Interplays of Complex and Symplectic Geometry

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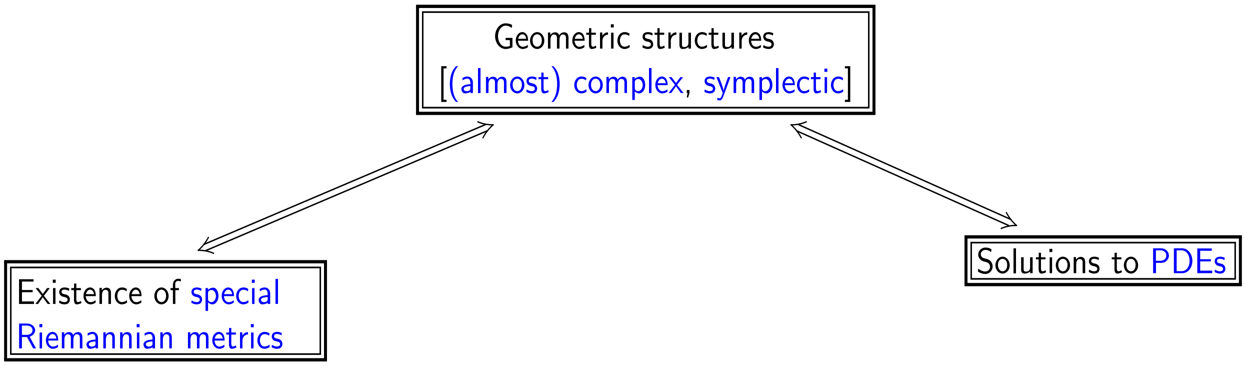
Links between different objects on a (smooth) manifold M^{2n}



Lecture 1: Taming condition and Pluriclosed Metrics

Lecture 2: Symplectic Calabi-Yau Problem

Lecture 3: Balanced Metrics and the Hull-Strominger System

Complex structures

Definition

An almost complex structure on M^{2n} is an endomorphism of TM^{2n} such that $J^2 = -Id$.

Theorem (Newlander-Nirenberg)

An almost complex structure J on M^{2n} is integrable $\iff N_J = 0$.

For a complex manifold (M^{2n}, J) the differential d splits as $d = \partial + \overline{\partial}$ and $d^2 = 0$ gives

$$\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$$

$\hookrightarrow$ Dolbeault cohomology.

Kähler metrics

Some complex manifolds are more complex than others!

Definition

A Riemannian metric g on (M^{2n}, J) is J-Hermitian (or compatible) if $g_p(Ju, Jv) = g_p(u, v)$, $\forall p \in M^{2n}, \forall u, v \in T_pM^{2n}$

Remark

$\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ is a differential 2-form of type $(1, 1)$.

Definition

A complex mfd (M^{2n}, J) is called Kähler if it admits a Hermitian metric g such that $d\omega = 0$.

Examples

Riemann surfaces, $\mathbb{C}^n, \mathbb{C}^n/\mathbb{Z}^n, \mathbb{CP}^n$.

Question: Are all complex manifolds Kähler? No

- $n = 1$ all Riemann surfaces are Kähler.
- $n \geq 2$ necessary topological conditions for compact manifolds (e.g. the odd Betti numbers have to be even; fundamental group of particular type; formality in the sense of rational homotopy theory..)

Theorem (Kodaira; Siu; Buchdal; Lamori)

A compact complex surface M is Kähler $\Leftrightarrow$ $b_1(M)$ is even.

A non-Kähler example

Example (Kodaira-Thurston manifold)

$M^4 = G/\mathbb{Z}^4$, with $G = H \times \mathbb{R}$ a Lie group of nilpotent matrices:

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \right\}$$

M^4 is complex and symplectic, but since $b_1(M^4) = 3$ it cannot admit a Kähler metric.

Remark

• M^4 is an example of compact nilmanifold ($\hookrightarrow$ compact locally homogeneous space).
• Every complex structure on M^4 is invariant, i.e. it is induced by a complex structure on the Lie algebra of G.

Taming condition

For a Kähler manifold (M^{2n}, J, ω) we have:

- $d\omega = 0$ and $\omega^n \neq 0 \hookrightarrow$ symplectic
- ω of type $(1,1)$, i.e. $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$
- $\omega > 0$
- $N_J = 0 \hookrightarrow$ complex

Definition (Gromov)

An almost cpx structure J on a symplectic manifold (M^{2n}, Ω) is tamed by Ω if $\Omega(X, JX) > 0$, $\forall X \neq 0$.

If J is tamed by Ω, then $g(X, Y) = \frac{1}{2}(\Omega(X, JY) - \Omega(JX, Y))$ is a J-Hermitian metric.

Theorem (Streets, Tian; Li, Zhang)

If a compact complex (M^4, J) admits a symplectic structure taming J, then (M^4, J) has a Kähler metric.

