Stability analysis of ecosystems with random modular interactions

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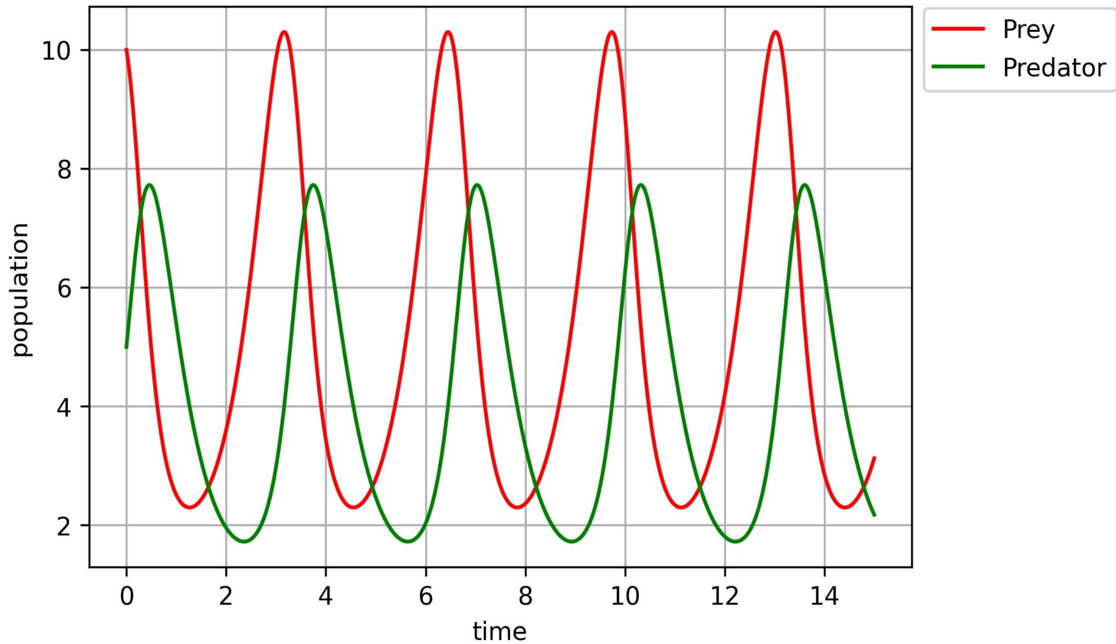
Outline of Presentation

I. Introduction
II. Description of the Model
III. Theoretical Analysis
IV. Numerical Simulations
V. Conclusions

Introduction: Population dynamics in ecosystems

$$\frac{dx_i}{dt} = f_i(\boldsymbol{x}(t))$$

x_i : population of species i



Introduction: Population dynamics in ecosystems

Get equilibrium points $\boldsymbol{x}^*$

$$\frac{dx_i}{dt}\Big|_{x^*} = f_i(\boldsymbol{x}^*) = 0$$

$\boldsymbol{x}^*$ can be either stable and unstable

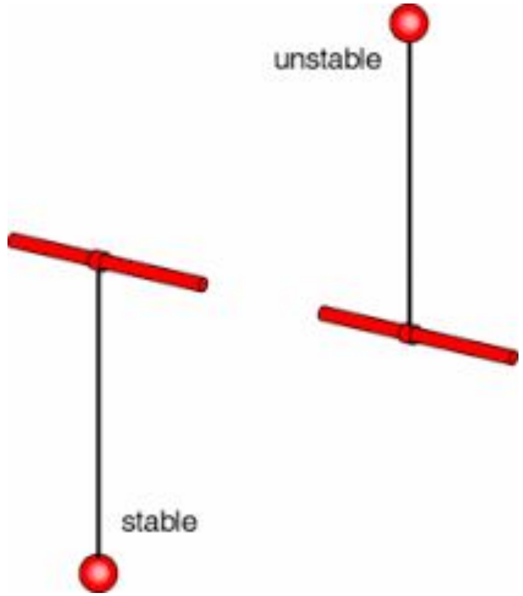


Image from: http://www.scholarpedia.org/article/Equilibrium

Introduction: The Community Matrix

Using Multivariate Taylor Series

$$\frac{d\boldsymbol{x}}{dt} = \cancel{\boldsymbol{f}(\boldsymbol{x}^*)} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\Big|_{\boldsymbol{x}^*}(\boldsymbol{x} - \boldsymbol{x}^*) + \ldots$$

Jacobian Matrix : $\tilde{J}_{ij} = \dfrac{\partial f_i(\boldsymbol{x})}{\partial x_j}$

Community Matrix: $A = \tilde{J}\Big|_{\boldsymbol{x}^*}$

Introduction: Using Eigenvalue to determine stability

if we let $\tilde{\boldsymbol{x}} = (\boldsymbol{x} - \boldsymbol{x}^*)$

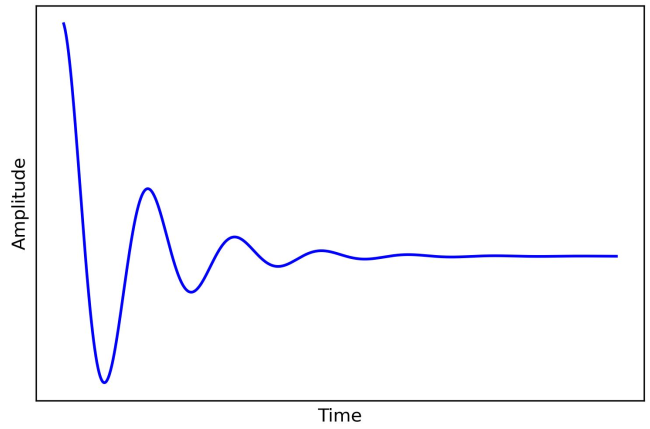
$$\frac{d\tilde{\boldsymbol{x}}}{dt} = \boldsymbol{A}\tilde{\boldsymbol{x}}$$

$$\tilde{\boldsymbol{x}}(t) = c_1 e^{\lambda_1 t} \boldsymbol{v}_1 + c_2 e^{\lambda_2 t} \boldsymbol{v}_2 + \ldots + c_n e^{\lambda_n t} \boldsymbol{v}_n$$

where $\mathrm{Re}\left(\lambda_1\right) > \mathrm{Re}\left(\lambda_2\right) > \ldots > \mathrm{Re}\left(\lambda_n\right)$

