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L^2 extension theorems and applications to algebraic geometry

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General plan of the lectures

(1) First lecture: a general qualitative extension theorem
– Setup and general statement
– Main ideas of the proof

(2) Second lecture: extension with optimal L^2 estimates
– Ohsawa residual measure
– Log canonical case, case of higher order jets
– Main L^2 estimate; solution of the Suita conjecture
– Approximation of quasi-psh functions and currents

(3) Third lecture: applications
– Solution of the strong openness conjecture (Guan and Zhou)
– Pham's strong semicontinuity theorem
– Generalized Nadel vanishing theorem by Junyan Cao
– Hard Lefschetz theorem with psef coefficients
(and a complement by Xiaojun Wu)

First lecture

First lecture: notation and main concepts

Let (X, ω) be a possibly noncompact n-dimensional Kähler manifold, and L a holomorphic line bundle on X, with a possibly singular hermitian metric $h = e^{-\varphi}$, $\varphi \in L^1_{\rm loc}$. The curvature current is

$$\Theta_{L,h} = i\,\partial\overline{\partial} \log h^{-1} = i\partial\overline{\partial}\varphi$$

computed in the sense of distributions.
Very often, one needs positivity assumptions for L.

Definition

- L is positive if $\exists h \in C^\infty$ such that $\Theta_{L,h} > 0$ ($\Leftrightarrow$ L ample);
- L is nef if $\forall \varepsilon > 0$, $\exists h_\varepsilon \in C^\infty$ such that $\Theta_{L,h_\varepsilon} \geq -\varepsilon\omega$;
- L is pseudoeffective (psef) if $\exists h$ singular such that $\Theta_{L,h} \geq 0$.

Now, let $\mathcal{J} \subset \mathcal{O}_X$ a coherent ideal sheaf, $Y = V(\mathcal{J})$ its zero variety and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$. Here Y may be non reduced, i.e. $\mathcal{O}_Y$ may have nilpotent elements.

The extension problem

Consider the exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to 0.$$

By twisting with $\mathcal{O}_X(K_X \otimes L)$, where $K_X = \Lambda^n T_X^*$, one gets the long exact sequence of cohomology groups

$$\cdots \to H^q(X, K_X \otimes L) \to H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X/\mathcal{J})$$
$$\to H^{q+1}(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{J}) \cdots$$

Surjectivity / extension problem

Under which conditions on X, $Y = V(\mathcal{J})$ and (L, h) is

$$H^q(X, K_X \otimes L) \to H^q(Y, (K_X \otimes L)_{|Y}) = H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X/\mathcal{J})$$

a surjective restriction morphism?

Equivalent injectivity problem

When is $H^{q+1}(X, K_X \otimes L \otimes \mathcal{J}) \to H^{q+1}(X, K_X \otimes L)$ injective ?

Multiplier ideal sheaves

Given a hermitian metric $h = e^{-\varphi}$ with φ quasi-psh (i.e. such that $\varphi = \text{psh} + C^\infty$), one defines the associated multiplier ideal sheaf $\mathcal{I}(h) = \mathcal{I}(e^{-\varphi}) \subset \mathcal{O}_X$ by

$$\mathcal{I}(e^{-\varphi})_{x_0} = \Big\{ f \in \mathcal{O}_{X,x_0} \,;\, \exists U \ni x_0\,,\ \int_U |f|^2 e^{-\varphi} d\lambda < +\infty \Big\}$$

Theorem (Nadel)

$\mathcal{I}(e^{-\varphi})$ is a coherent ideal sheaf.
Moreover, $\mathcal{I}(e^{-\varphi})$ is always integrally closed.

One says that a quasi-psh function φ has *analytic singularities*, i.e. locally on a neighborhood V of an arbitrary point $x_0 \in X$ we have

$$\varphi(z) = c \log \sum |g_j(z)|^2 + u(z),\quad g_j \in \mathcal{O}_X(V),\ c > 0,\ u \in C^\infty(V),$$

Example: $\varphi(z) = c \log |s(z)|^2_{h_E},\ c > 0,\ s \in H^0(X, E),\ h_E \in C^\infty.$

Nadel vanishing theorem

Theorem (Nadel vanishing theorem)

Let (X,ω) be a Kähler manifold that is weakly pseudoconvex, i.e. X admits a smooth psh exhaustion γ. Let $L \to X$ be a holomorphic line bundle equipped with a singular hermitian metric h such that

$$\Theta_{L,h} \geq \alpha\omega, \quad \alpha \text{ continuous} > 0 \text{ function.}$$

Then $H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0$ for $q \geq 1$.

Corollary

Assume instead that $(*)$ $\Theta_{L,h} + i\partial\overline{\partial}\psi \geq \alpha\omega$ for some quasi-psh function ψ on X. Then $H^q(X, K_X \otimes L \otimes \mathcal{I}(he^{-\psi})) = 0$ for $q \geq 1$, and for all $q \geq 0$, we have a surjective restriction morphism

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes L \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})).$$

Proof. $0 \to \mathcal{I}(he^{-\psi}) \to \mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})) \to 0$.

However, one would like to relax the strict positivity assumption $(*)$.

Motivation: abundance conjecture and MMP

One potential application would be to study the Minimal Model Program (MMP) for arbitrary projective – or even Kähler – varieties, whereas only the case of general type varieties is known.

For a line bundle L, one defines the Kodaira-Iitaka dimension $\kappa(L) = \limsup_{m\to+\infty} \log\dim H^0(X, L^{\otimes m})/\log m$ and the numerical dimension $\mathrm{nd}(L) =$ maximum exponent p of non zero "positive intersections" $\langle T^p\rangle$ of a positive current $T \in c_1(L)$ when L is psef (pseudoeffective), and $\mathrm{nd}(L) = -\infty$ otherwise. They always satisfy

$$-\infty \leq \kappa(L) \leq \mathrm{nd}(L) \leq n = \dim X.$$

Definition (abundance)

A line bundle L is said to be abundant if $\kappa(L) = \mathrm{nd}(L)$.

The fundamental abundance conjecture can be stated: for each nonsingular klt pair (X,Δ) the $\mathbb{Q}$-line bundle $K_X + \Delta$ is abundant.

Generalized base point free theorem ?

One can try to investigate the abundance of $L = K_X + \Delta$ by induction on the dimension $n = \dim X$, by extending sections of $K_X + L_m$, $L_m = (m-1)K_X + m\Delta$ from subvarieties (noticing that $K_X + \Delta$ psef implies L_m psef, and even $L_m - \Delta$ psef). Cf. BCHM and recent work of D-Hacon-Păun, Fujino, Gongyo, Takayama.

Standard base point free theorem

Let (X, Δ) be a projective klt pair, and L be a nef line bundle such that $L - (K_X + \Delta)$ is nef and big. Then L is semiample, i.e. $|mL|$ is base point free for some $m > 0$.

Question (weak positivity variant of the BPF property ?)

Assume that X is not uniruled, i.e. that K_X is pseudoeffective, and let L be a line bundle such that $L - \varepsilon K_X$ is pseudoeffective for some $0 < \varepsilon \ll 1$. Does there exist $G \in \mathrm{Pic}^0(X)$ such that $L + G$ is abundant ?

General (qualitative) extension theorem

The following very general statement was recently obtained by X. Zhou-L. Zhu as the culmination of many previous works: Ohsawa-Takegoshi, Ohsawa, ..., D, Cao-D-Matsumura (2017).

