Entanglement and criticality in the extended Bose-Hubbard model of cavity quantum electrodynamics

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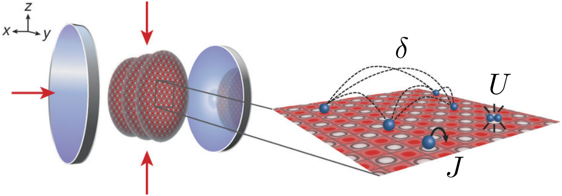
Tuesday 23rd March, 2021

Model: Cavity induced Long-range interacting system



ultracold bosons in an optical lattice (λ_0)

$+$

cavity with a standing wave mode (λ)

R. Landig et. al, Nature 532, 476-479 (2016)

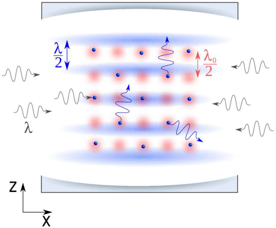
- In specific limit of time: Adiabatic elimination of cavity

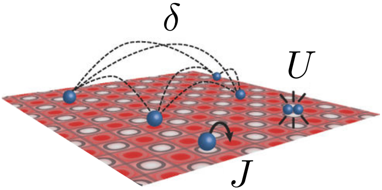
$$H_{LRI} = H_{\rm BH} + H_{\rm laser} + H_{\rm cavity}$$

$H_{\rm BH}$	$H_{\rm laser}$	$H_{\rm cavity}$
Bose-Hubbard Hamilt.	Pot. term due to laser	Long-range pot. mediated by cavity photons interacting with atoms

Cavity induced Long-range interacting model

Astrid E. Niederle, Phys. Rev. A 94, 033607 (2016)





$$H_{LRI} = H_{\rm BH} + H_{\rm laser} + H_{\rm cavity}$$

- $H_{\rm BH} = -t\sum_{\langle i,j\rangle}(\hat{b}_i^\dagger \hat{b}_j + {\rm h.c.}) + \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \mu\sum_i \hat{n}_i$
- $H_{\rm laser} = V_1\sum_i J_0^{(i)} n_i, \qquad J_0^{(i)} = \int d\mathbf{r}\, w_i(\mathbf{r})\cos^2(k\,x)\, w_i(\mathbf{r})$
- $H_{\rm cavity} = -\delta L^2\hat{\Phi}^2;$

$$\hat{\Phi} = \frac{1}{L^2}\sum_i Z_0^i \hat{n}_i; \quad Z_0^i = \int d\mathbf{r}\, w_i(\mathbf{r})\cos(k\,z)\cos(k\,x)\, w_i(\mathbf{r})$$

$$\hat{H}_{\rm LRI} = -\sum_{\langle i,j\rangle} J(\hat{b}_i^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{b}_i) + \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) + \sum_i (V_1 J_0^i - \mu)\hat{n}_i - \delta(\sum_i Z_i \hat{n}_i)^2$$

- Dynamic potential introduces correlated pattern:

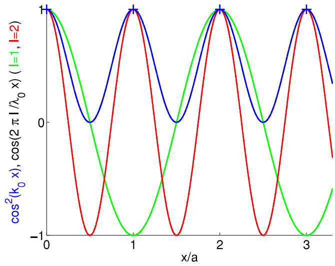
$$\cos(k\,x) = \cos(k\,i_x a) = \cos(i_x \pi \frac{\lambda_0}{\lambda}); \quad \cos(k\,z) = \cos(k\,i_z a) = \cos(i_z \pi \frac{\lambda_0}{\lambda})$$

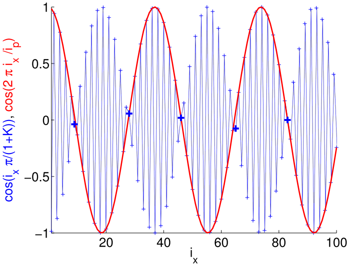
Commensurate Lattice:
$\lambda = \lambda_0/\ell, \quad \ell = 1, 2 \cdots$