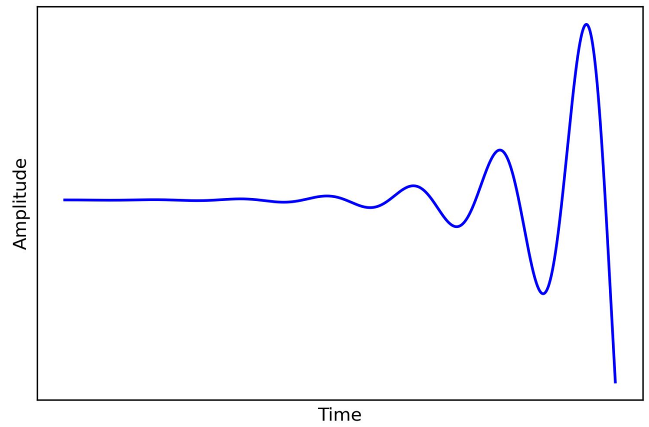
Introduction: Using Eigenvalue to determine stability

$$\mathrm{Re}(\lambda_1) < 0$$



stable

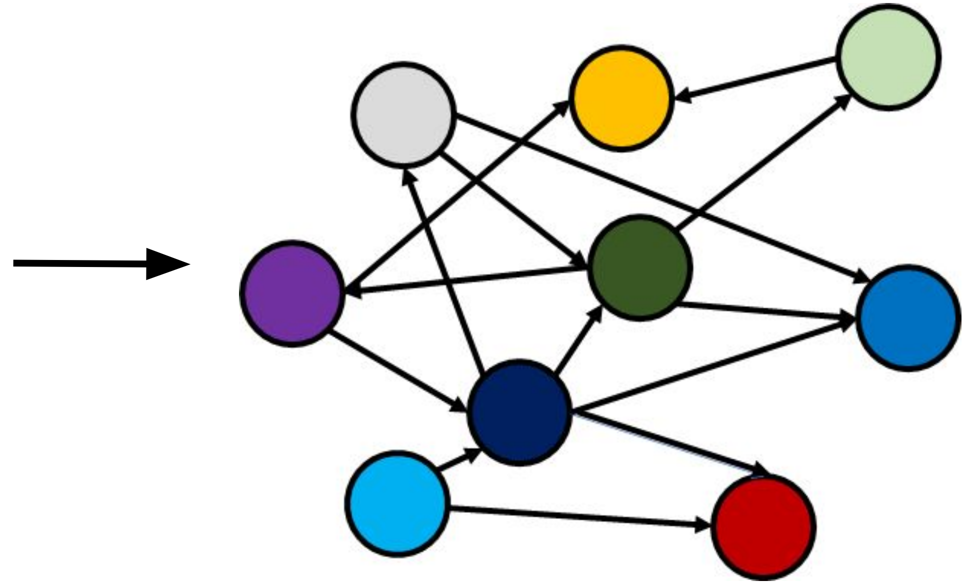
$$\mathrm{Re}(\lambda_1) > 0$$



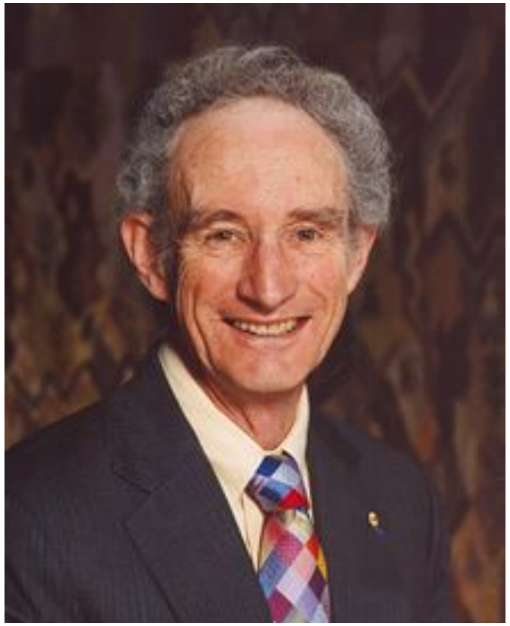
unstable

Introduction: Community matrix as the Adjacency Matrix

$$\boldsymbol{A} = \begin{pmatrix} 3.546 & 2.234 & \cdots & 1 \\ \vdots & \ddots & 0 & -0.2 \\ -1.456 & 0.45 & \cdots & 3 \end{pmatrix}$$



Introduction: Using Random matrices in ecological dynamics analysis



Robert May

- Robert May: Random Matrix as Community Matrix

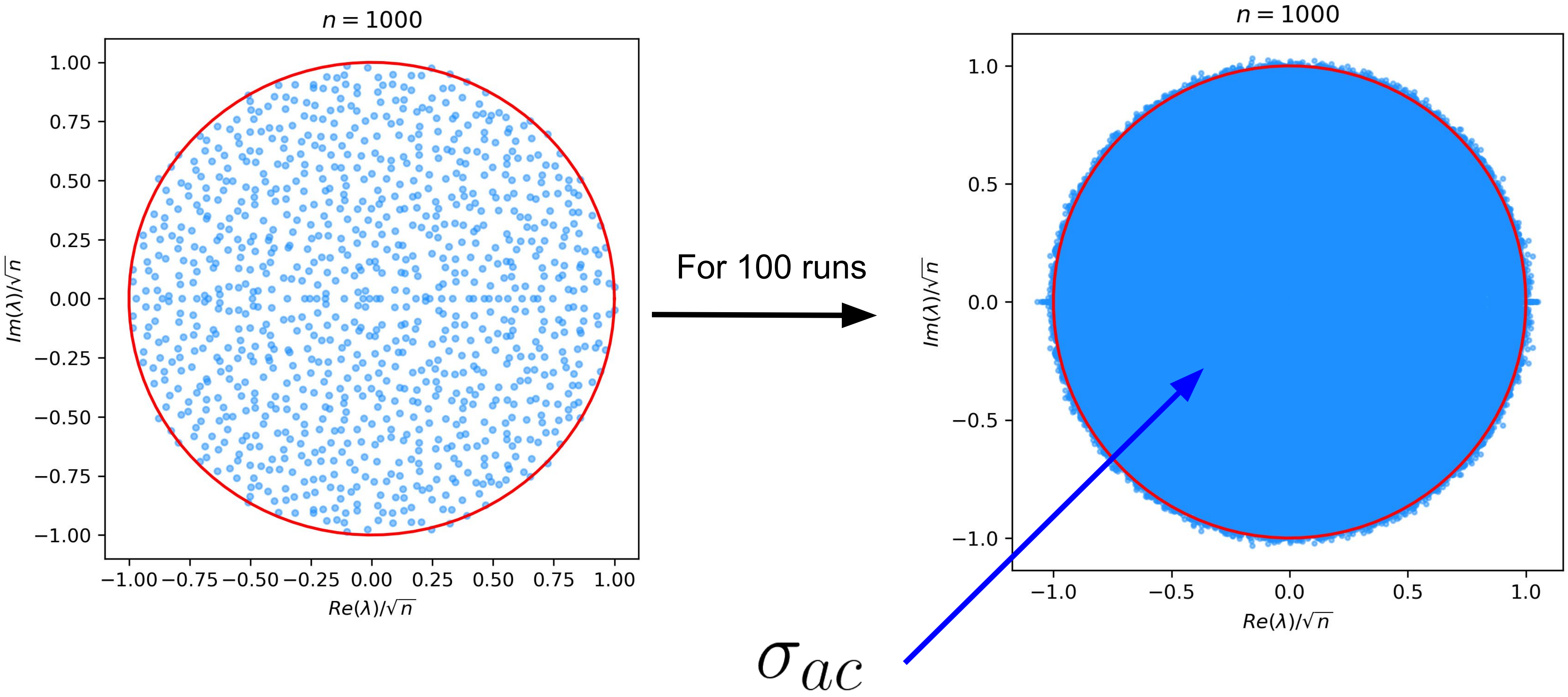
- Condition of stability for a general (fully-connected) random matrix:

$$\sqrt{SC\sigma^2} < d$$

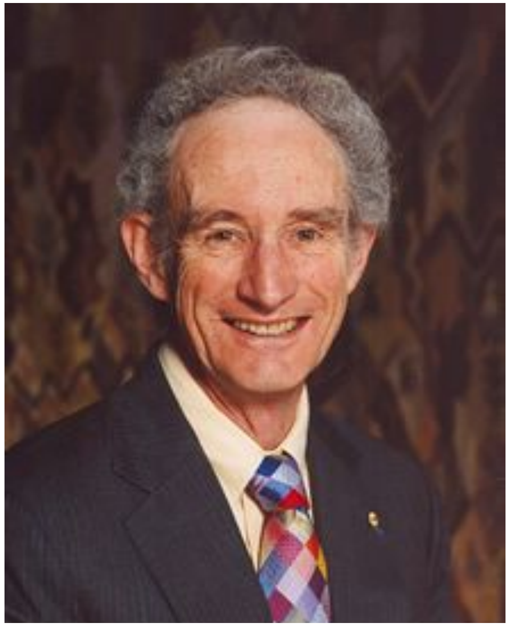
where
S : size of matrix
C : connentance of matrix
σ^2 : variance of the elements
d : diagonal elements

Image from: https://www.science.org.au/learning/general-audience/history/interviews-australian-scientists/lord-robert-may-physicist-and

Self-Averaging of Random Matrices



Introduction: Problems of May’s approach



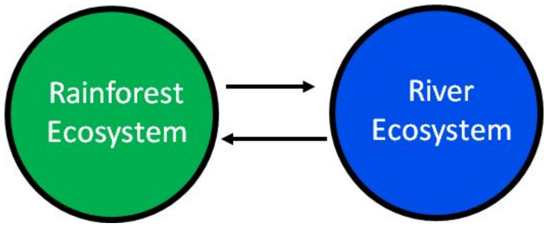
Robert May

- However, May emphasize the importance of studying different network structures to model ecological communities realistically.
- For very large food webs, the connection between nodes are sparse. What happens if the matrix becomes sparser and more structured?

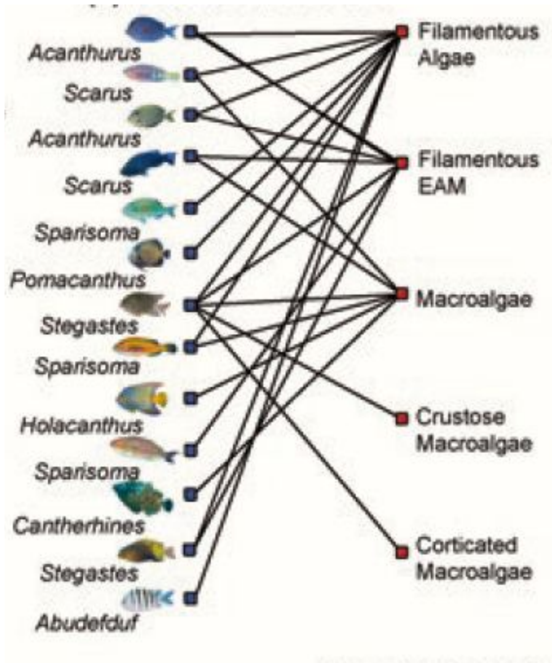
Image from: https://www.science.org.au/learning/general-audience/history/interviews-australian-scientists/lord-robert-may-physicist-and

Introduction: Network Structures in an Ecosystem

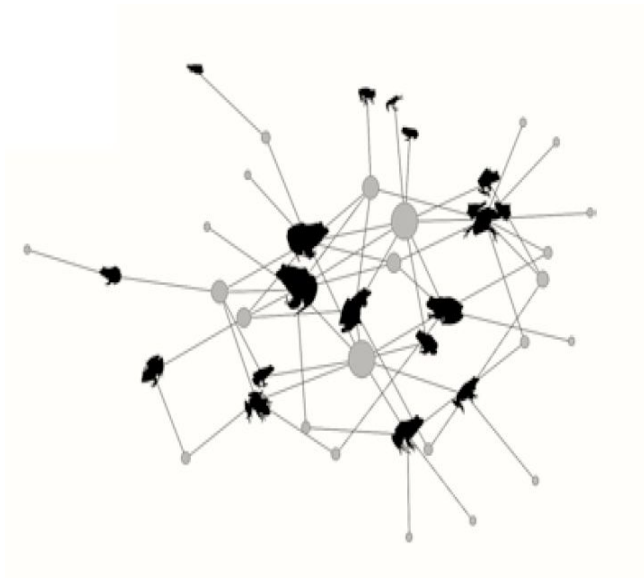
In reality, food webs can be of the following structure (but not limited to):



