Analogue of many-body Berry-Esseen theorem for critical systems

Karen Hovhannisyan

The Abdus Salam International Centre for Theoretical Physics

Adriatic Conference on Strongly Correlated Systems,
ICTP, 22 Mar 2021

More details at

arXiv:2011.10513 **[quant-ph]**

In collaboration with

Mathias Jørgensen (Technical University of Denmark)

Gabriel Landi (Universidade de São Paulo)

Álvaro Alhambra (Max-Planck-Institut für Quantenoptik)

Jonatan Brask (Technical University of Denmark)

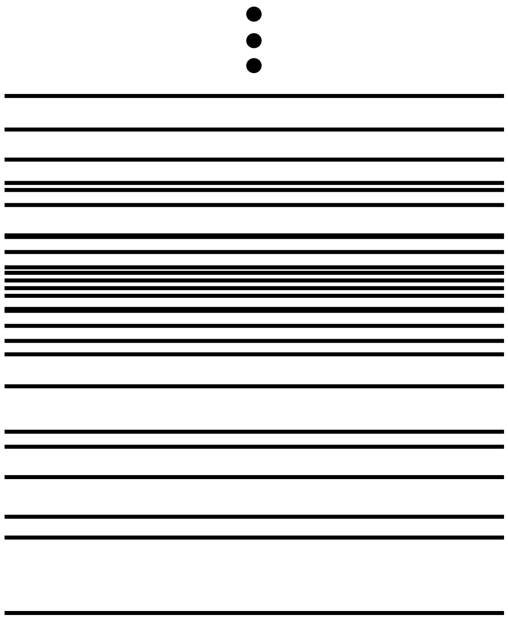
Martí Perarnau-Llobet (Université de Genève) many-body system

$$H = \sum_{\langle i,k \rangle} [atom_i] \otimes [atom_k] + \sum_i V(atom_i)$$

complicated Hamiltonian

$$H = \sum_{\langle i,k \rangle} [atom_i] \otimes [atom_k] + \sum_i V(atom_i)$$

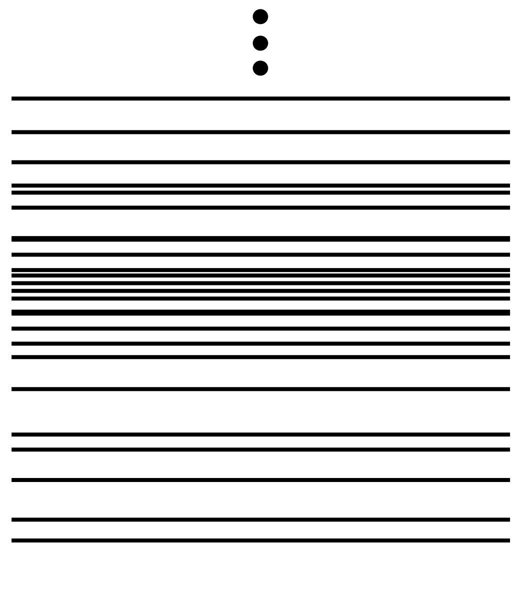
complicated spectrum



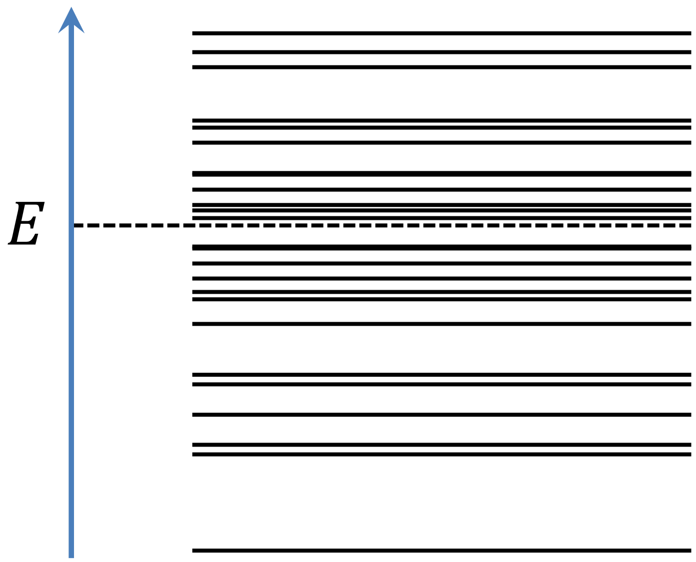
simple Hamiltonian

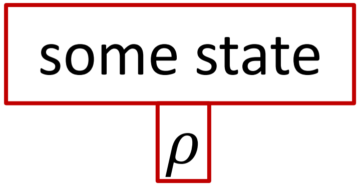
$$H = J \sum_{\langle i,i+1 \rangle} \sigma_x^i \otimes \sigma_x^{i+1} + h \sum_i \sigma_z^i$$