- checkerboard pattern

Incommensurate Lattice:
$\lambda = \lambda_0/\ell + \epsilon_\lambda, \quad \ell = 1, 2 \cdots$

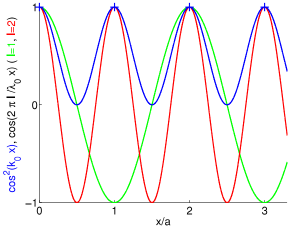
- quasi-perodic potential





$$\hat{H}_{\rm LRI} = -\sum_{\langle i,j\rangle} t(\hat{b}_i^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{b}_i) + \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) + \sum_i \hat{\epsilon}_i \hat{n}_i$$

Commensurate Lattice:
$\lambda = \lambda_0/\ell, \quad \ell = 1, 2\cdots$

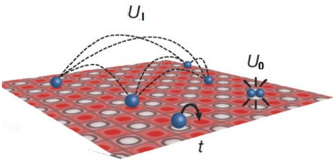


- leads to a checkerboard pattern
- **Phases**: Charge-density wave, Supersolid, Mott-insulator, Superfluid

Incommensurate Lattice:
$\lambda = \lambda_0/\ell + \epsilon_\lambda, \quad \ell = 1, 2\cdots$

- leads to a quasi-periodic potential
- **Phases**: Bose-glass, Super-glass, Mott-insulator, Superfluid

Model Hamiltonian (2D)



R. Landig et. al, Nature (2016))

$$H_{LRI} = H_{\mathrm{BH}} + H_{\mathrm{cavity}}$$

$$H_{\mathrm{BH}} = -t\sum_{\langle i,j\rangle}(\hat{b}_i^\dagger \hat{b}_j + \mathrm{h.c.}) + \frac{U_0}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \mu\sum_i \hat{n}_i$$

$$H_{\mathrm{cavity}} = -U_l L^2 \hat{\Phi}^2$$

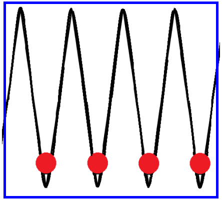
$\hookrightarrow \hat{\Phi} = \frac{1}{L^2}\sum_i (-1)^i \hat{n}_i \qquad \Rightarrow \lambda/\lambda_0 =$odd-integer

$\hookrightarrow$ Cavity output photon number, $\langle \hat{n}_{photon}\rangle \sim \langle \hat{\Phi}^2 \rangle$

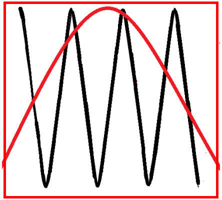
$$\hat{H} = -\sum_{\langle \mathbf{i},\mathbf{j}\rangle} t(\hat{b}_\mathbf{i}^\dagger \hat{b}_\mathbf{j} + \hat{b}_\mathbf{j}^\dagger \hat{b}_\mathbf{i}) + \frac{U_0}{2}\sum_\mathbf{i} \hat{n}_\mathbf{i}(\hat{n}_i - 1) - \mu\sum_\mathbf{i} \hat{n}_i - \frac{U_l}{L^2}\left(\sum_{\mathbf{i}\in\mathbf{e}} \hat{n}_\mathbf{i} - \sum_{\mathbf{i}\in\mathbf{o}} \hat{n}_\mathbf{i}\right)^2$$