General qualitative extension theorem

Let (X, ω) be Kähler holomorphically convex, L a holomorphic line bundle with a hermitian metric $h = h_0 e^{-\varphi}$, $h_0 \in C^\infty$, φ quasi-psh on X, and $\psi \in L^1_{\mathrm{loc}}(X)$. Assume $\exists \alpha > 0$ continuous such that

$$\Theta_{L,h} + (1 + \nu\alpha) i\partial\bar\partial\psi \geq 0 \quad \text{on } X, \quad \nu = 0, 1.$$

Then, for all $q \geq 0$, the following restriction map is surjective:

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes L \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})).$$

Remark. Here $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})$ is supported on the subvariety $(Y, \mathcal{O}_Y)$ where $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}_Y$ and $\mathcal{J}_Y$ is the conductor ideal:

$$\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h) \underset{\mathrm{def}}{=} \{f \in \mathcal{O}_X \,;\, f \cdot \mathcal{I}(h) \subset \mathcal{I}(he^{-\psi})\}.$$

Simple algebraic corollary

Assume that X is projective (or $\exists$ projective morphism $X \to S$ over S affine algebraic). Let $Y = \sum m_j Y_j$ be a simple normal crossing divisor, and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{O}_X(-Y)$. Then $\mathcal{O}_X(-Y) = \mathcal{I}(\psi)$ with

$$\psi(z) = \sum c_j \log |\sigma_{Y_j}|^2_{h_j}, \quad c_j > 0 \text{ such that } \lfloor c_j \rfloor = m_j,$$

for any choice of smooth hermitian metrics h_j on $\mathcal{O}_X(Y_j)$.
We have $i\partial\overline{\partial}\psi = \sum c_j(2\pi[Y_j] - \Theta_{\mathcal{O}(Y_j),h_j})$.

Corollary

Assume $\exists\,(G_\nu)_{\nu=0,1}$ semiample $\mathbb{Q}$-divisors such that

$$(**) \qquad L - (1+\nu\alpha)\sum c_j Y_j \equiv G_\nu \mod \mathrm{Pic}^0(X), \quad c_j > 0, \ \alpha > 0.$$

Then, for $Y = \sum m_j Y_j$, $m_j = \lfloor c_j \rfloor$, there is a surjective morphism

$$H^q(X, K_X \otimes L) \twoheadrightarrow H^q(Y, (K_X \otimes L)_{|Y}).$$

The case where ψ has analytic singularities can in fact always be reduced to the divisorial case by blowing up.

(1) Qualitatively, approximate solutions suffice

Assume X to be holomorphically convex. By the Cartan-Remmert theorem, this is the case iff X admits a proper holomorphic map $p : X \to S$ only a Stein complex space S.

Observation : cohomology is then always Hausdorff

Let X be a holomorphically convex complex space and $\mathcal{F}$ a coherent analytic sheaf over X. Then all cohomology groups $H^q(X, \mathcal{F})$ are Hausdorff with respect to their natural topology (local uniform convergence of holomorphic Čech cochains)

Proof. $H^q(X, \mathcal{F}) \simeq H^0(S, R^q p_* \mathcal{F})$ is a Fréchet space.
Consequence. Coboundary spaces are closed in cocycle spaces.

Corollary

To solve an equation $\overline{\partial} u = v$ on a holomorphically convex manifold X, it is enough to solve it approximately:

$$\overline{\partial} u_\varepsilon = v + w_\varepsilon, \qquad w_\varepsilon \to 0 \ \text{ as } \varepsilon \to 0.$$

(2) Twisted Bochner-Kodaira-Nakano inequality (Ohsawa-Takegoshi)

Let (X, ω) be a Kähler manifold and let $\eta,\ \lambda > 0$ be smooth functions on X.

For every compacted supported section $u \in \mathcal{C}^\infty_c(X, \Lambda^{p,q} T^*_X \otimes L)$ with values in a hermitian line bundle (L, h), one has

$$\|(\eta+\lambda)^{\frac12}\overline\partial^* u\|^2 + \|\eta^{\frac12}\overline\partial u\|^2 + \|\lambda^{\frac12}\partial u\|^2 + 2\|\lambda^{-\frac12}\partial\eta\wedge u\|^2 \geq \int_X \langle B^{p,q}_{L,h,\omega,\eta,\lambda} u, u\rangle dV_{X,\omega}$$

where $dV_{X,\omega} = \frac{1}{n!}\omega^n$ is the Kähler volume element and $B^{p,q}_{L,h,\omega,\eta,\lambda}$ is the Hermitian operator on $\Lambda^{p,q} T^*_X \otimes L$ such that

$$B^{p,q}_{L,h,\omega,\eta,\lambda} = [\eta\, i\Theta_L - i\,\partial\overline\partial\eta - i\lambda^{-1}\partial\eta\wedge\overline\partial\eta\ ,\ \Lambda_\omega].$$

In the sequel, we will apply this to the case of (n, q)-forms $(p = n)$, and choose $\eta, \lambda > 0$ so that $B^{p,q}_{L,h,\omega,\eta,\lambda}$ is ≥ 0 (or close).

(3) L^2 approximate solutions of $\overline\partial$-equations

L^2 existence theorem "with error term"

Let (X, ω) be a Kähler manifold possessing a complete Kähler metric let (E, h_E) be a Hermitian vector bundle over X. Assume that $B = B^{n,q}_{E,h,\omega,\eta,\lambda}$ satisfies $B + \varepsilon\,\mathrm{Id} > 0$ for some $\varepsilon > 0$ (so that B can be just semi-positive or slightly negative, e.g. $B \geq -\frac{\varepsilon}{2}\,\mathrm{Id}$).
Take a section $v \in L^2(X, \Lambda^{n,q} T^*_X \otimes E)$ such that $\overline\partial v = 0$ and

$$M(\varepsilon) := \int_X \langle (B+\varepsilon\,\mathrm{Id})^{-1} v, v\rangle\, dV_{X,\omega} < +\infty.$$

Then there exists an approximate solution $u_\varepsilon \in L^2(X, \Lambda^{n,q-1} T^*_X \otimes E)$ and a correction term $w_\varepsilon \in L^2(X, \Lambda^{n,q} T^*_X \otimes E)$ such that

$$\overline\partial u_\varepsilon = v + w_\varepsilon \quad \text{and}$$

$$\int_X (\eta+\lambda)^{-1}|u_\varepsilon|^2\, dV_{X,\omega} + \frac{1}{\varepsilon}\int_X |w_\varepsilon|^2\, dV_{X,\omega} \leq M(\varepsilon).$$

Moreover, notice that $\varepsilon\, M(\varepsilon)$ involves $\varepsilon(B+\varepsilon\,\mathrm{Id})^{-1} \leq 2\,\mathrm{Id}$.

(4) Represent cohomology classes as Čech cocycles

Every cohomology class in

$$H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$$

is represented by a holomorphic Čech q-cocycle with respect to a Stein covering $\mathcal{U} = (U_i)$, say $(c_{i_0\ldots i_q})$,

$$c_{i_0\ldots i_q} \in H^0(U_{i_0} \cap \ldots \cap U_{i_q}, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})).$$

By the standard sheaf theoretic isomorphism with Dolbeault cohomology, this class is represented by a smooth (n, q)-form

$$f = \sum_{i_0,\ldots,i_q} c_{i_0\ldots i_q}\, \xi_{i_0} \overline{\partial}\xi_{i_1} \wedge \ldots \overline{\partial}\xi_{i_q}$$

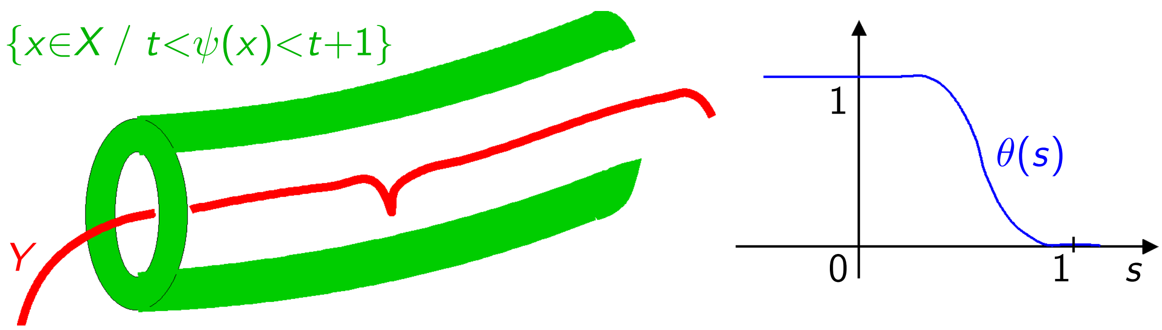
by means of a partition of unity (ξ_i) subordinate to (U_i). This form is to be interpreted as a form on the (non necessarily reduced) analytic subvariety Y associated with the conductor ideal sheaf $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

(5) Smooth lifting and associated $\overline{\partial}$ equation

We get an extension of f as a smooth (no longer $\overline{\partial}$-closed) (n, q)-form $\widetilde{f}$ on X by taking a lifting via $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$

$$\widetilde{f} = \sum_{i_0,\ldots,i_q} \widetilde{c}_{i_0\ldots i_q} \xi_{i_0} \overline{\partial}\xi_{i_1} \wedge \ldots \overline{\partial}\xi_{i_q},$$

where $\widetilde{c}_{i_0\ldots i_q} \in H^0(U_{i_0} \cap \ldots \cap U_{i_q}, K_X \otimes L \otimes \mathcal{I}(h))$.