complicated spectrum

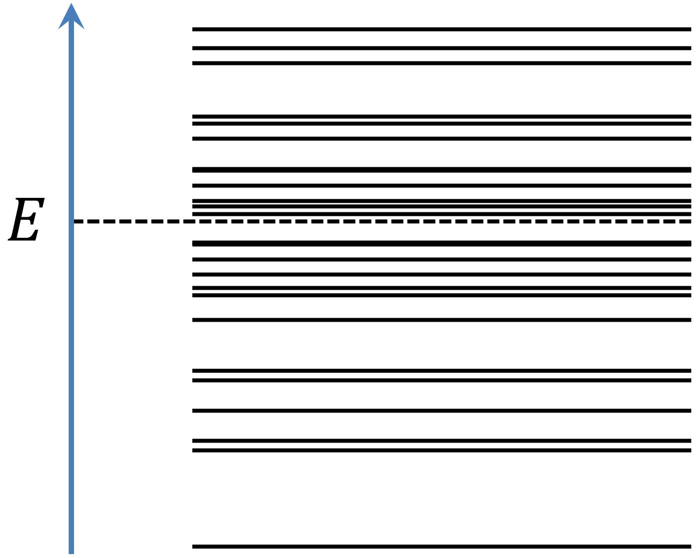


Energy distribution





Energy distribution

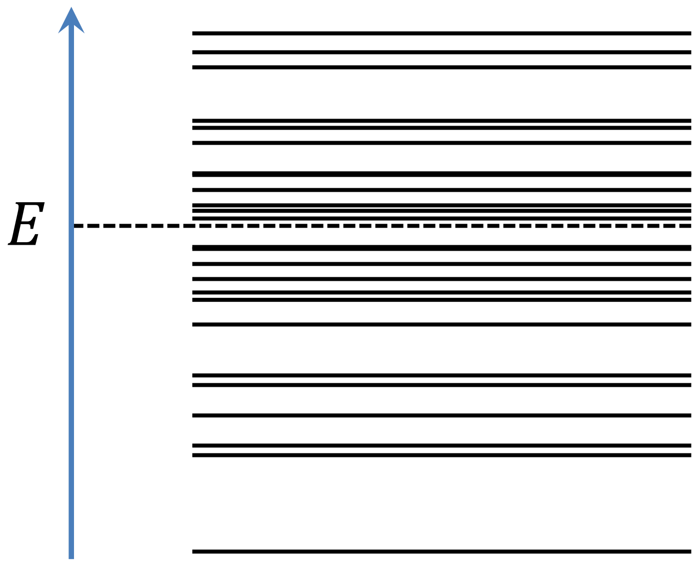


some state

ρ

$$J(E) = \sum_{E_i \leq E} \langle E_i | \rho | E_i \rangle$$

Energy distribution



some state

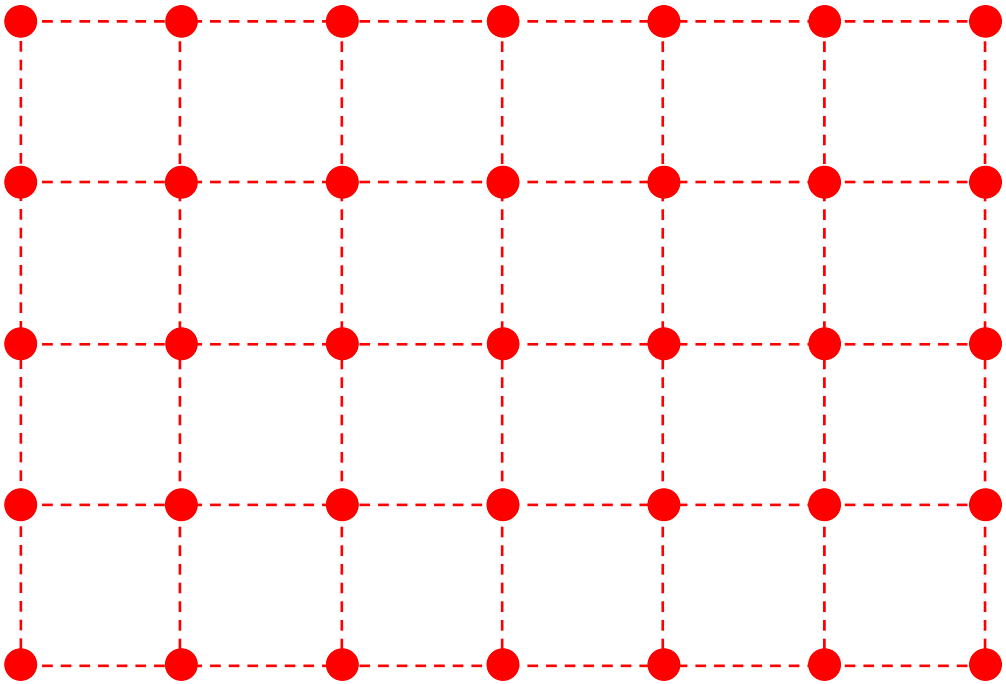
ρ

$$J(E) = \sum_{E_i \le E} \langle E_i | \rho | E_i \rangle$$

cumulative distribution of energy

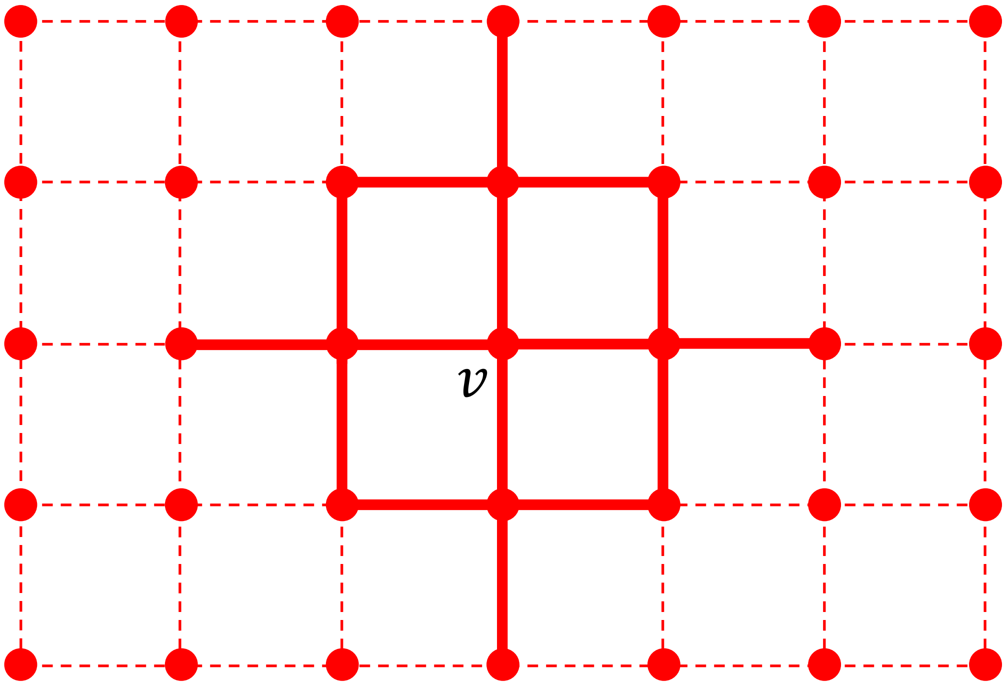
Noncritical Berry-Esseen theorem

Brandão & Cramer, arXiv:1502.03263 [quant-ph]