Expected phases of the model

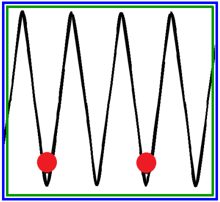
$$\hat{H} = -\underbrace{\sum_{\langle \mathbf{i},\mathbf{j}\rangle} t(\hat{b}_{\mathbf{i}}^{\dagger}\hat{b}_{\mathbf{j}} + \hat{b}_{\mathbf{j}}^{\dagger}\hat{b}_{\mathbf{i}})}_{} + \underbrace{\frac{U_0}{2}\sum_{\mathbf{i}} \hat{n}_{\mathbf{i}}(\hat{n}_i - 1)}_{} - \mu \sum_{\mathbf{i}} \hat{n}_i - \underbrace{\frac{U_l}{L^2}\left(\sum_{\mathbf{i}\in\mathbf{e}} \hat{n}_{\mathbf{i}} - \sum_{\mathbf{i}\in\mathbf{o}} \hat{n}_{\mathbf{i}}\right)^2}_{}$$



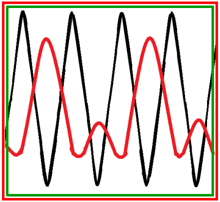
Mott-Insulator (MI)



Superfluid (SF)



Charge-Density Wave (CDW)



Super-solid (SS)

Mean-field Analysis:

$$\hat{H} = -\sum_{\langle \mathbf{i},\mathbf{j}\rangle} t(\hat{b}_{\mathbf{i}}^{\dagger}\hat{b}_{\mathbf{j}} + \hat{b}_{\mathbf{j}}^{\dagger}\hat{b}_{\mathbf{i}}) + \frac{U_0}{2}\sum_{\mathbf{i}} \hat{n}_{\mathbf{i}}(\hat{n}_i - 1) - \mu \sum_{\mathbf{i}} \hat{n}_i - \frac{U_l}{L^2}\Big(\sum_{\mathbf{i}\in\mathbf{e}} \hat{n}_{\mathbf{i}} - \sum_{i\in o} \hat{n}_{\mathbf{i}}\Big)^2$$

- $b_i^{\dagger} b_j \approx b_i^{\dagger}\langle b_j\rangle + \langle b_i^{\dagger}\rangle b_j - \langle b_i^{\dagger}\rangle\langle b_j\rangle$
  $\Rightarrow$ $\boxed{\varphi_i = \langle b_i\rangle}$ **SF order-parameter**

- $\hat{\Phi} = \frac{1}{L^2}\sum_i (-1)^i \hat{n}_i \Rightarrow \hat{\Phi}^2 \approx 2\langle\hat{\Phi}\rangle\hat{\Phi} - \langle\hat{\Phi}\rangle^2$
  $\rightarrow 2\langle\hat{\Phi}\rangle = \Theta \quad \Rightarrow \boxed{\Theta/2}$ **Order-parameter LRI**

$$H_{s\in\{\mathrm{e},\mathrm{o}\}} = -zt\varphi_{\bar{s}}(\hat{b}_s + \hat{b}_s^{\dagger} - \varphi_s) + \frac{U_0}{2}\hat{n}_s(\hat{n}_s - 1) - \mu\hat{n}_s - \sigma_s U_l \Theta \hat{n}_s + U_l\Theta^2/4$$

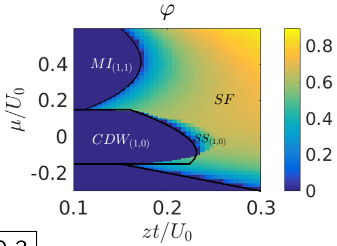
$\bar{\varphi}_s = z\varphi_{\bar{s}}$; $z = 4$; $\sigma_{\mathrm{e}} = +1$, $\sigma_{\mathrm{o}} = -1$

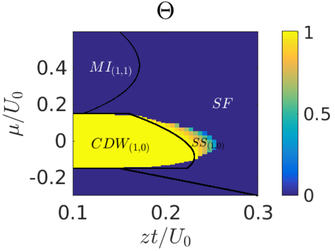
Phase	φ	$\lvert\Theta\rvert$
MI	0	0
SF	$\neq 0$	0
CDW	0	$\neq 0$
SS	$\neq 0$	$\neq 0$

Mean field results:

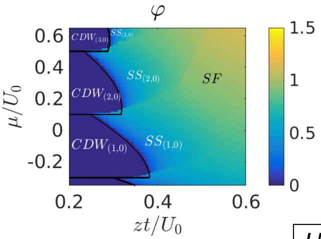
$$H_{s\in\{e,o\}} = -zt\varphi_{\bar{s}}(\hat{b}_s + \hat{b}_s^\dagger - \varphi_s) + \frac{U_0}{2}\hat{n}_s(\hat{n}_s - 1) - \mu\hat{n}_s - \sigma_s U_l \Theta \hat{n}_s + U_l \Theta^2/4$$

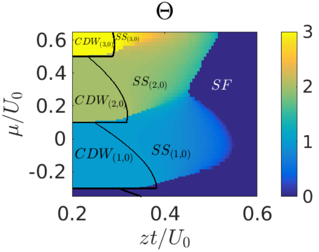
$U_l = 0.3$





$U_l = 0.6$

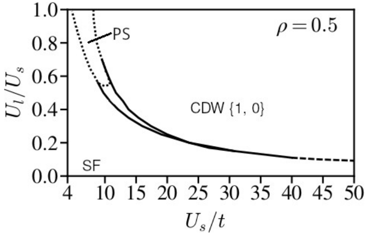


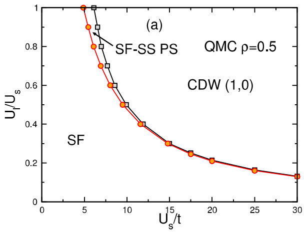


L. Himbert et. al Phys Rev A 99, 043633(2019)

Validity of Mean field results

Ground state phase diagram for fixed density ($\rho = 0.5$)





⇒ **MF results: L. Himbert et. al Phys Rev A 99, 043633(2019)**

⇒ **Quantum Monte Carlo result: T. Flottat et al., PRB 95, 144501 (2017)**

- Agrees qualitatively
- Transition boundaries may change at higher densities
- The type of phase transitions and the phases match well in most of the parameter space.

- **What is the behavior of ENTANGLEMANT ENTROPY in these phases?**

Mean Field with slave-boson approach

$$H^i_{\mathrm{MF}} = -zt\varphi_{\langle j\rangle}(\hat{b}_i + \hat{b}_i^\dagger - \varphi_i) + \frac{1}{2}\hat{n}_i(\hat{n}_i - 1) - \mu\hat{n}_i - U_l\Theta(-1)^i\hat{n}_i + KU_l\frac{\Theta^2}{4},$$

$$\Downarrow$$

NEW BASIS: $|\Psi_\alpha\rangle$, with $\alpha = 0, \cdots, n_{max}$

$$\hat{H}_{\mathrm{LRI}} = -\sum_{\langle i,j\rangle} t(\hat{b}_i^\dagger\hat{b}_j + \hat{b}_j^\dagger\hat{b}_i) + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \mu n_i - U_l L^2(\sum_i \Phi_i)^2$$

$\Rightarrow$ **Rotation to new basis**

$$\tilde{H}_{\mathrm{LRI}} = -\sum_{\langle i,j\rangle} t(\gamma_i^\dagger F_i\gamma_i\gamma_j^\dagger \tilde{F}_j^\dagger\gamma_j + h.c.) + \sum_i \gamma_i^\dagger\tilde{G}_i\gamma_i + \sum_{i,j} L^2\delta\gamma_i^\dagger\hat{\Phi}_i\gamma_i\gamma_j^\dagger\Phi_j\gamma_j$$

$$\gamma_{i,\alpha} = \sum_{n=0}^{n_{max}} \langle\Psi_\alpha|n_i\rangle|0\rangle\langle n_i|$$

$$F_i{}^{\alpha,\beta} = \langle\Psi_i{}^\alpha|b_i^\dagger|\Psi_i{}^\beta\rangle$$

$$G_i{}^{\alpha,\beta} = \langle\Psi_i{}^\alpha|\tfrac{1}{2}\hat{n}_i(\hat{n}_i - 1) - \mu\hat{n}_i|\Psi_i{}^\beta\rangle$$

$\Phi_i{}^{\alpha,\beta} = \langle\Psi_i{}^\alpha|\hat{\Phi}_i|\Psi_i{}^\beta\rangle = \Phi_{i,00}$ $\rightarrow$ on MF level!