Now, truncate $\widetilde{f}$ as $\theta(\psi - t)\cdot\widetilde{f}$ on the green hollow tubular neighborhood, and solve an approximate $\overline{\partial}$-equation

$$(*) \qquad \overline{\partial}u_{t,\varepsilon} = \overline{\partial}(\theta(\psi - t)\cdot\widetilde{f}) + w_{t,\varepsilon}, \quad 0 \le \theta \le 1, \ \ |\theta'| \le 1 + \varepsilon.$$

(6) L^2 bound and regularization of the metrics

Here we have

$$\overline{\partial}(\theta(\psi - t)\cdot \widetilde{f}) = \theta(\psi - t)\cdot \overline{\partial}\widetilde{f} + \theta'(\psi - t)\overline{\partial}\psi \wedge \widetilde{f}$$

where the second term vanishes near Y.

Moreover the image of $\overline{\partial}\widetilde{f}$ in $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})$ is $\overline{\partial}f = 0$, thus $\overline{\partial}\widetilde{f}$ has coefficients in $\mathcal{I}(he^{-\psi})$. Hence $\overline{\partial}\widetilde{f} \in L^2_{\rm loc}(he^{-\psi}) = L^2_{\rm loc}(h_0 e^{-\varphi-\psi})$.

Truncate $p : X \to S$ by taking $X' = p^{-1}(S')$, $S' \Subset S$ Stein. There are quasi-psh regularizations $\varphi_\delta \downarrow \varphi$, $\psi_\delta \downarrow \psi$ with analytic singularities, smooth on $X' \smallsetminus Z_\delta$, Z_δ analytic, and a complete Kähler metric ω_δ on $X' \smallsetminus Z_\delta$ such that

$$\int_{X'\smallsetminus Z_\delta} |\overline{\partial}\widetilde{f}|^2_{\omega_\delta,h_0} e^{-\varphi_\delta-\psi_\delta} dV_{\omega_\delta} \le \int_{X'} |\overline{\partial}\widetilde{f}|^2_{\omega,h_0} e^{-\varphi-\psi} dV_\omega < +\infty,$$

and we have an arbitrary small loss $O(\delta)$ of positivity in the curvature assumptions. Since ε errors are permitted, we take $\delta \ll \varepsilon$ and are reduced to the case where φ and ψ are smooth on X'.

(7) Bound of the error term in the $\overline{\partial}$-equation

We obtain an approximate L^2 solution $u_{t,\varepsilon}$ of the $\overline{\partial}$-equation $\overline{\partial}u_{t,\varepsilon} = v_t + w_{t,\varepsilon}$, $v_t := \theta(\psi - t)\cdot \overline{\partial}\widetilde{f} + \theta'(\psi - t)\overline{\partial}\psi \wedge \widetilde{f}$, with

$$\int_{X'} |w_{t,\varepsilon}|^2_{\omega,h_0}\, e^{-\varphi-\psi} dV_{X,\omega} \le 4\int_{X'\cap\{\psi<t+1\}} |\overline{\partial}\widetilde{f}|^2_{\omega,h_0}\, e^{-\varphi-\psi} dV_\omega$$
$$+\,4\int_{X'\cap\{t<\psi<t+1\}} \varepsilon\langle (B_t + \varepsilon\,\mathrm{Id})^{-1}\overline{\partial}\psi\wedge\widetilde{f}, \overline{\partial}\psi\wedge\widetilde{f}\rangle_{\omega,h_0}\, e^{-\varphi-\psi}.$$

The first integral in the right hand side tends to 0 as $t \to -\infty$.

The main point is to choose ad hoc factors $\eta = \eta_t$, $\lambda = \lambda_t$ in the twisted Bochner identity to get the last integral to converge to 0. As $X' \Subset X$, we can assume α constant and $\psi < 0$. For $u < 0$, set

$$\zeta(u) = \log \frac{\frac{1}{\alpha}+1}{\frac{1}{\alpha}+1-e^u},\quad \chi(u) = \frac{\frac{1}{\alpha^2}-1+e^u-(\frac{1}{\alpha}+1)u}{\frac{1}{\alpha}+1-e^u},\quad \beta = \frac{(\chi')^2}{\chi\zeta''-\chi''}.$$

One checks that $\varepsilon = e^{2t}$, $\sigma_t(u) = \log(e^u + e^t)$, $\eta_t = \chi(\sigma_t(\psi))$, $\lambda_t = \beta(\sigma_t(\psi))$ and $h_0 \mapsto h_t = h_0 e^{-\zeta(\sigma_t(\psi))}$ yield an $O(e^t)$ bound.

Second lecture

Second lecture: extension with optimal L^2 estimates

Setup. Let $L \to X$ be a holomorphic line bundle, equipped with a singular hermitian metric $h = h_0 e^{-\varphi}$, φ quasi-psh. Let $\psi \in L^1_{\rm loc}$ such that $\varphi + \psi$ is quasi-psh, and $Y \subset X$ the subvariety defined by the conductor ideal $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

For a section $f \in H^0(Y, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, the goal is to get an "extension" $F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$,

$$\textit{via}\ \ \mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi}), \quad F \mapsto f,$$

with an explicit L^2 estimate of F on X in terms of a suitable L^2 integral of f on the subvariety Y.

Additionally, it will be convenient to assume that X is weakly pseudoconvex (this is weaker than being holomorphically convex). This means that there exists a smooth psh exhaustion γ on X.

We first define the Ohsawa residual measure associated with f. As for f, this will be a measure supported on Y.

The Ohsawa residual measure

Given $f \in H^0(U, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, there exists a Stein covering (U_i) of X and liftings $\widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$ of f on U_i via $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$. We obtain in this way a C^∞ extension $\widetilde{f} = \sum \xi_i \widetilde{f}_i$ where (ξ_i) is a partition of unity.

Definition of the Ohsawa residual measure

For $g \in C_c(Y)$, $g \geq 0$, and $0 \leq \widetilde{g} \in C_c(X)$ extending g, we set

$$\int_Y g\, dV_Y[f^2, h, \psi] := \inf_{\widetilde{g}} \limsup_{t \to -\infty} \int_{\{t<\psi<t+1\}} \widetilde{g}\, |\widetilde{f}|^2_{\omega,h} e^{-\psi} dV_{X,\omega}.$$

Proposition

$dV_Y[f^2, h, \psi]$ is independent of the choice of $\widetilde{f}$ as well as of ω, and defines a positive measure on Y (but not necessarily locally finite).

Proof. When $\delta\widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi}))$, then $|\delta\widetilde{f}_i|^2_{\omega,h} e^{-\psi} \in L^1_{\mathrm{loc}}(X)$ and the $\limsup \to 0$ for $\mathrm{Supp}(\widetilde{g}) \subset U$.