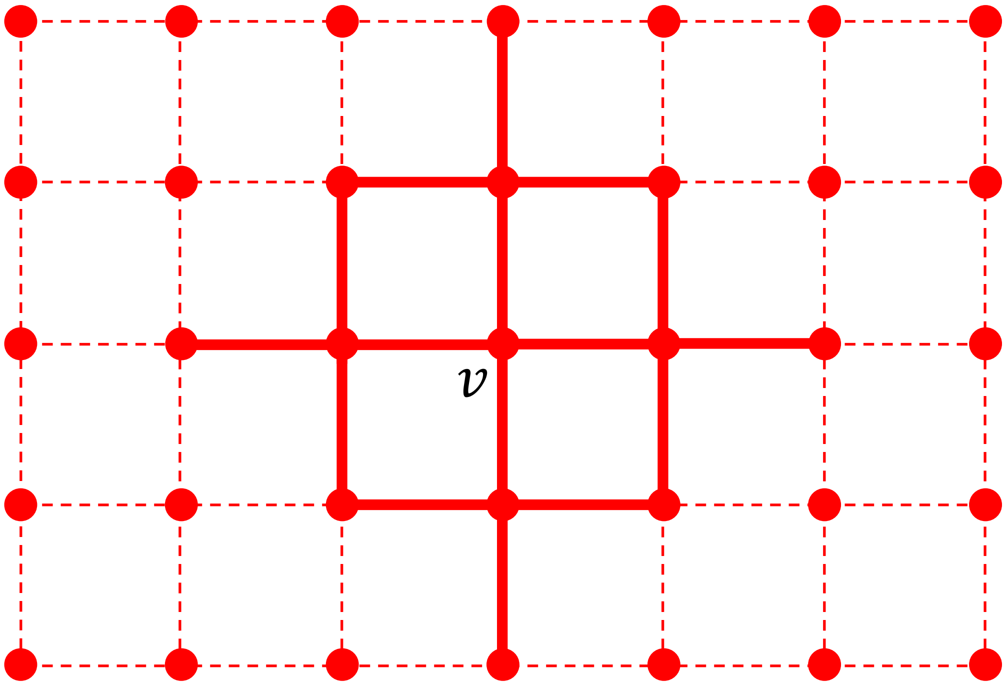
$$H = \sum_v H_v$$

Noncritical Berry-Esseen theorem



$$H = \sum_v H_v$$

Noncritical Berry-Esseen theorem

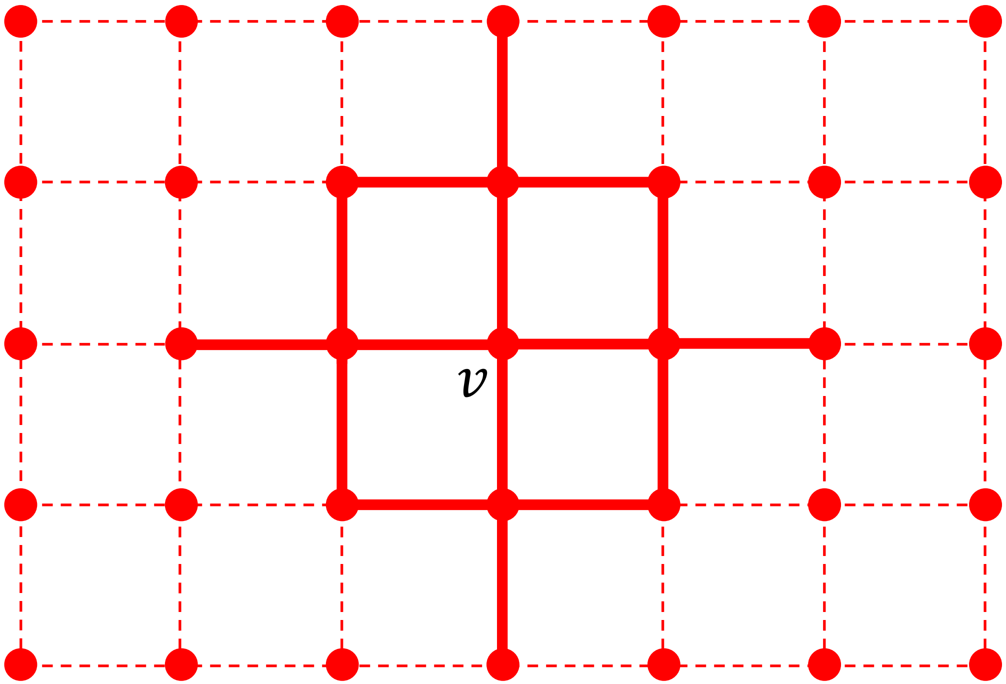


$$H = \sum_v H_v$$

H_v → finite range

$$\|H_v\| \leq h$$

Noncritical Berry-Esseen theorem



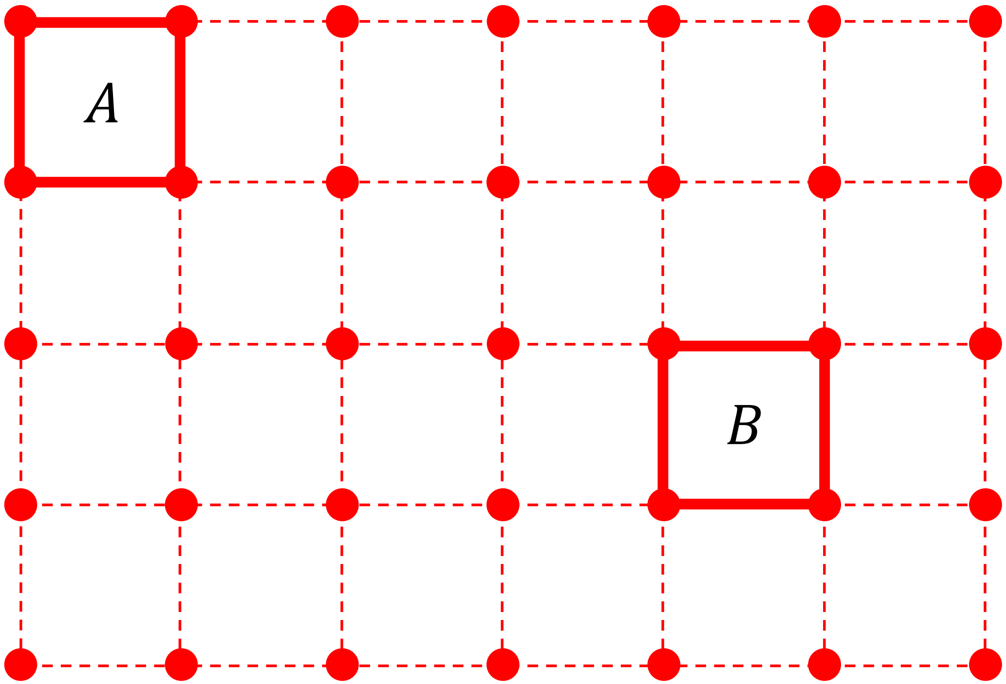
$$H = \sum_v H_v$$

finite range

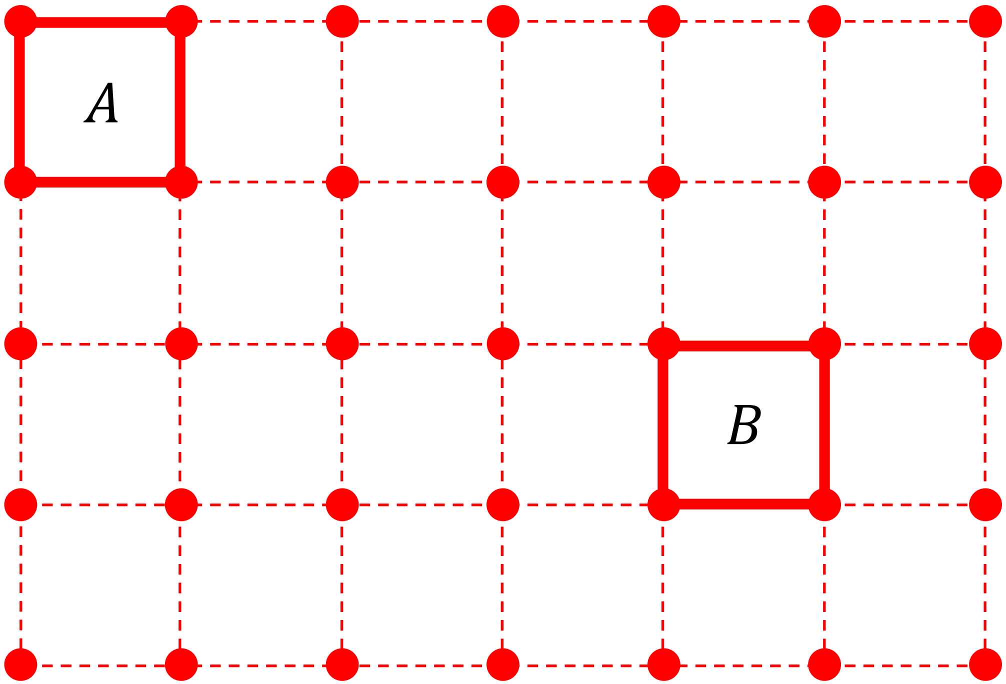
not necessarily translation invariant

$$\|H_v\| \leq h$$

Noncritical Berry-Esseen theorem

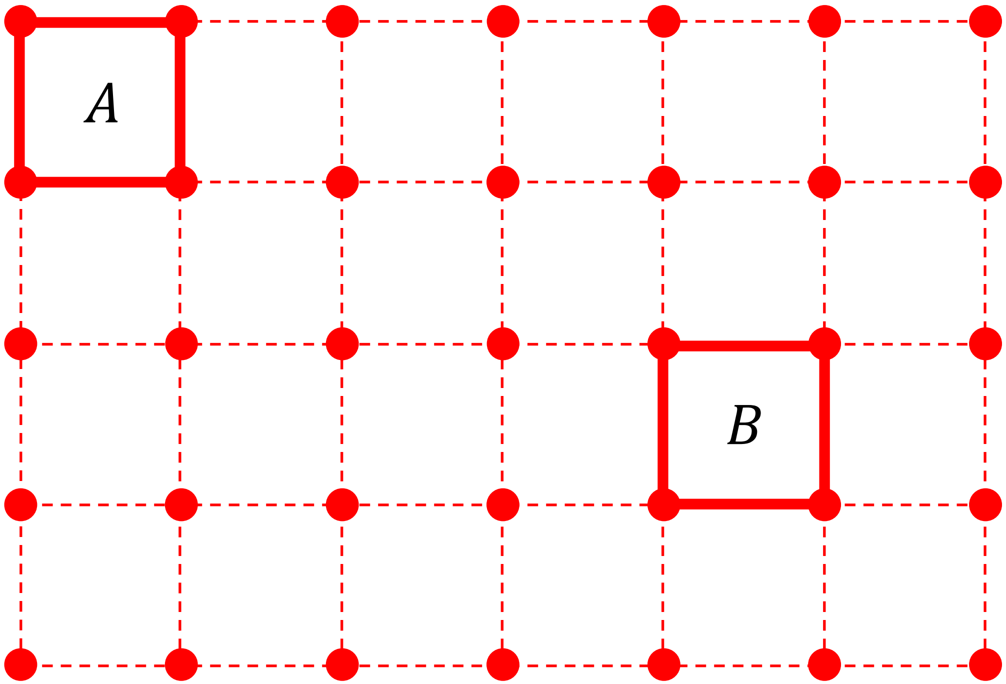


Noncritical Berry-Esseen theorem



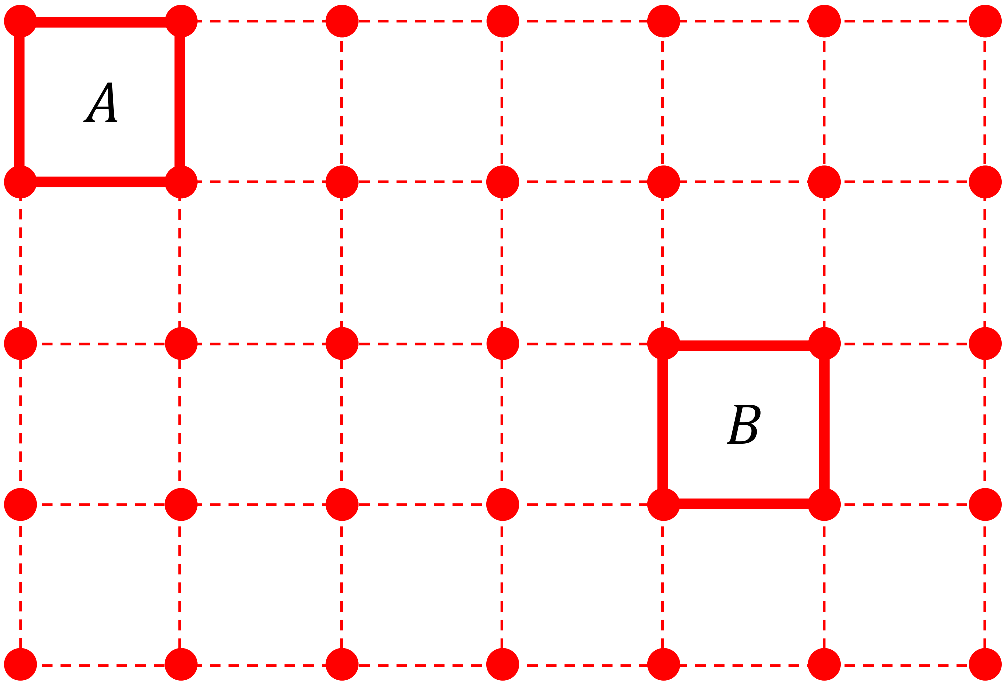
$$\frac{\mathrm{tr}(\rho A \otimes B) - \mathrm{tr}(\rho A)\,\mathrm{tr}(\rho B)}{\|A\| \cdot \|B\|} \leq e^{-D(A,B)/\xi}$$