Modular



Bipartite



Core-periphery

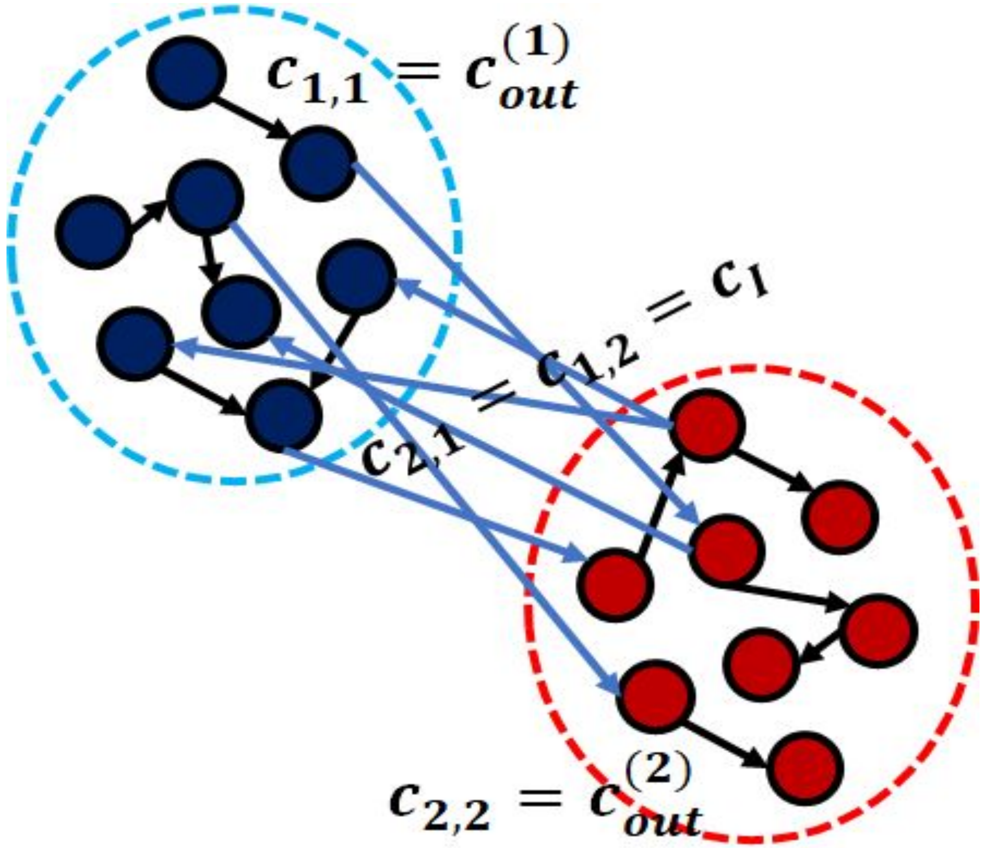
Image from: Dáttilo,W.(2018).Ecological Networks in the Tropics:An Integrative Overview of Species Interactions from Some of the Most SpeciesRich Habitats on Earth.Springer

Description of the model

$$A_{ij} = C_{ij} - D\delta_{ij}$$

where $C_{ij} \in \{0, 1\}$

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{C^{(1)}} & \boldsymbol{\alpha} \\ \boldsymbol{\beta} & \boldsymbol{C^{(2)}} \end{pmatrix}$$

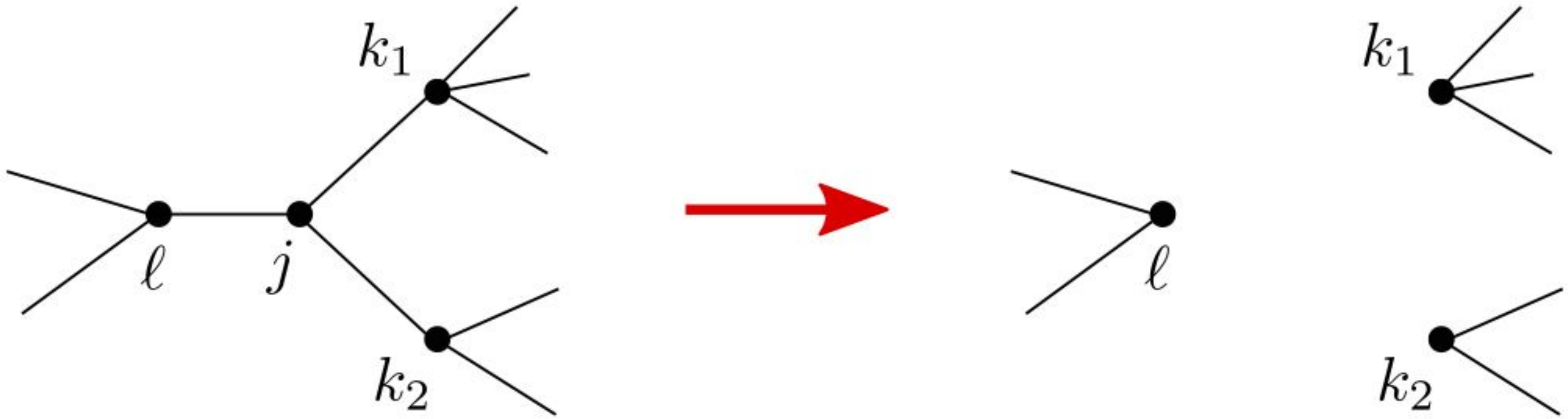


Each block is controlled by a connectivity parameter:

$$p_{u,v} = c_{u,v}/n$$

u, v : communities , n: no. of species per community

Right eigenvector of oriented sparse locally tree-like matrices



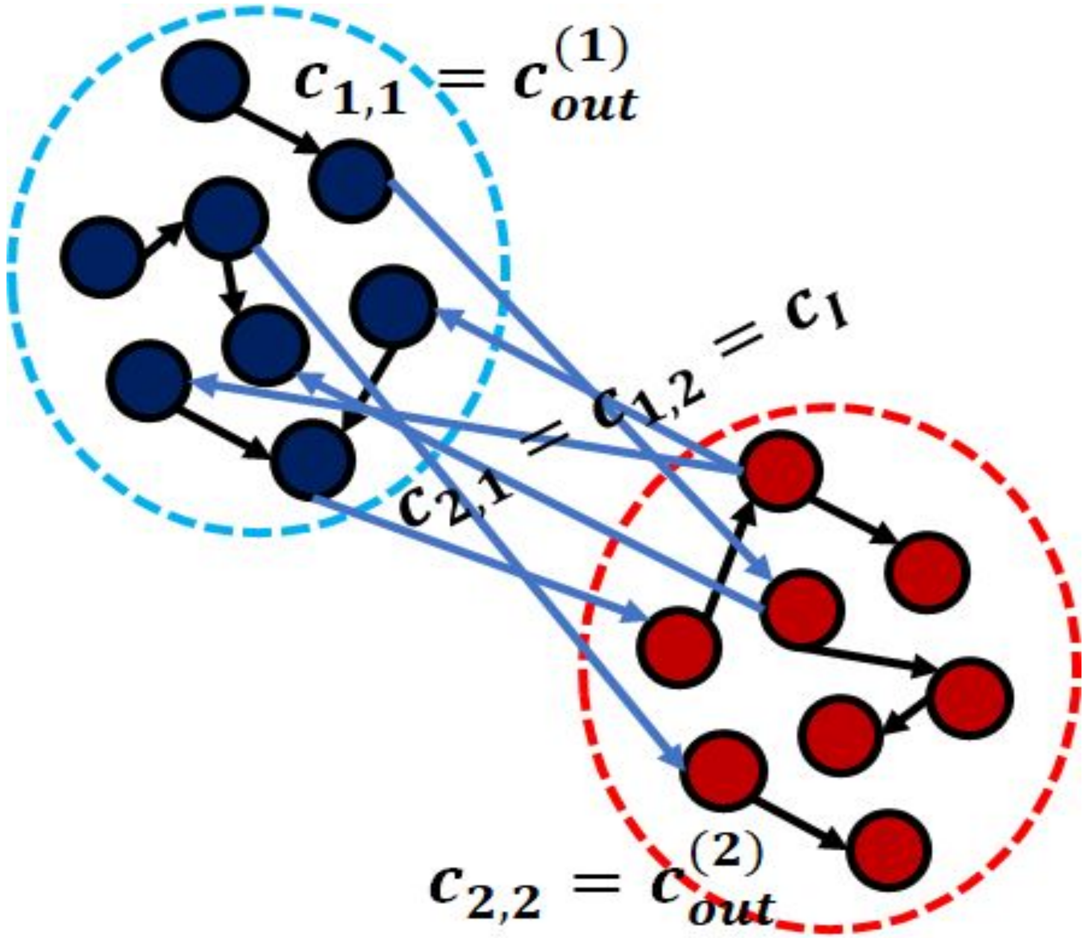
$$R_j = \frac{1}{z + D} \sum_{k \in \partial_j^{\mathrm{out}}} R_k \,, \qquad z \notin \sigma_{\mathrm{ac}}$$