The Ohsawa residual measure (2)

Example 1. Take $\psi(z) = r \log |s(z)|^2_{h_E}$, where $s \in H^0(X, E)$ and $r = \mathrm{rank}(E)$. Assume that $Y = s^{-1}(0)$ is of codimension r, that s is generically transverse to 0 on Y and $h \in C^\infty$. Then

$$dV_Y[f^2, h, \psi] = c_{n,r} \frac{|f|^2_{\omega,h}\, dV_{Y,\omega}}{|\Lambda^r(ds)|^2_{\omega,h_E}} \quad \text{on } Y \smallsetminus \{\Lambda^r(ds) = 0\}.$$

Proof. Near a regular point z_0 be can pick a holomorphic frame $(e_\lambda)_{1\leq\lambda\leq r}$ of E and coordinates $(z_1, \ldots, z_n)$ such that (e_λ) is h-orthornormal and $(\partial/\partial z_j)$ is ω-orthonormal at z_0, and $s(z) = \sum_{1\leq j\leq r} \lambda_j z_j e_j$, $\lambda_j \neq 0$. Then $\omega \sim i \sum dz_j \wedge d\overline{z}_j$ and $\psi(z) \sim r \log(|\lambda_1|^2|z_1|^2 + \ldots + |\lambda_r|^2|z_r|^2)$. This is an easy calculation of integrals on ellipsoids.

Example 2. Take now $\psi(z) = \sum c_j \log |s_{D_j}|^2_{h_j}$ where $D = \sum c_j D_j$ is a simple normal crossing divisor, $c_j > 0$, and h_j is a C^∞ metric on $\mathcal{O}_X(D_j)$. Also assume $h \in C^\infty$.

Ohsawa residual measure for s.n.c. singularities

By a change of coordinates, we are reduced to computing $dV_Y[f^2,h,\psi]$ for $\psi(z)=\sum c_j\log|z_j|^2+u(z)$, $u\in C^\infty$. However

$$dV_Y[f^2,h,\psi+u]=e^{-u}\,dV_Y[f^2,h,\psi],$$

thus we may assume $u=0$. At a regular point of $D_j\smallsetminus\bigcup_{k\neq j}D_k$, (and $j=1$, say) we apply the Fubini theorem with $z=(z_1,z')$, $z'=(z_2,\ldots,z_n)$. We have to compute limits of the form

$$\lim_{t\to-\infty}\int_{e^t<|z_1|^{2c_1}<e^{t+1}}\frac{\widetilde g(z)|\widetilde f(z)|^2}{|z_1|^{2c_1}}\,idz_1\wedge d\overline z_1=\frac{2\pi}{m_1}\,g(0,z')|\widetilde h(0,z')|^2$$

when $c_1=m_1\in\mathbb{N}^*$ and $\widetilde f(z)=z_1^{m_1-1}\widetilde h(z)$. However, if $c_j<1$, we get 0, and in general, if $c_j\notin\mathbb{N}^*$ and $c_j>1$, we can get only 0 or ∞ values, according to the divisibility of f by $z_j^{m_j-1}$, $m_j=\lfloor c_j\rfloor\in\mathbb{N}^*$.

As a consequence, we can capture an interesting (i.e. locally finite, non zero) residual measure $dV_Y[f^2,h,\psi]$ only in the case where one of the coefficients c_j is an integer.

Ohsawa residual measure for analytic singularities

One general case of interests is when ψ has analytic singularities, i.e. locally $\psi(z)=c\log\sum|g_j(z)|^2+u(z)$, $g_j\in\mathcal{O}_X(V)$, $u\in C^\infty(V)$.

Then, it is interesting to look at the family of multiplier ideal sheaves $\mathcal{I}(e^{-s\psi})$ when $s\in\mathbb{R}_+$, which decrease as s increases. Assume without loss of generality that $c=1$.

By Hironaka, we know that there exists a composition of blow-ups $\mu:\widetilde X\to X$ such that the pull-back ideal $\mu^*(g_j)=(g_j\circ\mu)$ is an invertible ideal sheaf $\mathcal{O}_{\widetilde X}(-\sum m_jD_j)$ associated with a simple normal crossing divisor. The direct image formula implies

$$\mathcal{I}(e^{-s\psi})=\mu_*(K_{\widetilde X/X}\otimes\mathcal{I}(e^{-s\,\psi\circ\mu}))=\mu_*\mathcal{O}_{\widetilde X}\Big(\sum(a_j-\lfloor sm_j\rfloor)D_j\Big)$$

where $K_{\widetilde X/X}=\mathcal{O}_{\widetilde X}(\sum a_jD_j)$. This implies that $\mathcal{I}(e^{-s\psi})$ “jumps” precisely for a discrete sequence of rational numbers $0=s_0<s_1<\ldots<s_k<\ldots$ such that $s_km_j\in\mathbb{N}$ for some j.

For $f\in\mathcal{I}(e^{-s_{k-1}\psi})$, the measure $dV_Y[f^2,h,s_k\psi]$ will be interesting.

Restricted multiplier ideals

We first have to introduce a suitable sheaf of integrable functions on the subvariety Y associated with $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

Definition of the restricted multiplier ideal

For $x \in Y$, we define $\mathcal{I}'_\psi(h)_x \subset \mathcal{I}(h)_x$ to be the ideal of germs of functions $\widetilde{f} \in \mathcal{I}(h)_x$ associated with $f = \widetilde{f} \mod \mathcal{I}(he^{-\psi})_x$ in $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})_x$, for which $dV[f^2, h, \psi]$ is locally finite near x on Y.

Clearly, $\mathcal{I}(he^{-\psi}) \subset \mathcal{I}'_\psi(h) \subset \mathcal{I}(h)$.

Typical case of application. Assume that $h = e^{-\varphi}$ and ψ have analytic singularities, and that $s_k = 1$ is one of jumping values for $s \mapsto \mathcal{I}(e^{-s\psi})$ (case of log canonical singularities: $s_1 = 1$).

Then $\mathcal{I}'_\psi(h) \subset \mathcal{I}(he^{-s_{k-1}\psi})$ on X, and $\mathcal{I}'_\psi(h) = \mathcal{I}(he^{-s_{k-1}\psi})$ on a Zariski open subset $X_0 = X \smallsetminus Z$, $Z \subsetneq Y$ (however, the ideals may differ on Z).

Use of more "flexible" weights

The next issue is that we need special and rather flexible weights. Let $\alpha \in]0,1[$ and $A = \sup_X \psi \in]-\infty, +\infty]$. We consider functions $\rho : [-\infty, A] \to \mathbb{R}_+^*$, such as

$$\rho(u) = 1 - (A + 1 + \alpha^{-1/2} - u)^{-1},$$

that are continuous strictly decreasing, with the property that ρ is concave near $-\infty$.

We assume moreover that

$$\int_t^A \rho(u)\, du + \frac{\rho(A)}{\alpha} \le \frac{\rho(t)^2}{|\rho'(t)|} \quad \text{for all } t \in]-\infty, A].$$

The L^2 estimates will involve integrals of the form $\int_X |F|^2_{\omega,h} e^{-\psi} |\rho'(\psi)|\, dV_{X,\omega}$, where $|\rho'(\psi)| = (C - \psi)^{-2}$ in the above example, so that $e^{-\psi}|\rho'(\psi)|$ is locally sommable when ψ has log canonical singularities.

General L^2 extension theorem

Theorem (X. Zhou-L. Zhu 2019)

Let (X,ω) be a weakly pseudoconvex Kähler manifold, L a holomorphic line bundle with a hermitian metric $h = h_0 e^{-\varphi}$, $h_0 \in C^\infty$, φ quasi-psh on X, and $\psi \in L^1_{\rm loc}(X)$. Assume $\exists\alpha > 0$ constant such that

$$\Theta_{L,h} + (1+\nu\alpha) i\partial\overline{\partial}\psi \geq 0 \quad \text{on } X, \quad \nu = 0, 1.$$

Then, for every $f \in H^0(Y, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}'_\psi(h)/\mathcal{I}(he^{-\psi}))$ s.t.

$$\int_Y dV_Y[f^2, h, \psi] < +\infty,$$

there exists $F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}'_\psi(h)$ that is mapped to f by the morphism $\mathcal{I}'_\psi(h) \to \mathcal{I}'_\psi(h)/\mathcal{I}(he^{-\psi})$, such that

$$\int_X |F|^2_{\omega,h} e^{-\psi} |\rho'(\psi)|\, dV_{X,\omega} \leq \rho(-\infty) \int_Y dV_Y[f^2, h, \psi].$$

(1) Construction of a smooth extension

Every section $f \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$ admits a C^∞ lifting

$$\widetilde{f} = \sum \xi_i \widetilde{f}_i, \quad \widetilde{f}_i \in H^0\big(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

by means of a Stein covering (U_i) of X and a partition of unity (ξ_i) subordinate to (U_i).