• **Expand about the Mean-Field ground state** $\rightarrow$ Quadratic Hamiltonian

$$f(x) = f(x_0) + f'(x)|_{x=x_0}x + \tfrac{1}{2}f''(x)|_{x=x_0}x^2 + \cdots$$

Phys. Rev. Lett. **57**, 1362 (1986); Phys. Rev. A **68**, 043623 (2003); Phys. Rev. B **75**, 085106 (2007);Phys. Rev. Lett. **116**, 190401 (2016).

von Neumann Entanglement entropy

- **Quadratic Hamiltonian**

$$H = \sum_{i,j} \left[c_i^\dagger A_{i,j} c_j + \frac{1}{2}(c_i^\dagger B_{i,j} c_j^\dagger + H.c.) \right] + \frac{1}{2} \sum_i A_{i,i}$$

- **Correlation matrix** $\rightarrow$ one-body correlations $\langle c_i^\dagger c_j \rangle$ and $\langle c_i c_j \rangle$

- Divide the system in 2 parts and ask how much one part is correlated with other?
- **Correlation matrix of sub-system A** (avarage over whole system's ground state but $(i,j) \in A$)

$$r_A = \begin{pmatrix} \langle c_i^\dagger c_j \rangle_{i,j=1,N_A} & \langle c_i^\dagger c_j^\dagger \rangle_{i,j=1,N_A} \\ \langle c_i c_j \rangle_{i,j=1,N_A} & \langle c_i c_j^\dagger \rangle_{i,j=1,N_A} \end{pmatrix}$$

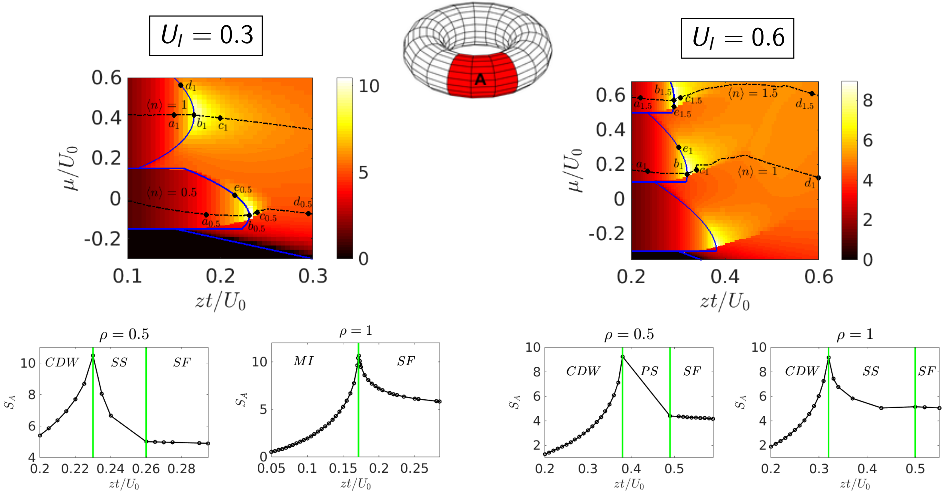
$$\Downarrow$$

$$\lambda_i$$

Entanglement Entropy

$$S_{\mathrm{A}} = \sum_i (1+\lambda_i) \log(1+\lambda_i) - \lambda_i \log \lambda_i$$

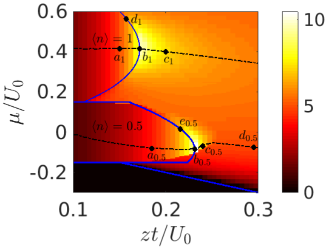
Entanglement entropy



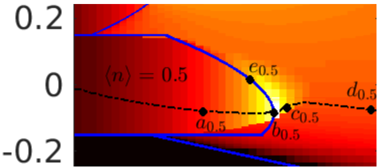
- ALWAYS A PEAK AT FIXED DENSITY!
- Appears at transition from INCOMPRESSIBLE PHASE TO COMPRESSIBLE PHASE!