Noncritical Berry-Esseen theorem



$$\frac{\langle A \otimes B \rangle_\rho - \langle A \rangle_\rho \langle B \rangle_\rho}{\|A\| \cdot \|B\|} \leq e^{-D(A,B)/\xi}$$

Noncritical Berry-Esseen theorem

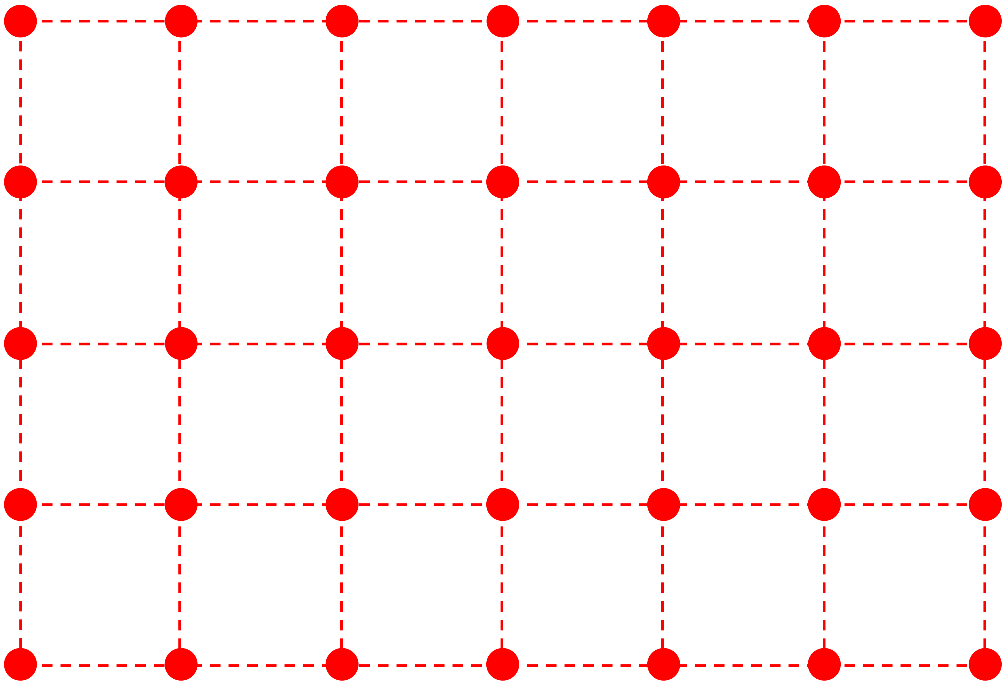


$$\frac{\langle A \otimes B\rangle_\rho - \langle A\rangle_\rho \langle B\rangle_\rho}{\|A\| \cdot \|B\|} \leq e^{-D(A,B)/\xi}$$

$$\mu^2 \coloneqq \mathrm{var}(H) \coloneqq \langle H^2\rangle_\rho - \langle H\rangle_\rho^2 = s^2 N$$

Noncritical Berry-Esseen theorem

Brandão & Cramer, arXiv:1502.03263 [quant-ph]

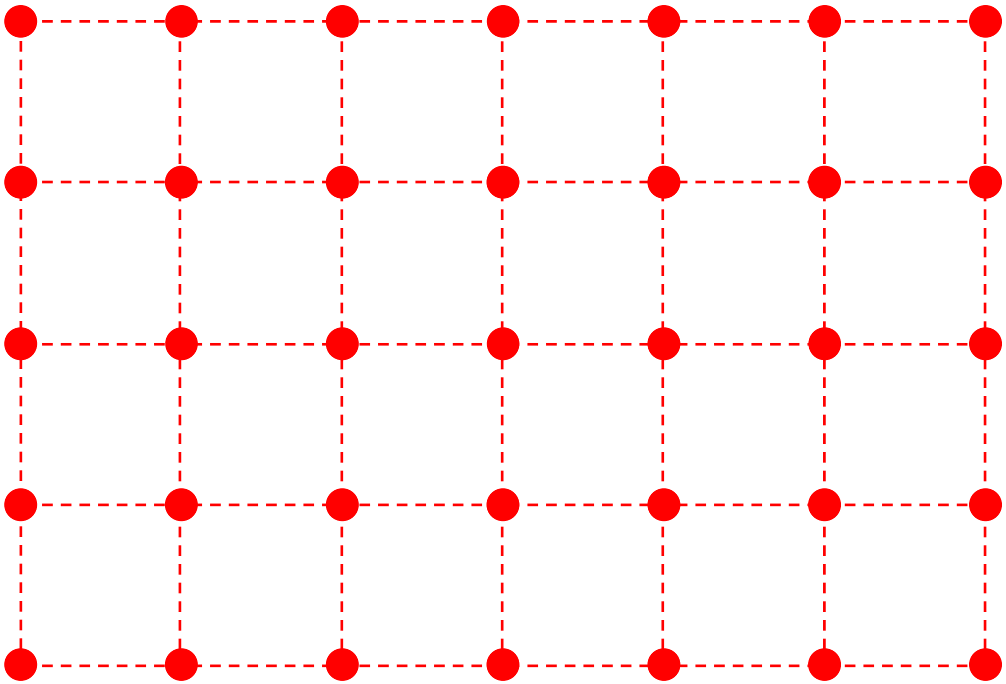


$$\sup_E |J(E) - G(E)| \leq O\left(\frac{\ln^{2d} N}{\sqrt{N}}\right)$$

$$G(E) = \frac{1}{\sqrt{2\pi\mu^2}} \int_{-\infty}^{E} dE\, e^{-(E-\langle H \rangle)^2/(2\mu^2)}$$