Since $\sum \overline{\partial}\xi_i = 0$, we have $\overline{\partial}\widetilde{f} = \sum \overline{\partial}\xi_i(\widetilde{f}_i - \widetilde{f}_j)$ on U_j, and since $\widetilde{f}_i - \widetilde{f}_j$ has coefficients in $\mathcal{I}(he^{-\psi})$, we see that $\overline{\partial}\widetilde{f}$ is valued in

$$\mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi}) \otimes_{\mathcal{O}_X} C^\infty.$$

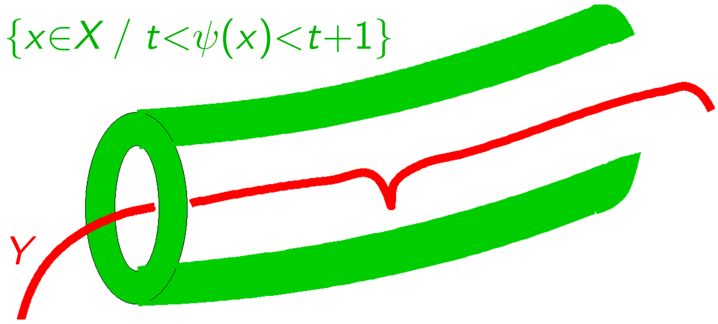
As X is assumed to be weakly pseudoconvex, we can consider $X_c = \{z \in X\,;\ \gamma(z) < c\} \Subset X$, $\forall c \in \mathbb{R}$, and get by compactness

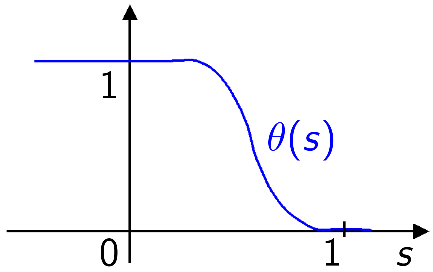
$$\int_{X_c} |\overline{\partial}\widetilde{f}|^2_{\omega,h} e^{-\psi} dV_{X,\omega} < +\infty.$$

It will be enough to get estimates on X_c, and then let $c \to +\infty$.

(2) Solving the $\overline{\partial}$ equation

The next idea is to truncate $\widetilde{f}$ by multiplying $\widetilde{f}$ with a cut-off function $\theta(\psi - t)$ equal to 1 near $Y \subset \psi^{-1}(-\infty)$.





We next solve the approximate $\overline{\partial}$-equation

$$(*) \qquad \overline{\partial} u_{t,\varepsilon} = v_t + w_{t,\varepsilon}$$

with $v_t := \overline{\partial}(\theta(\psi - t)\cdot \widetilde{f}) = \theta(\psi - t)\cdot \overline{\partial}\widetilde{f} + \theta'(\psi - t)\overline{\partial}\psi \wedge \widetilde{f}$.

It the weights ψ and φ of $h = h_0 e^{-\varphi}$ are not smooth, we use regularizations $\varphi_\delta \downarrow \varphi$, $\psi_\delta \downarrow \psi$ and complete Kähler metrics $\omega_\delta \downarrow \omega$ on $X \smallsetminus Z_\delta$. (We omit details here).

(3) L^2 estimates for solution and error term

The existence theorem with twisting factors $\eta_{t,\varepsilon}$, $\lambda_{t,\varepsilon}$ yields

$$\begin{aligned}\int_{X_c} (\eta_{t,\varepsilon} + \lambda_{t,\varepsilon})^{-1} |u_{t,\varepsilon}|^2_{\omega,h_0} e^{-\varphi-\psi} dV_{X,\omega} + \frac{1}{\varepsilon}\int_{X_c} |w_{t,\varepsilon}|^2_{\omega,h_0} e^{-\varphi-\psi} dV_{X,\omega} \\ \le 4\int_{X_c\cap\{\psi<t+1\}} |\overline{\partial}\widetilde{f}|^2_{\omega,h_0} e^{-\varphi-\psi} dV_\omega \\ + 4\int_{X_c\cap\{t<\psi<t+1\}} \langle (B_t + \varepsilon\,\mathrm{Id})^{-1} \overline{\partial}\psi\wedge\widetilde{f}, \overline{\partial}\psi\wedge\widetilde{f}\rangle_{\omega,h_0} e^{-\varphi-\psi}.\end{aligned}$$

The first integral in the right hand side tends to 0 as $t \to -\infty$.

Again, the main point is to choose ad hoc factors η_t, λ_t, and we want here the last integral to converge to a finite limit. One can check that this works with

$$\zeta(u) = \log\frac{\rho(-\infty)}{\rho(u)}, \quad \chi(u) = \frac{\int_u^A \rho(v)dv + \frac{1}{\alpha\rho(A)}}{\rho(u)}, \quad \beta = \frac{(\chi')^2}{\chi\zeta'' - \chi''},$$

$$\sigma_{t,\varepsilon}(u) = \max{}_\varepsilon(u,t), \quad \eta_{t,\varepsilon} = \chi(\sigma_{t,\varepsilon}(\psi)), \quad \lambda_{t,\varepsilon} = \beta(\sigma_{t,\varepsilon}(\psi)).$$

Extension from hypersurface (Stein case)

In the hypersurface case, one gets the following simpler statement.

Theorem

Let X be a Stein manifold of dimension n. Let φ and ψ be plurisubharmonic functions on X. Assume that w is a holomorphic function on X such that $\sup_X(\psi + 2\log|w|) \le 0$ and dw does not vanish identically on any branch of $w^{-1}(0)$.

Denote $Y = w^{-1}(0)$ and $Y_0 = \{x \in Y : dw(x) \ne 0\}$.

Then for any holomorphic $(n-1)$-form f on Y_0 satisfying

$$\int_{Y_0} \mathrm{e}^{-\varphi-\psi} i^{(n-1)^2} f \wedge \bar{f} < +\infty,$$

there exists a holomorphic n-form F on X satisfying $F_{|Y_0} = dw \wedge f$ and an optimal estimate

$$\int_X \mathrm{e}^{-\varphi} i^{n^2} F \wedge \bar{F} \le 2\pi \int_{Y_0} \mathrm{e}^{-\varphi-\psi} i^{(n-1)^2} f \wedge \bar{f}.$$

The Suita conjecture

The Suita conjecture was posed originally on open Riemann surfaces in 1972. The motivation was to answer a question posed by Sario and Oikawa about the relation between the Bergman kernel B_Ω for holomorphic $(1,0)$ forms on an open Riemann surface Ω which admits a Green function G_Ω.

Recall that the logarithmic capacity $c_\beta(z)$ is locally defined by

$$c_\beta(z) = \exp \lim_{\xi\to z} \big(G_\Omega(\xi, z) - \log|\xi - z|\big) \ \text{ on } \Omega.$$

Suita conjecture

$(c_\beta(z))^2|dz|^2 \le \pi B_\Omega(z)$, for every $z \in \Omega$.

Theorem

The Suita conjecture holds true (planar case: Błocki 2013; general case: Guan-Zhou 2014). Moreover (Guan-Zhou 2014), equality holds iff Ω biholomorphic to disc minus a closed polar set.

Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a Kähler current T is a closed (1,1)-current T such that $T \geq \delta\omega$ for a smooth $(1,1)$ form $\omega > 0$ and $\delta \ll 1$.

Easy observation

$\alpha \in \mathcal{E}^\circ$ (interior of $\mathcal{E}$) $\iff$ $\alpha = \{T\}$, $T =$ a Kähler current.
We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

Theorem on approximate Zariski decomposition (D, 1992)

Any Kähler current can be written $T = \lim T_m$ where $T_m \in \{T\}$ has analytic singularities & logarithmic poles, i.e. $\exists$ modification $\mu_m : \widetilde{X}_m \to X$ such that $\mu_m^\star T_m = [E_m] + \beta_m$, where $E_m \geq 0$ is a $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coeff. in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on $\widetilde{X}_m$.