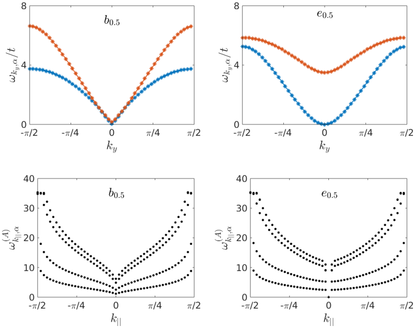
Irénée Frérot, Tommaso Roscilde, Phys. Rev. Lett. 116, 190401 (2016).

$U_I = 0.3$, $\langle n \rangle = 0.5$: Physical and entanglement spectra



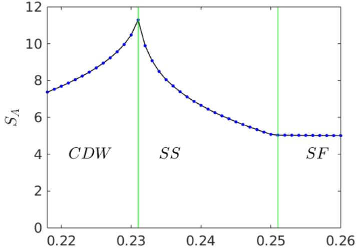
$\Rightarrow$

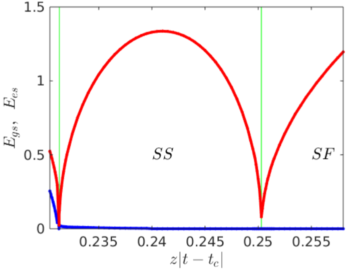


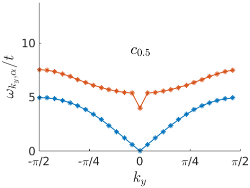


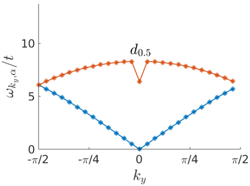
- In compressible phases higher order of entanglement modes contribute leading to non-zero (/higher value of) EE.
- Constant density phase transition tips are special points.
- At Tips: both amplitude and phase mode goes to zero.
- Tips ($b_{0.5}$): belong to different universality class than $e_{0.5}$.
- Point $b_{0.5}$ has $z = 1$ and point $e_{0.5}$ has $z = 2$.

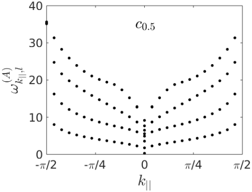
Long-range interaction: Roton mode ($U_l = 0.3$)

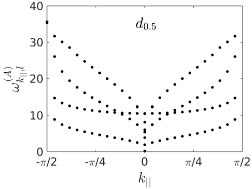




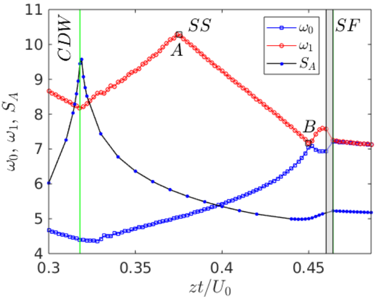


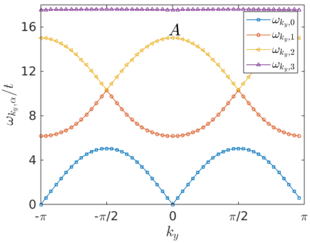


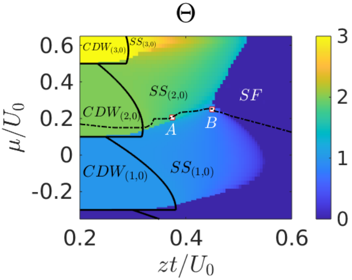


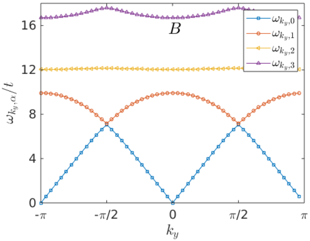


$U_l = 0.6$, $\rho = 1$: Physical spectrum ($k_x = 0$)