Noncritical Berry-Esseen theorem

Brandão & Cramer, arXiv:1502.03263 [quant-ph]



$$\sup_E |J(E) - G(E)| \leq O\left(\frac{\ln^{2d} N}{\sqrt{N}}\right) \xrightarrow{N\to\infty} 0$$

$$G(E) = \frac{1}{\sqrt{2\pi\mu^2}} \int_{-\infty}^{E} dE \; e^{-(E-\langle H\rangle)^2/(2\mu^2)}$$

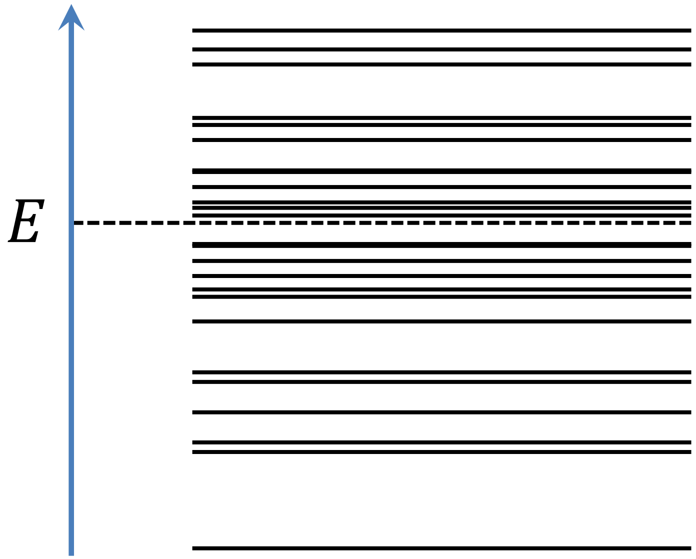
What about critical systems?

An analogue of Berry-Esseen theorem holds

… but is valid only for

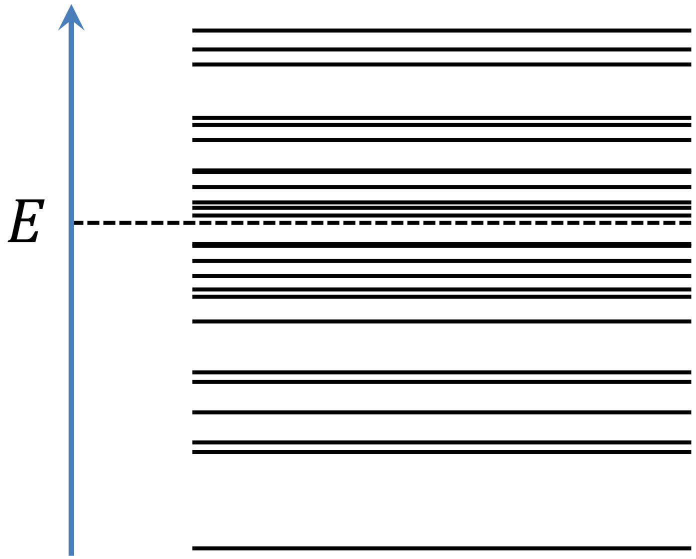
- Thermal states
- Finite-temperature phase transitions
- Translation-invariant lattices

Bird's-eye view on the proof: Spectral density



$$\Gamma(E) = \#\{E_i \leq E\}$$

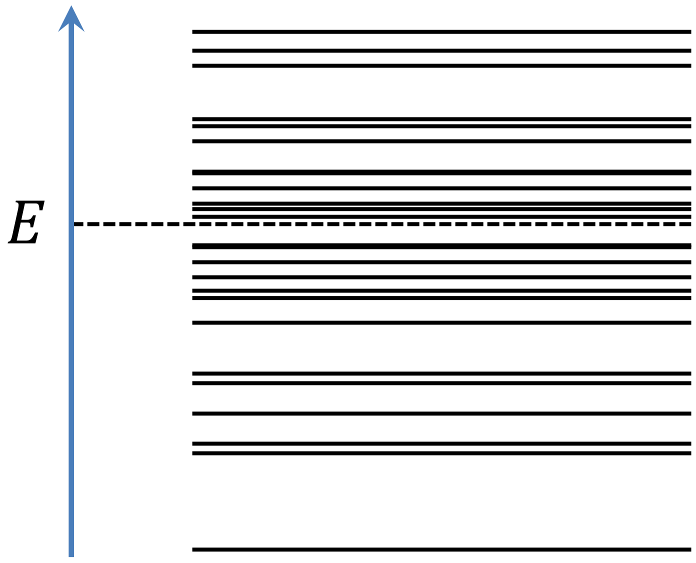
Bird's-eye view on the proof: Spectral density



$$\Gamma(E) = \#\{E_i \leq E\}$$

cumulative spectral density

Bird's-eye view on the proof: Spectral density

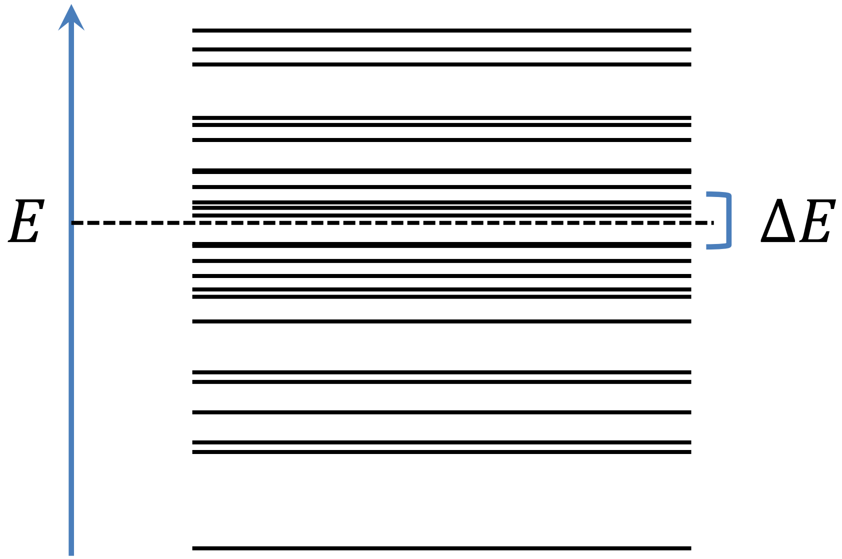


$$\Gamma(E) = \#\{E_i \leq E\}$$

$$\Omega(E) = \frac{d\Gamma(E)}{dE}$$

spectral density

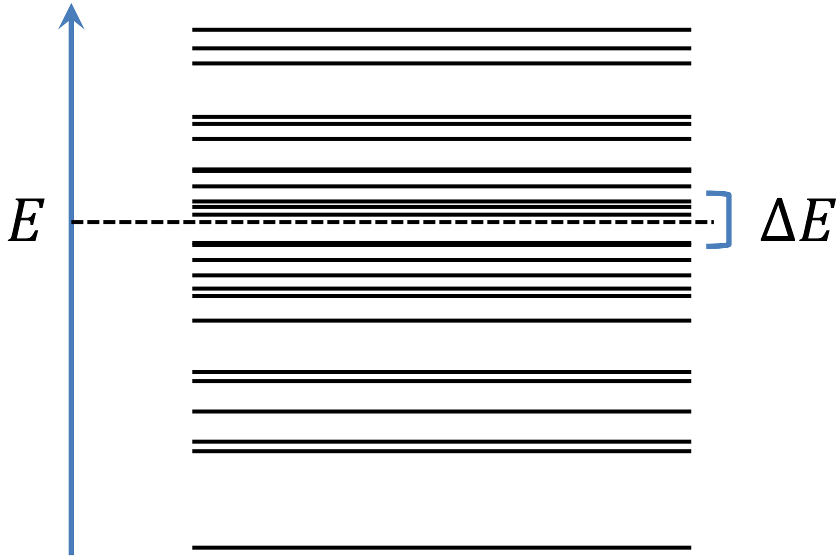
Bird's-eye view on the proof: Spectral density