Moreover (Boucksom), $\mathrm{Vol}(\beta_m) = \int_{\widetilde{X}_m} \beta_m^n \to \mathrm{Vol}(T)$ as $m \to +\infty$.

Proof of the analytic Zariski decomposition

- Write locally on any coordinate ball $\Omega \subset X$

$$T = i\partial\overline{\partial}\varphi$$

for some strictly plurisubharmonic psh potential φ on X.

- Approximate T on Ω by

$$T_m = i\partial\overline{\partial}\varphi_m, \quad \text{where} \quad \varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \Big\{f \in \mathcal{O}(\Omega)\,;\ \|f\|^2_{m\varphi} := \int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty\Big\}.$$

- We have $\varphi_m(z) = \dfrac{1}{2m} \sup_{\|f\|_{m\varphi} \leq 1} \log|f(z)|^2$.

The mean value inequality implies

$$|f(z)|^2 \leq \frac{1}{\pi^n r^{2n}/n!} \sup_{B(z,r)} e^{2m\varphi(z)} \Rightarrow \varphi_m(z) \leq \sup_{B(z,r)} \varphi + \frac{n}{m}\log\frac{C}{r}.$$

Use of the pointwise Ohsawa-Takegoshi theorem

• The Ohsawa-Takegoshi L^2 extension theorem (extension from a single isolated point) implies that for every $z_0 \in \Omega$, there exists $f \in \mathcal{O}(\Omega)$ such that $f(z_0) = c\, e^{m\varphi(z_0)}$ ($c > 0$ small), such that

$$\|f\|^2_{m\varphi} = \int_\Omega |f|^2 e^{-2m\varphi} dV \le C \int_{\{z_0\}} |f|^2 e^{-2m\varphi} \delta_{z_0} = 1$$

for $c = C^{-1/2}$. As a consequence $\varphi_m(z) \ge \varphi(z) + \frac{1}{2m} \log c$.

• By the above inequalities one easily concludes that the Lelong number at any point $z_0 \in \Omega$ satisfies

$$\nu(\varphi, z_0) - \frac{n}{m} \le \nu(\varphi_m, z_0) \le \nu(\varphi, z_0).$$

This implies Siu's analyticity result for Lelong upper level sets $E_c(T)$.

• The case of a global current $T = \alpha + dd^c\varphi$ is obtained by using a covering of X by balls Ω_j, and gluing the local approximations $\varphi_{j,m}$ of φ into a global one φ_m by a partition of unity.

Third lecture

Third lecture

Log canonical thresholds

The goal is to explain a proof of the strong openness conjecture for log canonical thresholds. Let Ω be a domain in $\mathbb{C}^n$, $f \in \mathcal{O}(\Omega)$ a holomorphic function, and $\varphi \in \mathrm{PSH}(\Omega)$ a psh function on Ω.

The log canonical threshold $c_{z_0}(\varphi) \in]0, +\infty]$ (or complex singularity exponent) is defined to be

$$c_{z_0}(\varphi) = \sup\big\{c > 0\,;\ e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\big\}.$$

A well known theorem of Skoda asserts that

$$\frac{1}{n}\nu(\varphi, z_0) \le c_{z_0}(\varphi)^{-1} \le \nu(\varphi, z_0).$$

For every holomorphic function f on Ω, we also introduce the weighted log canonical threshold $c_{f,z_0}(\varphi) \in]0, +\infty]$ of φ with weight f at z_0 to be

$$c_{f,z_0}(\varphi) = \sup\big\{c > 0\,;\ |f|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\big\}.$$

Semi-continuity theorem / strong openness

Theorem (Guan-Zhou 2013, version due to Pham H. Hiep 2014))

Let f be a holomorphic function on an open set Ω in $\mathbb{C}^n$ and let φ be a psh function on Ω.

(i) ("Semicontinuity theorem") Assume that $\int_{\Omega'} e^{-2c\varphi} dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$ and let $z_0 \in \Omega'$. Then there exists $\delta = \delta(c, \varphi, \Omega', z_0) > 0$ such that for every $\psi \in \mathrm{PSH}(\Omega')$, $\|\psi - \varphi\|_{L^1(\Omega')} \le \delta$ implies $c_{z_0}(\psi) > c$. Moreover, as ψ converges to φ in $L^1(\Omega')$, the function $e^{-2c\psi}$ converges to $e^{-2c\varphi}$ in L^1 on every relatively compact open subset $\Omega'' \Subset \Omega'$.

(ii) ("Strong effective openness") Assume that $\int_{\Omega'} |f|^2 e^{-2c\varphi} dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$. When $\psi \in \mathrm{PSH}(\Omega')$ converges to φ in $L^1(\Omega')$ with $\psi \le \varphi$, the function $|f|^2 e^{-2c\psi}$ converges to $|f|^2 e^{-2c\varphi}$ in L^1 norm on every relatively compact open subset $\Omega'' \Subset \Omega'$.

Consequences of the semi-continuity theorem

Corollary 1 (Strong openness, Guan-Zhou 2013)

For any plurisubharmonic function φ on a neighborhood of a point $z_0 \in \mathbb{C}^n$, the set $\{c > 0 : \ |f|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\}$ is an open interval $]0, c_{f,z_0}(\varphi)[$.

Proof. After subtracting a large constant to φ, we can assume $\varphi \le 0$. Then Cor. 1 is a consequence of assertion (ii) of the main theorem by taking Ω' small enough and $\psi = (1+\delta)\varphi$ with $\delta \searrow 0$.

Application to multiplier ideal sheaves (Guan-Zhou 2013)

Let $h = e^{-\varphi}$ a singular hermitian metric with φ quasi-psh. The "upper semicontinuous regularization" of $\mathcal{I}(h)$ is defined to be

$$\mathcal{I}_+(h) = \lim_{\varepsilon \to 0} \mathcal{I}(h^{1+\varepsilon}) = \lim_{\varepsilon \to 0} \mathcal{I}((1+\varepsilon)\varphi) = \lim_{k \to +\infty} \mathcal{I}((1+1/k)\varphi)$$

(by Noetherianity, this increasing sequence is stationary on all compact subsets). Then $\mathcal{I}_+(h) = \mathcal{I}(h)$.

Convergence from below / idea of the proof

Corollary 2 (Convergence from below)

If $\psi \le \varphi$ converges to φ in a neighborhood of $z_0 \in \mathbb{C}^n$, then $c_{f,z_0}(\psi) \le c_{f,z_0}(\varphi)$ converges to $c_{f,z_0}(\varphi)$.

Proof. We have by definition $c_{f,z_0}(\psi) \le c_{f,z_0}(\varphi)$ for $\psi \le \varphi$, but again (ii) shows that $c_{f,z_0}(\psi)$ becomes $\ge c$ for any given value $c \in (0, c_{f,z_0}(\varphi))$, when $\|\psi - \varphi\|_{L^1(\Omega')}$ is sufficiently small.

Phams's theorem is proved by induction on n ($n = 0, 1$ are easy).

Aassume that the theorem holds for dimension $n-1$. Let $f \in \mathcal{O}(\Delta_R^n)$ be holomorphic on a n-dimensional polydisc, such that $\int_{\Delta_R^n} |f(z)|^2 e^{-2c\varphi(z)} dV_{2n}(z)$ converges. The idea is to restrict f to a generic hyperplane $z_n = w_n$. By induction, the integral of the restriction still converges after increasing c to $c + \varepsilon$ (shrinking R). By the Ohsawa-Takegoshi theorem, the restriction can be extended to a function F and one proceeds by comparing f and F.