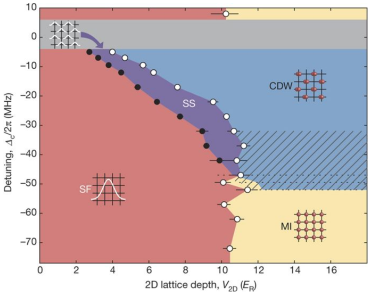


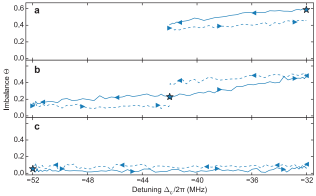




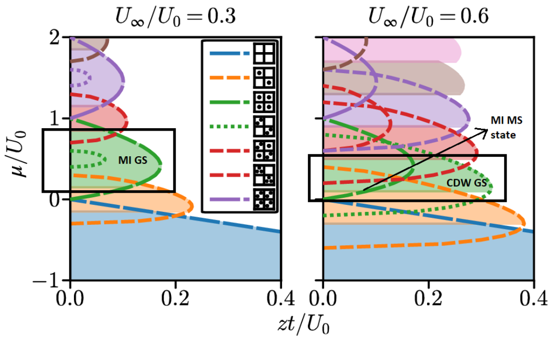


Metastable state in the incompressible regions



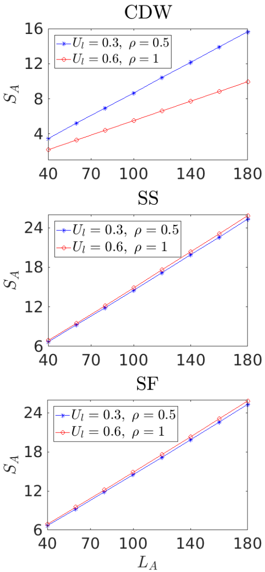


R. Landig et. al, Nature 532, 476-479 (2016)



MI metastable states in CDW region

Scaling: Entanglement Entropy



$$S_A = aL_A^{d-1} + b \log L_A + cL_A^d + const.$$

$U_l = 0.3, \ \rho = 0.5$			
	a	b	c
CDW	0.08686	-8.064×10^{-5}	-1.564×10^{-8}
SS	0.1151	0.541	5.484×10^{-5}
SF	0.07798	0.6368	5.994×10^{-5}
$U_l = 0.6, \ \rho = 1$			
	a	b	c
CDW	0.05525	-4.78×10^{-5}	-9.268×10^{-9}
SS	0.117	0.5705	5.641×10^{-5}
SF	0.1096	0.1874	-6.471×10^{-6}

- **Log scaling for fixed density** incompressible phases especially in Supersolid phase.
- Coefficient of Log term depends on number of number of Goldstone modes.
- The **Supersolid phase** (same as superfluid phase) shows an **area law scaling and not volume law scaling**

Conclusion and Outlook

- We have used a SB-MF approach to probe EE of a LRI BHM with phases CDW, SS, MI and SF.
- EE shows a peak at the fixed density in-compressible to compressible phase transition.
- This fixed density may or may not be an integer.
- These fixed density transition points are special where both the ground and first excited state goes linearly to zero.
- Fixed density in-compressible phases (SS and SF) have an additional log dependent scaling
- We observe Area-law scaling for SS phase using SB mean-field approach
- How EE would behave for the ICL case?

Tommaso Roscilde

Thank You