Metz et.al. (2019)

Two-community structure

Splitting the right eigenvector into $R^{(1)}$ and $R^{(2)}$

$$R = \begin{pmatrix} R^{(1)} \\ R^{(2)} \end{pmatrix}$$



Two-community structure

$$R_j^{(1)} = \frac{1}{z+D} \sum_{k \in \partial_j^{\text{out}}, k \in \mathbf{C}^{(1)}} R_k^{(1)} + \frac{1}{z+D} \sum_{k' \in \partial_j^{\text{out}}, k' \in \mathbf{C}^{(2)}} R_{k'}^{(2)}$$

$$R_j^{(2)} = \frac{1}{z+D} \sum_{k \in \partial_j^{\text{out}}, k \in \mathbf{C}^{(1)}} R_k^{(1)} + \frac{1}{z+D} \sum_{k' \in \partial_j^{\text{out}}, k' \in \mathbf{C}^{(2)}} R_{k'}^{(2)}$$

Equations for the first moments

$$\langle R^{(1)}\rangle = \frac{1}{z+D}\left[c_{\rm out}^{(1)}\langle R^{(1)}\rangle\right] + \frac{1}{z+D}\left[c_I\langle R^{(2)}\rangle\right]$$

$$\langle R^{(2)}\rangle = \frac{1}{z+D}\left[c_I\langle R^{(1)}\rangle\right] + \frac{1}{z+D}\left[c_{\rm out}^{(2)}\langle R^{(2)}\rangle\right]$$

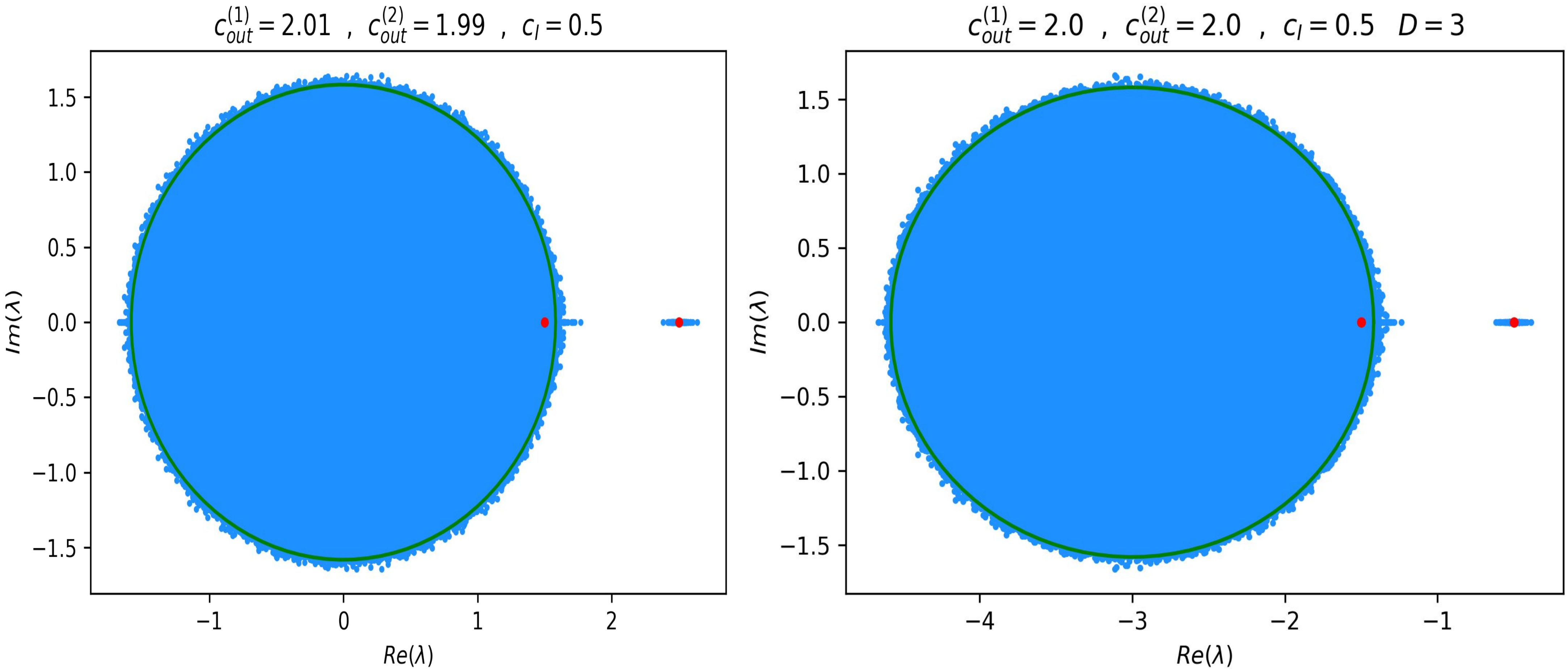
The outlier

$$z = -D + \frac{c_{\text{out}}^{(2)} + c_{\text{out}}^{(1)}}{2} \pm \frac{\sqrt{\left(c_{\text{out}}^{(2)} - c_{\text{out}}^{(1)}\right)^2 + 4c_I^2}}{2},$$