Key lemma in Pham's proof

Lemma (Pham)

Let $\varphi \le 0$ be psh and f be holomorphic on the polydisc Δ_R^n of center 0 and (poly)radius $R > 0$ in $\mathbb{C}^n$, such that for some $c > 0$

$$\int_{\Delta_R^n} |f(z)|^2 e^{-2c\,\varphi(z)} dV_{2n}(z) < +\infty.$$

Let $\psi_j \le 0$, $j \in \mathbb{N}$, be psh functions on Δ_R^n with $\psi_j \to \varphi$ in $L^1_{\rm loc}(\Delta_R^n)$, and assume that $f \equiv 1$ or $\psi_j \le \varphi$ for all $j \ge 1$.
Then for every $r < R$ and $\varepsilon \in \,]0, \frac{1}{2}r]$, there exist a value $w_n \in \Delta_\varepsilon \smallsetminus \{0\}$ (in a set of measure > 0), an index $j_0 = j_0(w_n)$, a constant $\tilde c = \tilde c(w_n) > c$ and holomorphic functions F_j on Δ_r^n, $j \ge j_0$, such that $F_j(z) = f(z) + (z_n - w_n)\sum a_{j,\alpha} z^\alpha$ with $|w_n||a_{j,\alpha}| \le r^{-|\alpha|}\varepsilon$ for all $\alpha \in \mathbb{N}^n$, $\underline{\rm IM}(F_j) \le \underline{\rm IM}(f)$, and

$$\int_{\Delta_r^n} |F_j(z)|^2 e^{-2\tilde c\,\psi_j(z)} dV_{2n}(z) \le \frac{\varepsilon^2}{|w_n|^2} < +\infty, \quad \forall j \ge j_0.$$

[Here $\underline{\rm IM}(F)$ = Initial Monomial in lexicographic order at 0].

Idea of proof of the key lemma

By Fubini's theorem we have

$$\int_{\Delta_R}\left[\int_{\Delta_R^{n-1}} |f(z', z_n)|^2 e^{-2c\,\varphi(z',z_n)} dV_{2n-2}(z')\right] dV_2(z_n) < +\infty.$$

Since the integral extended to a small disc $z_n \in \Delta_\eta$ tends to 0 as $\eta \to 0$, it will become smaller than any preassigned value, say $\varepsilon_0^2 > 0$, for $\eta \le \eta_0$ small enough. Therefore we can choose a set of positive measure of values $w_n \in \Delta_\eta \smallsetminus \{0\}$ such that

$$\int_{\Delta_R^{n-1}} |f(z', w_n)|^2 e^{-2c\,\varphi(z',w_n)} dV_{2n-2}(z') \le \frac{\varepsilon_0^2}{\pi\eta^2} < \frac{\varepsilon_0^2}{|w_n|^2}.$$

Since the main theorem is assumed to hold for $n-1$, for any $\rho < R$ there exist $j_0 = j_0(w_n)$ and $\tilde c = \tilde c(w_n) > c$ such that

$$\int_{\Delta_\rho^{n-1}} |f(z', w_n)|^2 e^{-2\tilde c\,\psi_j(z',w_n)} dV_{2n-2}(z') < \frac{\varepsilon_0^2}{|w_n|^2}, \quad \forall j \ge j_0.$$

Idea of proof of the key lemma (2)

By Ohsawa-Takegoshi, there exists a holomorphic function F_j on $\Delta_\rho^{n-1} \times \Delta_R$ such that $F_j(z', w_n) = f(z', w_n)$ for all $z' \in \Delta_\rho^{n-1}$, and

$$\int_{\Delta_\rho^{n-1}\times\Delta_R} |F_j(z)|^2 e^{-2\tilde{c}\,\psi_j(z)} dV_{2n}(z)$$
$$\le C_n R^2 \int_{\Delta_\rho^{n-1}} |f(z', w_n)|^2 e^{-2\tilde{c}\,\psi_j(z', w_n)} dV_{2n-2}(z') \le \frac{C_n R^2 \varepsilon_0^2}{|w_n|^2},$$

where C_n is a constant which only depends on n (the constant is universal for $R = 1$ and is rescaled by R^2 otherwise).
Taking $\rho = \frac{1}{2}(r+R)$, the mean value inequality implies

$$\|F_j\|_{L^\infty(\Delta_r^n)} \le \frac{2^n C_n^{\frac{1}{2}} R\varepsilon_0}{\pi^{\frac{n}{2}}(R-r)^n |w_n|}.$$

Since $F_j(z', w_n) - f(z', w_n) = 0$, $\forall z' \in \Delta_r^{n-1}$, we can write $F_j(z) = f(z) + (z_n - w_n) g_j(z)$ for some holomorphic function $g_j(z) = \sum_{\alpha\in\mathbb{N}^n} a_{j,\alpha} z^\alpha$ on $\Delta_r^{n-1} \times \Delta_R$. Then analyze $\underline{\mathrm{IM}}(F_j)$ ...

Volume and numerical dimension of currents

Definition

let (X, ω) be a compact Kähler manifold, and $T \ge 0$ a closed $(1,1)$-current on X. The positive intersection $\langle T^p\rangle \in H^{p,p}_{\ge 0}(X)$ (in the sense of Boucksom) is

$$\lim_{\varepsilon\to 0} \Big(\limsup (\mu_{m,\varepsilon})_*(\beta^p_{m,\varepsilon})\Big), \quad \mu_{m,\varepsilon} : \widetilde{X}_{m,\varepsilon} \to X$$

for the Zariski decomposition $\mu^*_{m,\varepsilon} T_{m,\varepsilon} = \beta_{m,\varepsilon} + [E_{m,\varepsilon}]$ of Bergman approximations $T_{m,\varepsilon}$ of $T + \varepsilon\omega$. The volume is $\mathrm{Vol}(T) = \langle T^n\rangle$.

Numerical dimension of a current

$$\mathrm{nd}(T) = \max\big\{p \in \mathbb{N};\ \langle T^p\rangle \ne 0 \quad \text{in } H^{p,p}_{\ge 0}(X)\big\}.$$

Numerical dimension of a hermitian line bundle (L, h)

If $\Theta_{L,h} \ge 0$, one defines $\mathrm{nd}(L, h) = \mathrm{nd}(\Theta_{L,h})$.

Generalized Nadel vanishing theorem

Theorem (Junyan Cao, PhD thesis 2012)

Let X be compact Kähler, and (L,h) be s.t. $\Theta_{L,h} \geq 0$ on X. Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}_+(h)) = 0 \ \text{ for } q \geq n - \mathrm{nd}(L,h) + 1,$$

Moreover we have in fact $\mathcal{I}_+(h) = \mathcal{I}(h)$ by Guan-Zhou.

Remark 1. There is also a concept of numerical dimension of a class $\alpha \in H^{1,1}(X)$: one defines $\mathrm{nd}(L)$ to be $-\infty$ if L is not psef, and

$$\mathrm{nd}(L) = \max\{p \in \mathbb{N};\ \lim_{\varepsilon \to 0} \sup_{\{T \in C_1(L),\ T \geq -\varepsilon\omega\}} \langle (T + \varepsilon\omega)^p \rangle \neq 0$$

when L is psef. In general, we have $\mathrm{nd}(L,h) \leq \mathrm{nd}(L)$, but it may happen that $\sup_{\{h,\ \Theta_{L,h} \geq 0\}} \mathrm{nd}(L,h) < \mathrm{nd}(L)$.

Remark 2. In the projective case, one can use a hyperplane section argument, using Tsuji's algebraic expression of $\mathrm{nd}(L,h)$:

$$\mathrm{nd}(L,h) = \max\big\{p \in \mathbb{N};\ \exists Y^p \subset X,\ h^0(Y, (L^{\otimes m} \otimes \mathcal{I}(h^m))_{|Y}) \geq cm^p\big\}.$$

Proof of generalized Nadel vanishing (projective case)

Hyperplane section argument (projective case). Take A = very ample divisor, $\omega = \Theta_{A,h_A} > 0$, and $Y = A_1 \cap \ldots \cap A_{n-p}$, $A_j \in |A|$. Then

$$\langle \Theta^p_{L,h} \rangle \cdot Y = \int_X \langle \Theta^p_{L,h} \rangle \cdot Y = \int_X \langle \Theta^p_{L,h} \rangle \wedge \omega^{n-p} > 0.$$

From this one concludes that $(\Theta_{L,h})_{|Y}$ is big.