$$\Gamma(E) = \#\{E_i \leq E\}$$

$$\Omega(E) = \frac{d\Gamma(E)}{dE} = \frac{\Delta N}{\Delta E}$$

Bird's-eye view on the proof: Spectral density



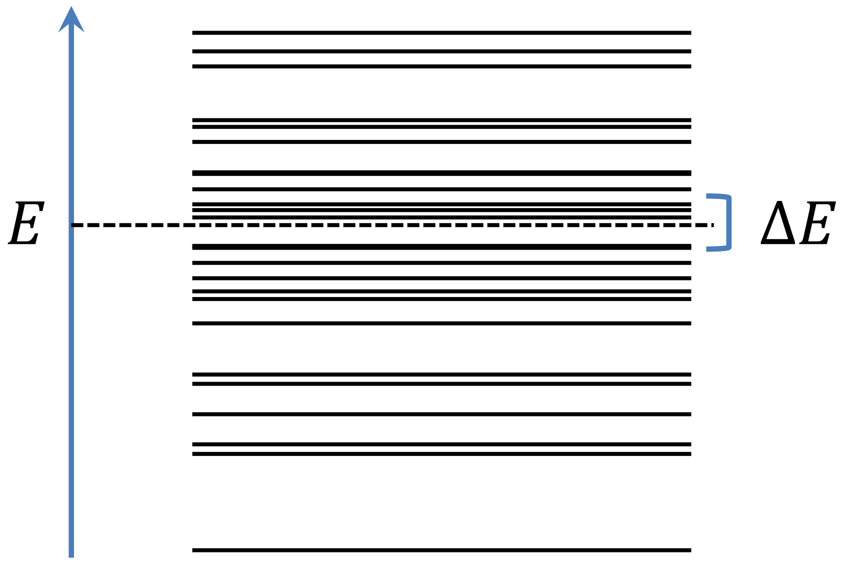
$$\Gamma(E) = \#\{E_i \leq E\}$$

$$\Omega(E) = \frac{d\Gamma(E)}{dE} = \frac{\Delta N}{\Delta E}$$

thermal state

$$\tau = \frac{e^{-\beta H}}{Z}$$

Bird's-eye view on the proof: Spectral density



$$\Gamma(E) = \#\{E_i \leq E\}$$

$$\Omega(E) = \frac{d\Gamma(E)}{dE} = \frac{\Delta N}{\Delta E}$$

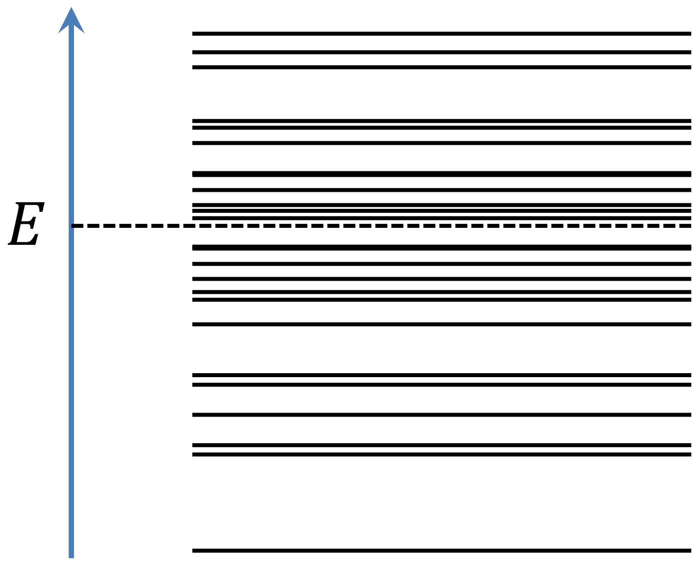
thermal state

$$\tau = \frac{e^{-\beta H}}{Z}$$

$$q(E) := \frac{dJ(E)}{dE} = \frac{e^{-\beta E}}{Z}\Omega(E)$$

Bird's-eye view on the proof: Equivalence of ensembles

Müller, Adlam, Masanes, & Wiebe, Commun. Math. Phys. **340**, 499 (2015)



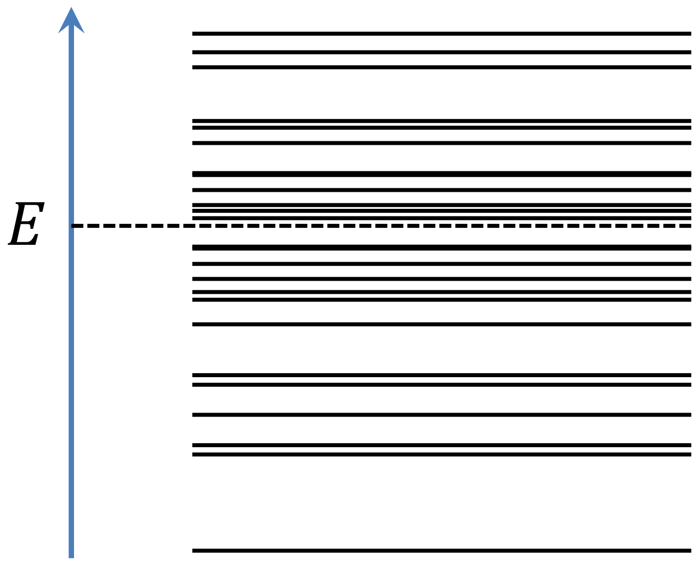
$$\Gamma(E) = \#\{E_i \leq E\}$$

$$\tau = \frac{e^{-\beta H}}{Z}$$

$$\lim_{N\to\infty} \frac{\ln \Gamma(uN)}{N} = s(u)$$

Bird's-eye view on the proof: Equivalence of ensembles

Müller, Adlam, Masanes, & Wiebe, Commun. Math. Phys. **340**, 499 (2015)



$$\Gamma(E) = \#\{E_i \leq E\}$$

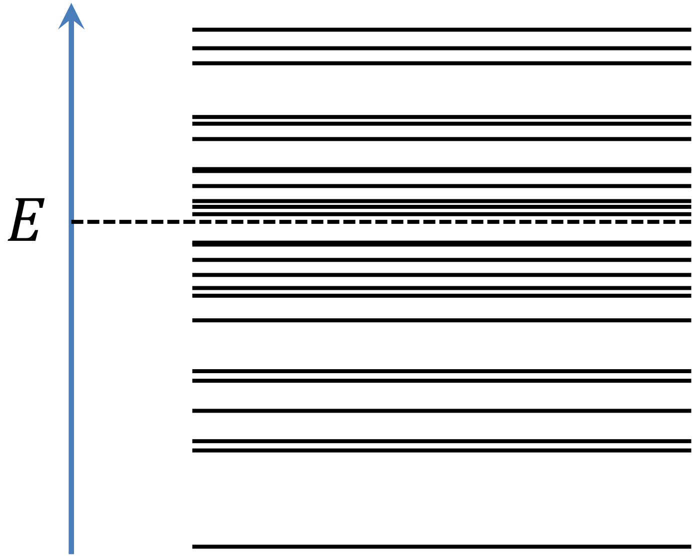
$$\tau = \frac{e^{-\beta H}}{Z}$$

$$\lim_{N\to\infty} \frac{\ln \Gamma(uN)}{N} = s(u)$$

holds at and above critical point, but not below.