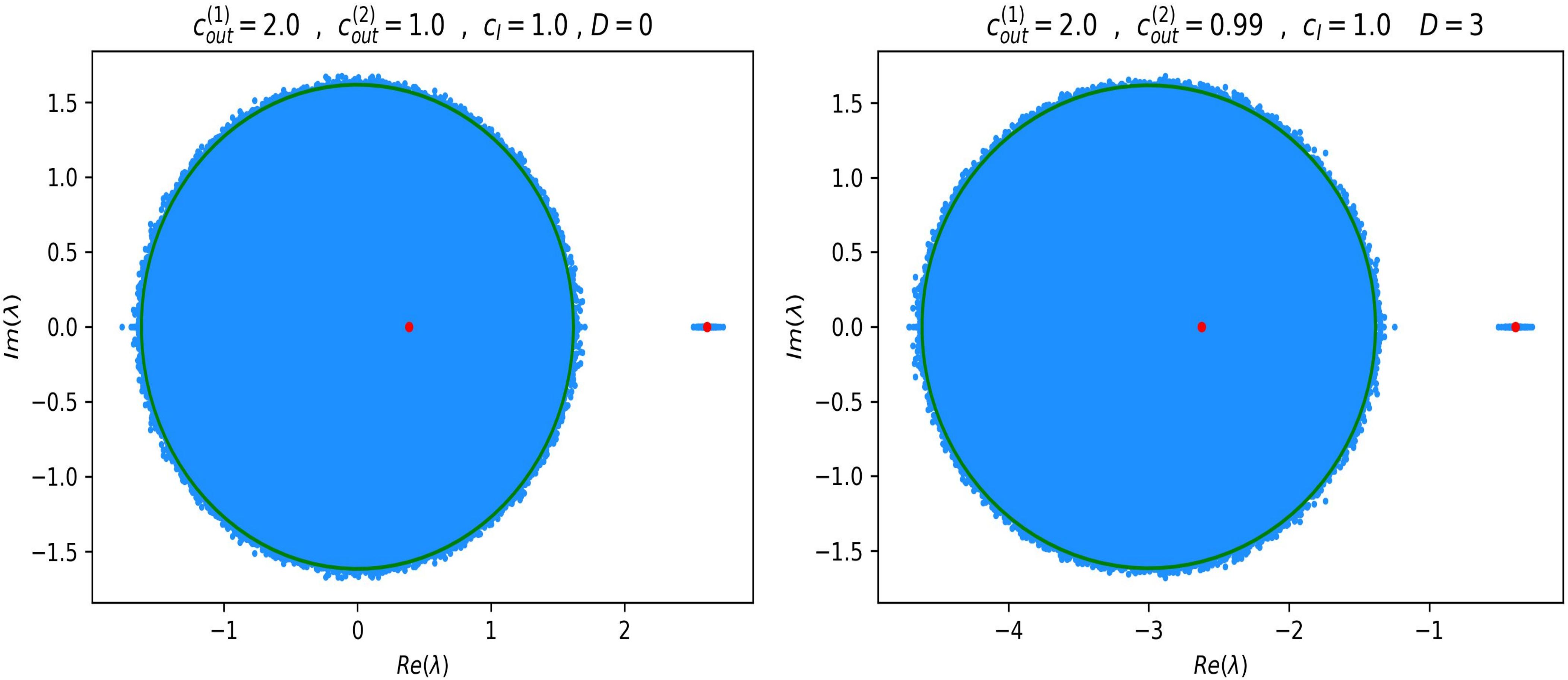
The boundary of the continuous part

$$|z + D|^2 = \frac{c_{\text{out}}^{(2)} + c_{\text{out}}^{(1)}}{2} \pm \frac{\sqrt{\left(c_{\text{out}}^{(2)} - c_{\text{out}}^{(1)}\right)^2 + 4c_I^2}}{2},$$

