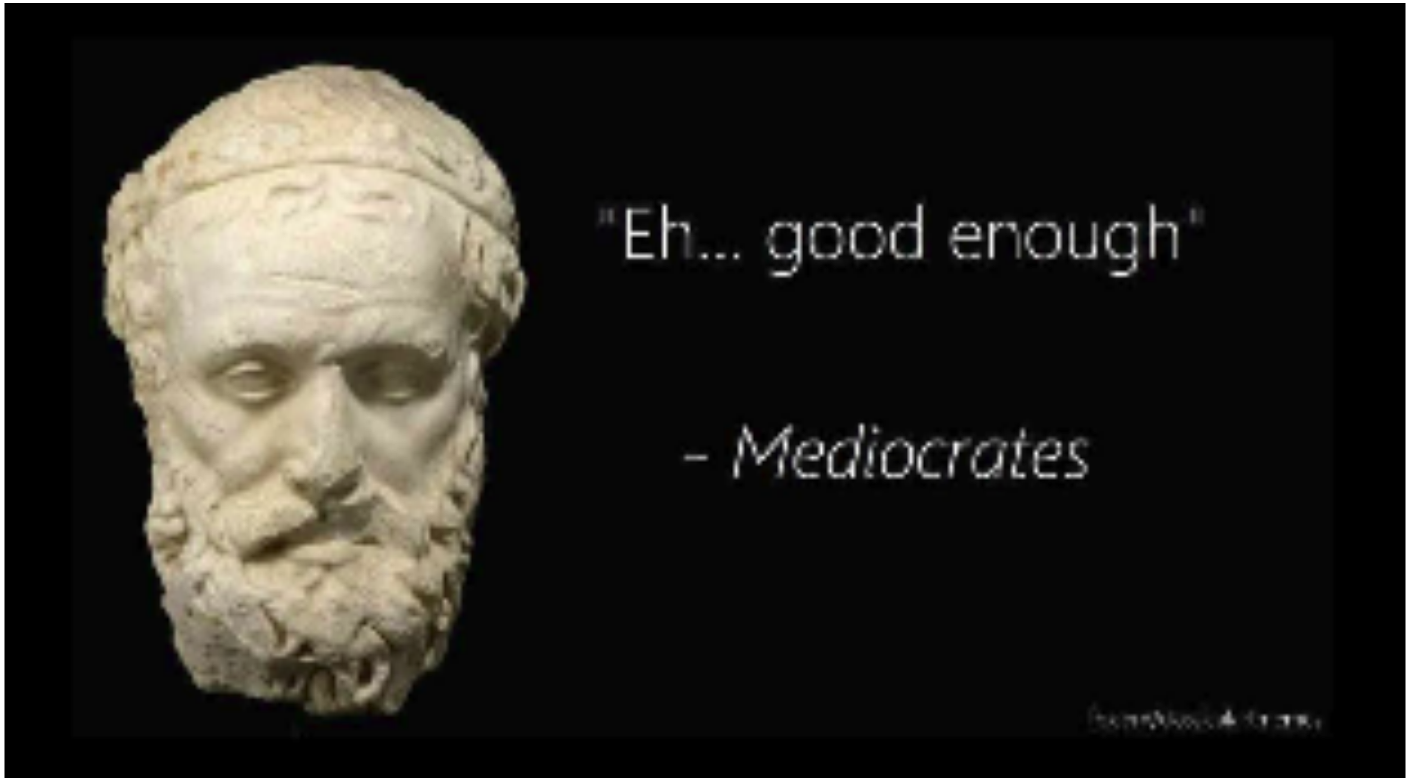
c=5



$\langle I_N(x_1) I_N(x_2)\rangle_c$
6
4
2
0
−2
−4
−6
x_2
x_1
0.0000100
0.0000075
0.0000050
0.0000025
0.0000000
−0.0000025
−0.0000050
−0.0000075
−0.0000100

Summary

- Developing Functional Rate theory.
- Mean for Index distribution.
- Correlation for Index distribution.
- Pending Theoretical verification.


"Eh... good enough"
- Mediocrates