Lemma (J. Cao)

When (L,h) is big, i.e. $\langle \Theta^n_{L,h} \rangle > 0$, there exists a metric $\widetilde{h}$ such that $\mathcal{I}(\widetilde{h}) = \mathcal{I}_+(h)$ with $\Theta_{L,\widetilde{h}} \geq \varepsilon\omega$ [Riemann-Roch].

Then Nadel $\Rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}_+(h)) = 0$ for $q \geq 1$.

Conclude by induction on $\dim X$ and the exact cohomology sequence for the restriction to a hyperplane section.

Proof of generalized Nadel vanishing (Kähler case)

Kähler case. By the regularization theorem, one finds an approximation $\tilde{h}_\varepsilon = h_0 e^{-\tilde{\varphi}_\varepsilon}$ with analytic singularities of the metric h of L, such that $\Theta_{L,\tilde{h}_\varepsilon} \geq -\frac{1}{2}\varepsilon\omega$.

Then, by blowing-up X to achieve divisorial singularities for $\tilde{h}_\varepsilon$ and using Yau's theorem, one solves on X a singular Monge-Ampère equation: $\exists h_\varepsilon = h_0 e^{-\varphi_\varepsilon}$ with logarithmic poles, such that

$$(\Theta_{L,h_\varepsilon} + \varepsilon\omega)^n = C_\varepsilon \omega^n.$$

where $C_\varepsilon \geq \binom{n}{p}\langle\Theta_{L,h}^p\rangle \cdot (\varepsilon\omega)^{n-p} \sim C\varepsilon^{n-p}$, $p = \mathrm{nd}(L,h)$.

Another important fact is that one can ensure the equalities $\mathcal{I}_+(h) = \mathcal{I}(h^{1+\varepsilon}) = \mathcal{I}(h_\varepsilon)$ (looking deeper in the regularization).

Ch. Mourougane argument (PhD thesis 1996). Let $\lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of $\Theta_{L,h} + \varepsilon\omega$ with respect to ω at each point $x \in X$. Then

$$\lambda_1 \ldots \lambda_n = C_\varepsilon \geq \text{Const } \varepsilon^{n-p}.$$

Final step: use Bochner-Kodaira formula

Moreover

$$\int_X \lambda_{q+1} \ldots \lambda_n \, \omega^n = \int_X \Theta_{L,h}^{n-q} \wedge \omega^q \leq \text{Const}, \quad \forall q \geq 1,$$

so $\lambda_{q+1} \ldots \lambda_n \leq C$ on a large open set $U \subset X$ and

$$\lambda_q^q \geq \lambda_1 \ldots \lambda_q \geq c\varepsilon^{n-p} \quad \Rightarrow \quad \lambda_q \geq c\varepsilon^{(n-p)/q} \text{ on } U,$$

$$\Rightarrow \sum_{j=1}^{q}(\lambda_j - \varepsilon) \geq \lambda_q - q\varepsilon \geq c\varepsilon^{(n-p)/q} - q\varepsilon > 0 \text{ for } q > n-p.$$

$\lambda_j =$ eigenvalues of $(\Theta_{L,h_\varepsilon} + \varepsilon\omega) \Rightarrow$ (eigenvalues of $\Theta_{L,h_\varepsilon}) = \lambda_j - \varepsilon$

and the Bochner-Kodaira formula yields

$$\|\overline{\partial}u\|_\varepsilon^2 + \|\overline{\partial}^* u\|_\varepsilon^2 \geq \int_U \Big(\sum_{j=1}^{q}(\lambda_j - \varepsilon)\Big)|u|^2 e^{-\varphi_\varepsilon} dV_\omega.$$

The fact that U has almost full volume allows to take the limit as $\varepsilon \to 0$ and conclude that $u = 0$. QED

Hard Lefschetz theorem with psef coefficients

Hard Lefschetz theorem (D-Peternell-Schneider 2001)

Let (L, h) be a psef line bundle on a compact n-dimensional Kähler manifold (X, ω), $\Theta_{L,h} \geq 0$. Then, the Lefschetz map : $u \mapsto \omega^q \wedge u$ induces a surjective morphism :

$$\Phi^q_{\omega,h} : H^0(X, \Omega^{n-q}_X \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h)).$$

The proof is based on using approximated metrics $h_\nu = h_0 e^{-\varphi_\nu}$, $\varphi_\nu \downarrow \varphi$, that are smooth on $X \smallsetminus Z_\nu$, with an increasing sequence of analytic sets Z_ν, such that $\Theta_{L,h_\nu} \geq -\varepsilon_\nu \omega$. We also consider Kähler metrics $\omega_\nu \downarrow \omega$ that are complete on $X \smallsetminus Z_\nu$.

Any cohomology class $\{u\}$ is represented by a (ω_ν, h_ν)-harmonic (n, q) form u_ν with values in $K_X \otimes L \otimes \mathcal{I}(h_\nu)$. One gets a unique $(n-q, 0)$-form v_ν s.t. $\omega^q_\nu \wedge v_\nu = u_\nu$, and a Bochner type formula

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^*_{h_\nu} u\|^2 = \|\overline{\partial} v\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j \in J} \lambda_{\nu,j}\Big) |u_{IJ}|^2 e^{-\varphi_\nu} dV_{\omega_\nu}.$$

Proof of the Hard Lefschetz theorem

Here the $\lambda_{\nu,j}$ are the curvature eigenvalues of Θ_{L,h_ν}, so $\lambda_{\nu,j} \geq -\varepsilon_\nu$. Taking u_ν = harmonic representative, we get $\overline{\partial} u_\nu = \overline{\partial}^*_{h_\nu} u_\nu = 0$, hence

$$\|\overline{\partial} v_\nu\|^2 = \int_X |\overline{\partial} v_\nu|_{\omega_\nu} e^{-\varphi_\nu} dV_{\omega_\nu} \leq q\varepsilon_\nu \int_X |u_\nu|^2_{\omega_\nu} e^{-\varphi_\nu} dV_{\omega_\nu}$$
$$\leq q\varepsilon_\nu \int_X |u|^2_{\omega_\nu} e^{-\varphi_\nu} dV_{\omega_\nu} \leq q\varepsilon_\nu \int_X |u|^2_\omega e^{-\varphi} dV_\omega.$$

We need the following consequence of the Ohsawa-Takegoshi theorem:

Equisingular approximation theorem

Writing $h = h_0 e^{-\varphi}$, there exists a decreasing sequence $\varphi_\nu \downarrow \varphi$ $\Rightarrow$ $h = \lim h_\nu$ with $h_\nu = h_0 e^{-\varphi_\nu}$, such that

- $\varphi_\nu \in C^\infty(X \smallsetminus Z_\nu)$, where Z_ν is an increasing sequence of analytic sets,
- $\mathcal{I}(h_\nu) = \mathcal{I}(h)$, $\forall \nu$,
- $\Theta_{L,h_\nu} \geq -\varepsilon_\nu \omega$.

Important complement by Xiaojun Wu

Theorem (Xiaojun Wu, PhD thesis 2020)

Let (L, h) be a psef line bundle on a compact Kähler manifold (X, ω), $\Theta_{L,h} \geq 0$. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces an isomorphism

$$H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \cap \mathrm{Ker}(\partial_h) \longrightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h)).$$

Moreover, each section $v \in H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \cap \mathrm{Ker}(\partial_h)$ is ∇_h-parallel, and gives rise to a holomorphic foliation of X by considering the subsheaf $\mathcal{F}_v = \{\xi \in \mathcal{O}(T_X)\,;\ i_\xi v = 0\} \subset \mathcal{O}(T_X)$.

Proof. In fact, with $c_q = i^{(n-q+1)^2}$, a formal integration by parts gives

$$\int_X |\partial_h v|_h^2 dV_\omega = \int_X c_q \{\partial_h v, \partial_h v\}_h \wedge \omega^{q-1} = -\int_X c_q \{i\bar\partial\partial_h v, v\}_h \wedge \omega^{q-1}$$
$$= -\int_X c_q \{\Theta_{L,h} v, v\}_h \wedge \omega^{q-1} \leq 0 \quad \Rightarrow \quad \partial_h v = 0.$$

One can check that this is meaningful in the sense of distributions.

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