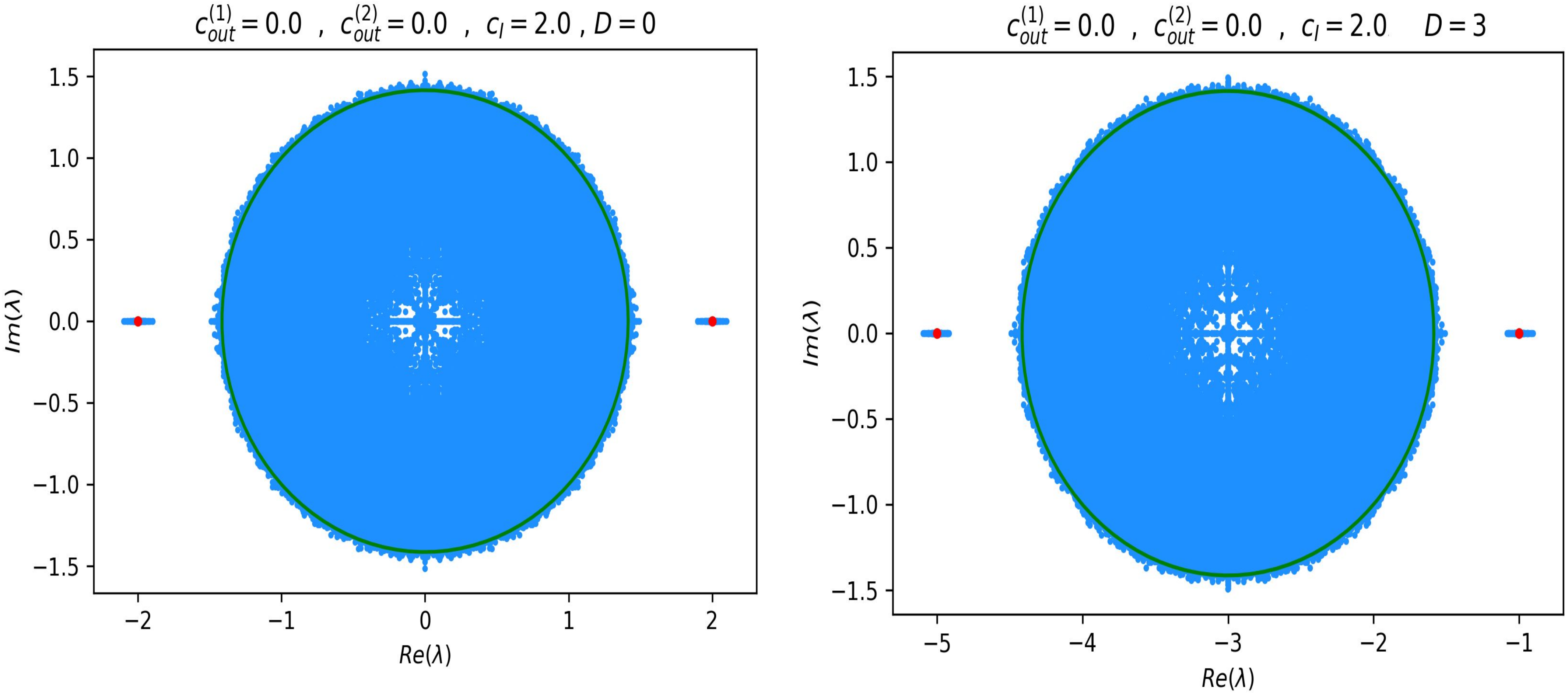
Two-modular networks



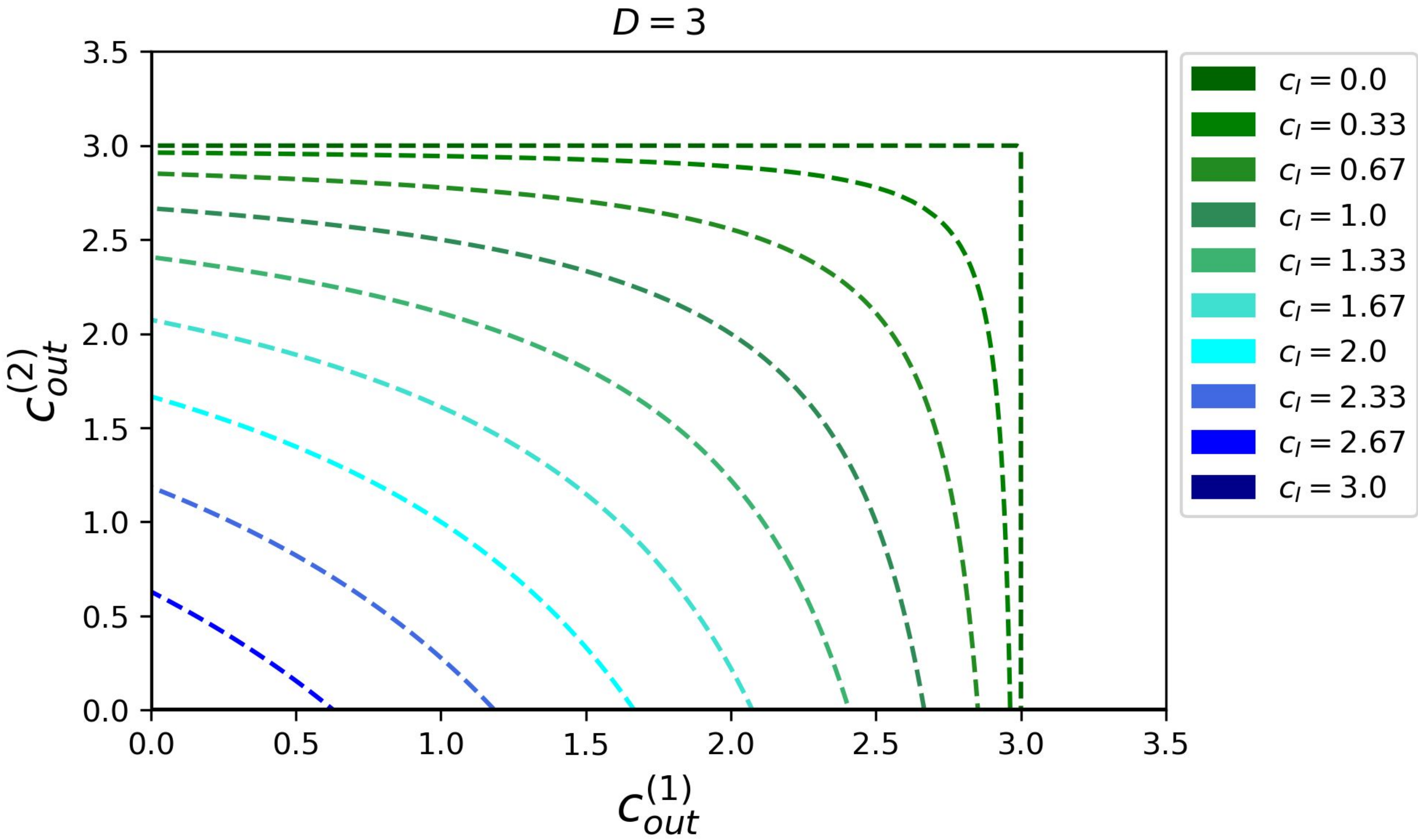
Core-periphery networks



Bi-partite networks

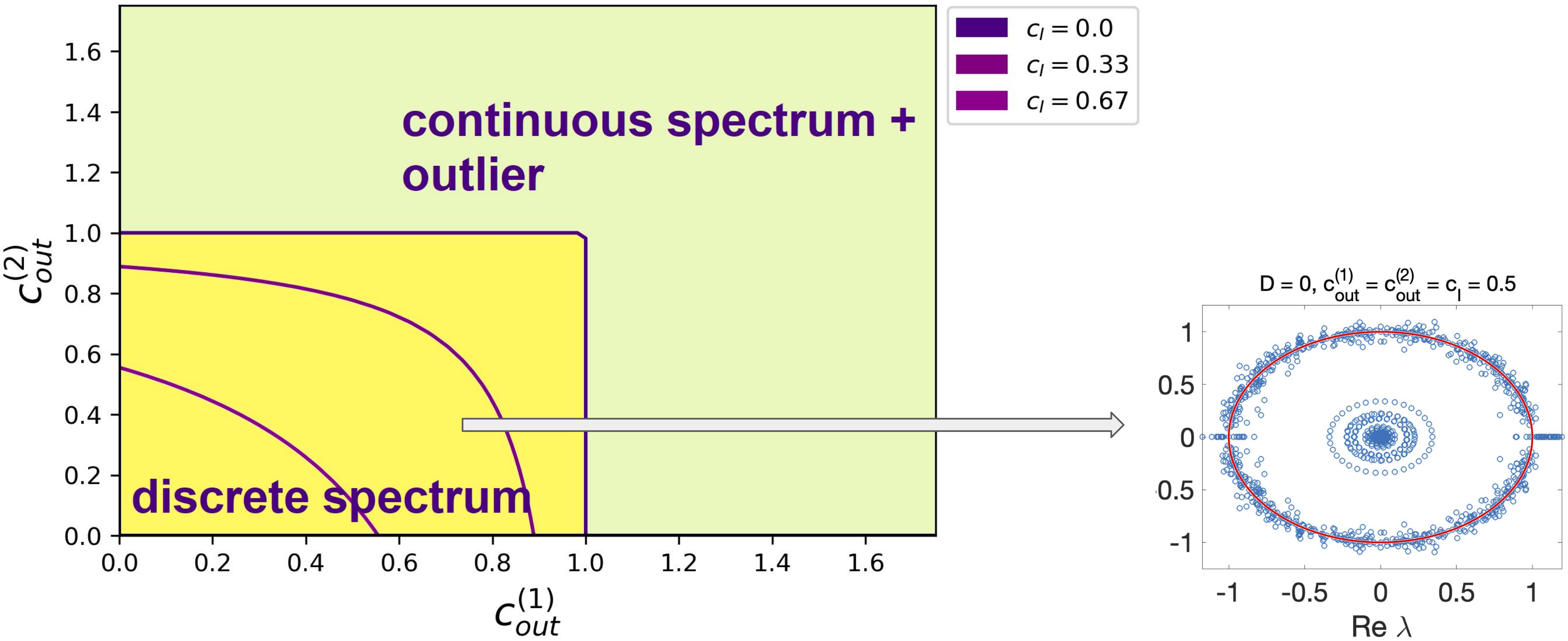


Stability -- Instablity diagram

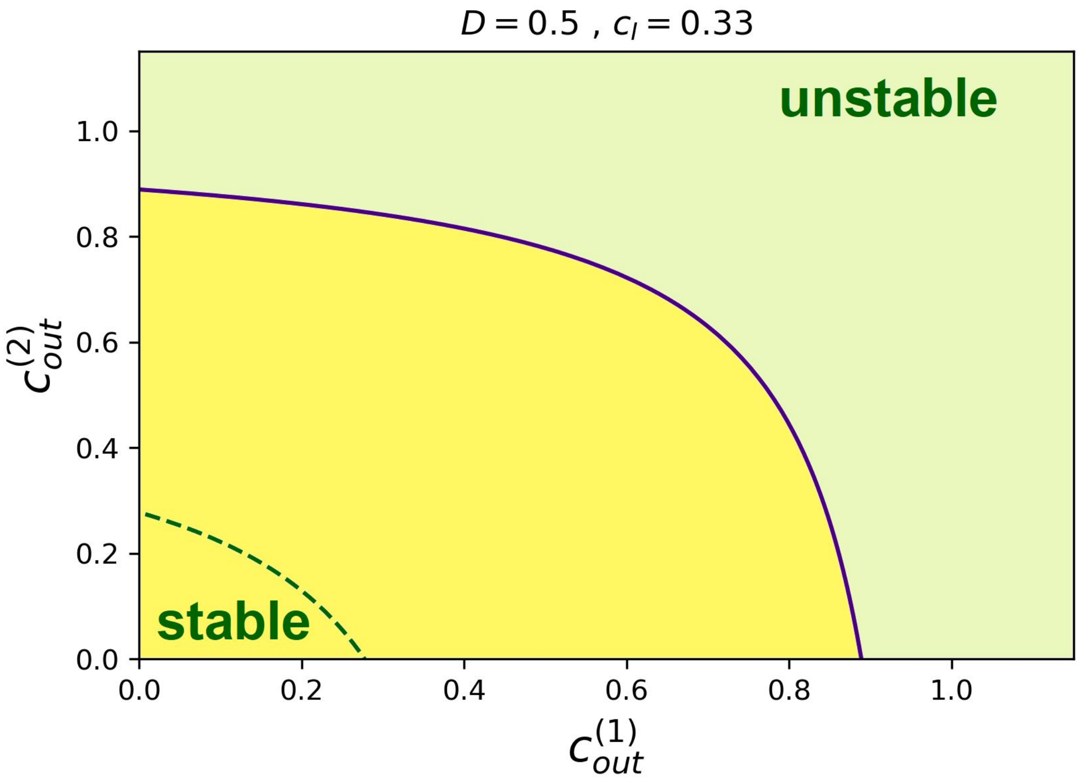


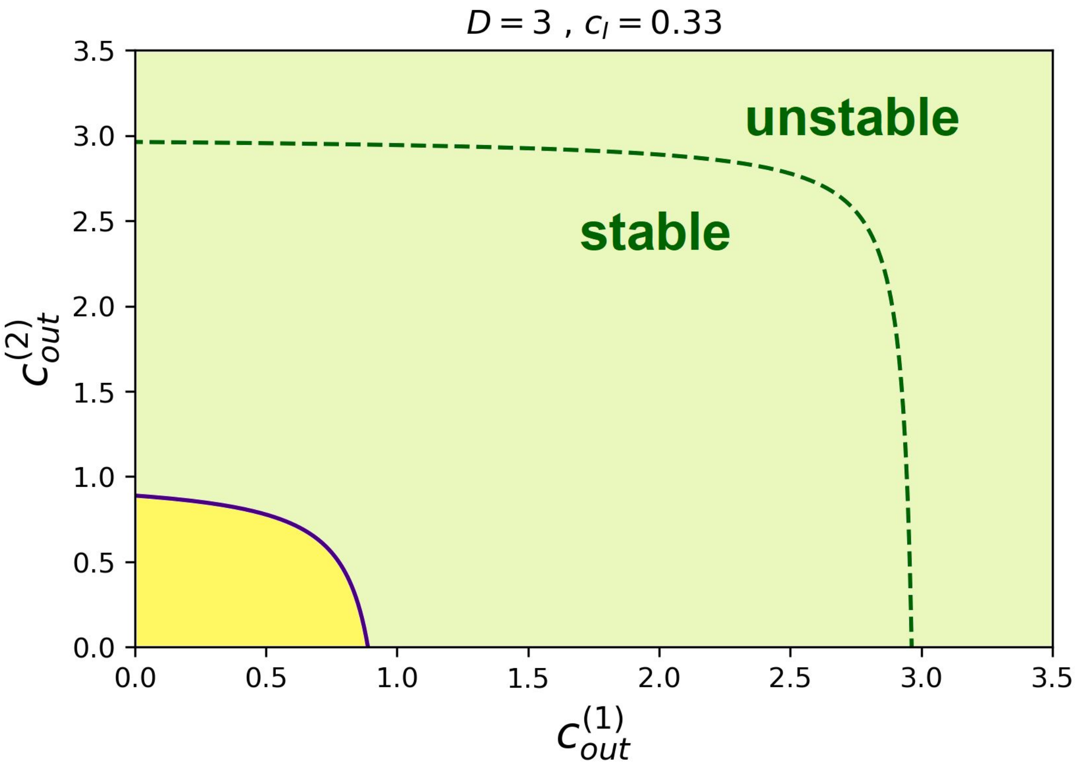
$$c_I^2 - c_{\mathrm{out}}^{(1)} c_{\mathrm{out}}^{(2)} + D(c_{\mathrm{out}}^{(1)} + c_{\mathrm{out}}^{(2)}) < D^2$$

The existence of spectral gap



Combination of the two diagrams





Conclusion & Future work

- Outlier and boundary of the bulk of the spectrum are found analytically
- Condition for the existence of gap between the outlier and the bulk
- Condition for the system stability
- Modular structure is more stable than the core-periphery one

Future work

Localisation of the eigenvectors

Multimodular structure

Weighted and degree-correlated graphs