Bird's-eye view on the proof: Equivalence of ensembles

Müller, Adlam, Masanes, & Wiebe, Commun. Math. Phys. **340**, 499 (2015)



$$\Gamma(E) = \#\{E_i \leq E\}$$

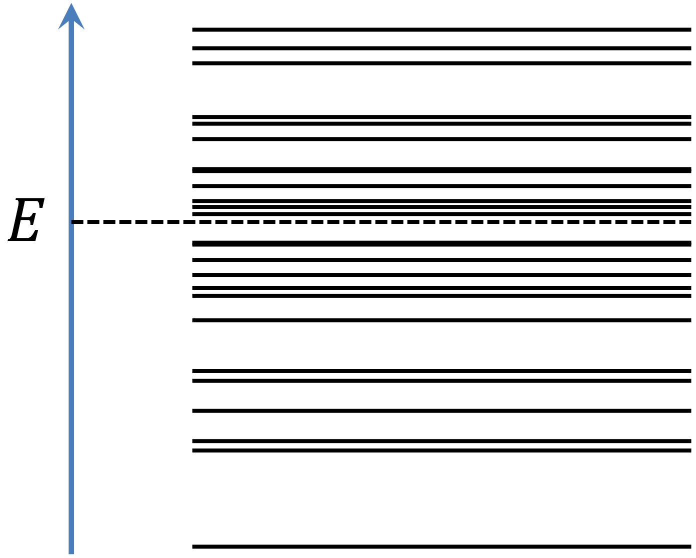
$$\tau = \frac{e^{-\beta H}}{Z}$$

$$\lim_{N\to\infty} \frac{\ln \Gamma(uN)}{N} = s(u)$$

energy density

Bird's-eye view on the proof: Equivalence of ensembles

Müller, Adlam, Masanes, & Wiebe, Commun. Math. Phys. **340**, 499 (2015)



$$\Gamma(E) = \#\{E_i \leq E\}$$

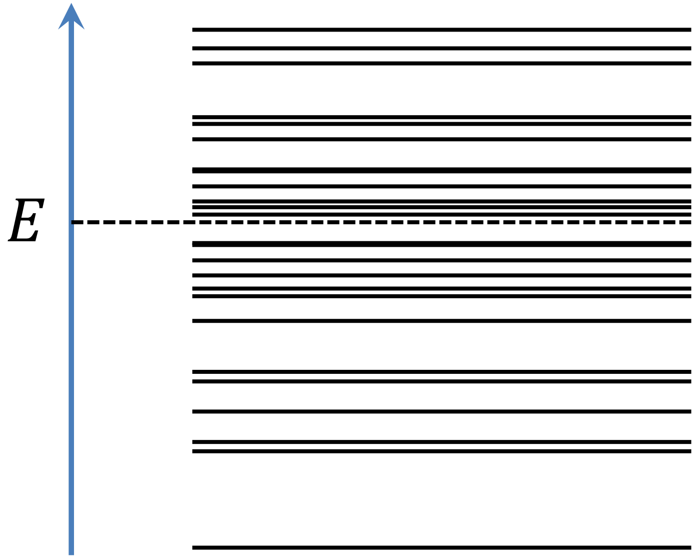
$$\tau = \frac{e^{-\beta H}}{Z}$$

$$\lim_{N\to\infty} \frac{\ln \Gamma(uN)}{N} = s(u)$$

density of canonical entropy corresponding to average energy density u

Bird's-eye view on the proof: Equivalence of ensembles

Müller, Adlam, Masanes, & Wiebe, Commun. Math. Phys. **340**, 499 (2015)



$$\Gamma(E) = \#\{E_i \leq E\}$$

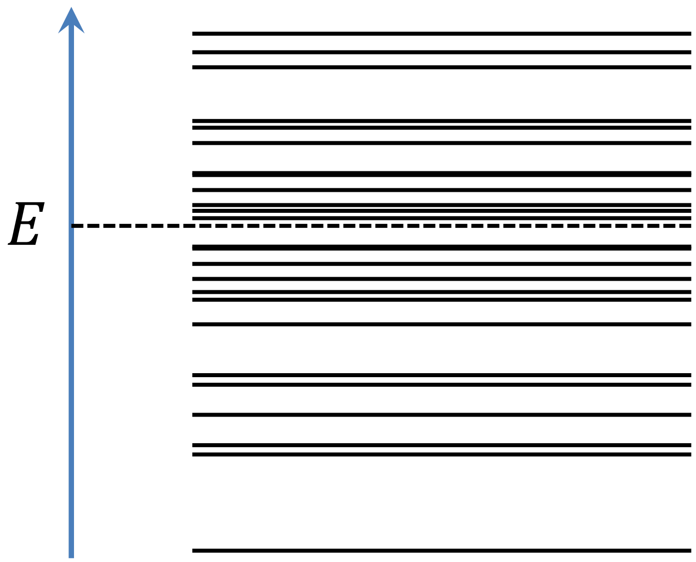
$$\tau = \frac{e^{-\beta H}}{Z}$$

$$\lim_{N\to\infty} \frac{\ln \Gamma(uN)}{N} = s(u)$$

same value, different things

Bird's-eye view on the proof: Equivalence of ensembles

Müller, Adlam, Masanes, & Wiebe, Commun. Math. Phys. **340**, 499 (2015)



$$\Gamma(E) = \#\{E_i \leq E\}$$

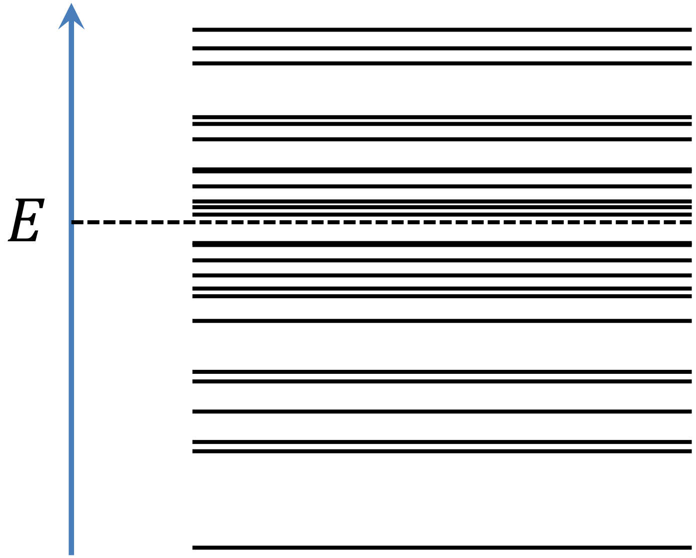
$$\tau = \frac{e^{-\beta H}}{Z}$$

$$\lim_{N\to\infty} \frac{\ln \Gamma(uN)}{N} = s(u)$$

$\approx$ microcanonical entropy density at u

Bird's-eye view on the proof: Equivalence of ensembles

Müller, Adlam, Masanes, & Wiebe, Commun. Math. Phys. **340**, 499 (2015)



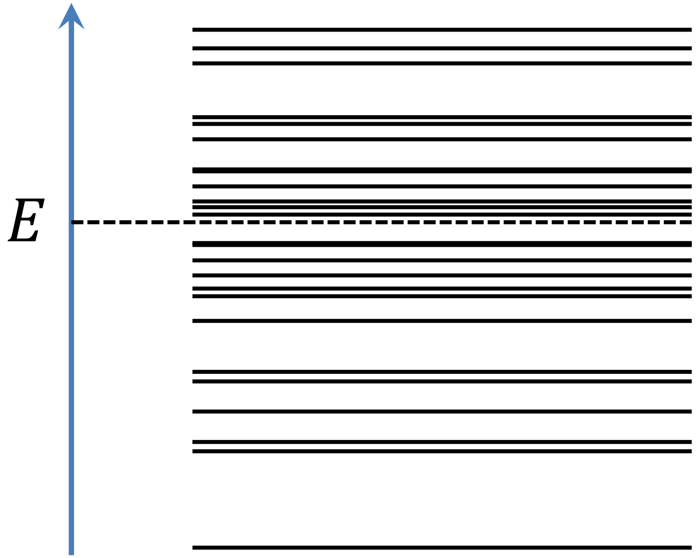
$$\Gamma(E) = \#\{E_i \leq E\}$$

$$\tau = \frac{e^{-\beta H}}{Z}$$

$$\lim_{N\to\infty} \frac{\ln \Gamma(uN)}{N} = s(u)$$

expresses the equivalence between canonical and microcanonical ensembles

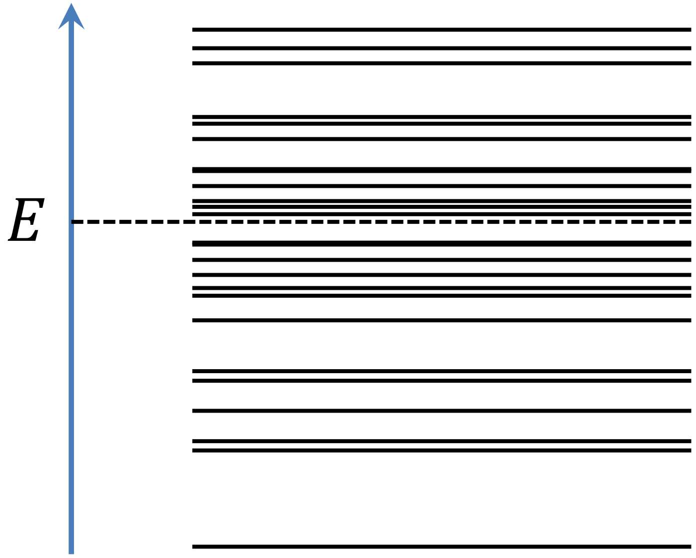
Bird's-eye view on the proof: Energy distribution



$$q(E) \coloneqq \frac{dJ(E)}{dE} = \frac{e^{-\beta E}}{Z}\Omega(E)$$

$$q(uN) = e^{\beta N f_N(\beta) - \beta N[u - T s_N(u)] + O(1)}$$

Bird's-eye view on the proof: Energy distribution

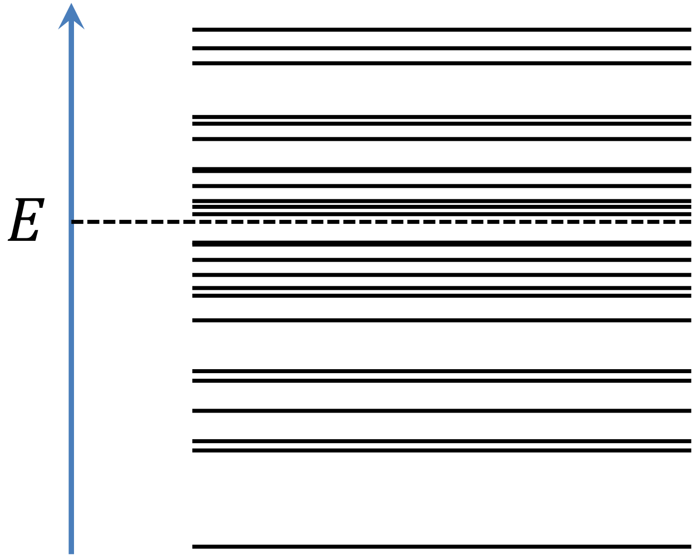


$$q(E) := \frac{dJ(E)}{dE} = \frac{e^{-\beta E}}{Z}\Omega(E)$$

$$q(uN) = e^{\beta N f_N(\beta) - \beta N[u - T s_N(u)] + O(1)}$$

$$\min_{\mathrm{u}}[u - T s_N(u)] = f_N(\beta)$$

Bird's-eye view on the proof: Energy distribution



$$q(E) \coloneqq \frac{dJ(E)}{dE} = \frac{e^{-\beta E}}{Z}\Omega(E)$$

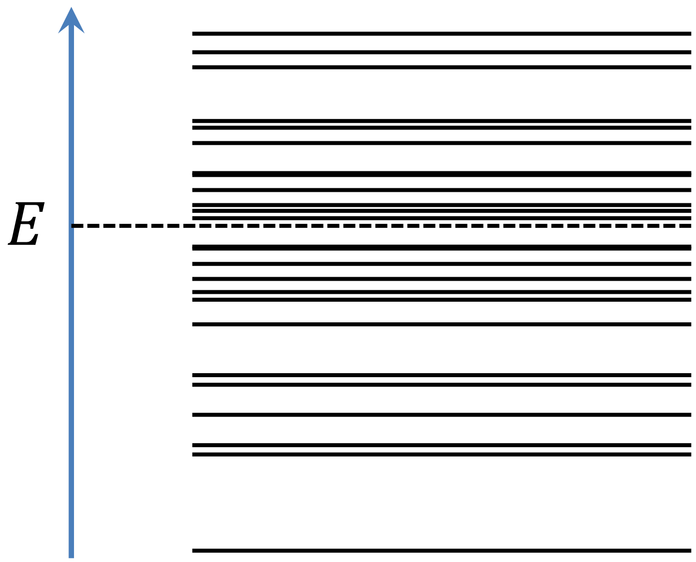
$$q(uN) = e^{\beta N f_N(\beta) - \beta N[u - T s_N(u)] + O(1)}$$

$$\min_{\mathrm{u}}[u - T s_N(u)] = f_N(\beta)$$

$$u - T s_N(u) = f_N(\beta) - \frac{\beta}{2 c_N(\beta)}[u - u_N(\beta)]^2 + \cdots$$

specific heat

Bird's-eye view on the proof: Energy distribution



$$q(E) := \frac{dJ(E)}{dE} = \frac{e^{-\beta E}}{Z}\Omega(E)$$

$$q(uN) = e^{\beta N f_N(\beta) - \beta N[u - T s_N(u)] + O(1)}$$

$$\min_{\mathrm{u}}[u - T s_N(u)] = f_N(\beta)$$

$$u - T s_N(u) = f_N(\beta) - \frac{\beta}{2c_N(\beta)}[u - u_N(\beta)]^2 + \cdots$$

canonic energy density

Critical exponents

$$c(\beta) \propto |\beta - \beta_c|^{-\alpha}$$

The scaling of $c(\beta_c)$ with N:

When $\alpha = 0$,

$$c(\beta) \propto \ln N$$

When $\alpha > 0$,

$$c(\beta) \propto N^{\alpha/(2-\alpha)}$$

Berry-Esseen analogue for $\alpha = 0$

Whenever

$$\frac{|E - \langle H \rangle|}{\sqrt{\mathrm{var}(H)}} \leq \sqrt{\ln N}$$

$$q(E) \propto e^{-\frac{(E - \langle H \rangle)^2}{2\mathrm{var}(H)} + O\left(\frac{1}{\ln N}\right)}$$

Berry-Esseen analogue for $\alpha = 0$

Whenever

$$\frac{|E - \langle H \rangle|}{\sqrt{\mathrm{var}(H)}} \leq \sqrt{\ln N}$$

$$q(E) \propto e^{-\frac{(E-\langle H \rangle)^2}{2\mathrm{var}(H)} + O\left(\frac{1}{\ln N}\right)}$$

Whenever

$$\frac{|E - \langle H \rangle|}{\sqrt{\mathrm{var}(H)}} > \sqrt{\ln N}$$

$$\int_{|E-\langle H \rangle| > \sqrt{N \mathrm{var}(H)}} dE\, q(E) \propto \frac{1}{N}$$

Berry-Esseen analogue for $\alpha = 0$

2D Ising model is a paradigmatic example of $\alpha = 0$.

Its free energy can be computed exactly, and therefore one has access to all moments of the energy.

$$\frac{|\kappa_n|^{1/n}}{\sqrt{\mathrm{var}(H)}} \propto \frac{1}{\sqrt{\ln N}}$$

Berry-Esseen analogue for $\alpha = 0$

2D Ising model is a paradigmatic example of $\alpha = 0$.

Its free energy can be computed exactly, and therefore one has access to all moments of the energy.

$$\frac{|\kappa_n|^{1/n}}{\sqrt{\mathrm{var}(H)}} \propto \frac{1}{\sqrt{\ln N}}$$

n-th cumulant of energy

Berry-Esseen analogue for $\alpha > 0$

Not much can be said except that $q(E)$ is a unimodal distribution peaked around $\langle H \rangle$ and decaying exponentially in the tails.

Thank you for your attention!