Problem

Does there exist an example of a compact complex (M^{2n}, J), with $n > 2$, admitting a symplectic form Ω taming J, but no Kähler structures?

We will give some negative answer to the problem by using that Ω tames $J \iff \partial\Omega^{1,1} = \overline{\partial}\beta$, for some ∂-closed $(2,0)$-form β.

$\hookrightarrow$ in particular $\omega = \Omega^{1,1}$ defines a pluriclosed metric.

Pluriclosed metrics

Definition

A Hermitian metric g on a complex manifold (M^{2n}, J) is called pluriclosed (or SKT) if

$$i\partial\overline{\partial}\omega = dd^c\omega = 0,$$

where $d^c = -J^{-1}dJ = -i(\overline{\partial} - \partial)$.

Remark

The pluriclosed condition is essentially the only weakening of the Kähler condition which is linear in the fundamental form!

Theorem (Gauduchon)

(M^{2n}, J, g) *compact Hermitian. Then* $\exists!$ $u \in \mathcal{C}^\infty(M^{2n})$ *such that*

$$\partial\overline{\partial}(e^{2u}\omega)^{n-1} = 0, \quad \int_{M^{2n}} u \, dV_g = 0.$$

$\hookrightarrow$ Every conformal hermitian structure on a compact complex (M^{2n}, J) contains an hermitian metric $\tilde{\omega}$ such that $\partial\overline{\partial}\tilde{\omega}^{n-1} = 0$
$\Rightarrow$ every compact complex surface admits pluriclosed metrics!

Theorem (Gauduchon)

On any Hermitian manifold (M^{2n}, J, g) there exists an affine line of canonical Hermitian connections ∇^τ ($\nabla^\tau J = 0$, $\nabla^\tau g = 0$), completely determined by their torsion

$$T(X, Y, Z) := g(T(X, Y), Z).$$

The family includes:

- the Chern connection ∇^C (T^C has trivial $(1,1)$-component)
- the Bismut (or Strominger) connection ∇^B (T^B is a 3-form)

Bismut and Chern connections

Remark

∇^C and ∇^B are related to the Levi-Civita connection ∇^{LC} by

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d^c\omega(X, Y, Z),$$
$$g(\nabla^C_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d\omega(JX, Y, Z).$$

Remark

- g is pluriclosed if and only if $dT^B = 0$.
- The trace of the torsion of ∇^C is equal to the Lee form

$\theta := Jd^*\omega$, which is the unique 1-form satisfying

$$d\omega^{n-1} = \theta \wedge \omega^{n-1}.$$

Even-dimensional compact Lie groups

$\mathfrak{t}^{\mathbb{C}}$:= Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$

• Left-invariant cpx structures J on G $\Longleftrightarrow$ pairs $(J_{\mathfrak{t}}, P)$, with $J_{\mathfrak{t}}$ any cpx structure on $\mathfrak{t}$ and $P \subseteq \Delta$ is a system of positive roots:

$$\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \bigoplus_{\alpha \in P} \mathfrak{g}_{\alpha}^{\mathbb{C}}$$

• Left-invariant pluriclosed metrics g on G are obtained by extending the negative of the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ to a J-compatible positive definite inner product:

$$\nabla^{LC}_X Y = \frac{1}{2}[X, Y], \qquad \nabla^{B}_X Y = 0, \quad X, Y \in \mathfrak{g},$$

with $T^B(X, Y, Z) = g([X, Y], Z)$ a closed 3-form!

Compact locally homogeneous spaces

Compact $(\Gamma\backslash G, J)$ with J invariant complex structure

• Classification results for the existence of pluriclosed metrics on nilmanifolds [F, Parton, Salamon; Enrietti, F, Vezzoni]

Conjecture: Every nilmanifold admitting a pluriclosed metric has to be 2-step and the total space of a holomorphic torus bundle over a torus!

• Classification results for the existence of pluriclosed metrics on solvmanifolds [F, Otal, Ugarte; F, Paradiso; Freibert, Swann]

Theorem (F, Tardini, Vezzoni)

The existence of a left-invariant pluriclosed metric on a unimodular Lie group G with a left-invariant abelian complex structure J forces the group G to be 2-step nilpotent.

Other examples which are not Bismut flat

• Characterization of the existence of pluriclosed metrics on Oeljkelaus-Toma (OT) manifolds $X(K, U) := \mathbb{H}^s \times \mathbb{C}^t / U \times \mathcal{O}_K$, where $\mathbb{Q} \subseteq K$ is an algebraic number field, $\mathcal{O}_K$ is the ring of algebraic integers of K and U is an admissible subgroup of the group of totally positive units $\mathcal{O}^{*,+}$ [Otiman].

• For any positive integer $k \geq 1$, $(k-1)(S^2 \times S^4) \#_k (S^3 \times S^3)$ has a pluriclosed metric [D. Grantcharov, G. Grantcharov, Y. Poon].

• Total spaces E of principal bundles over a projective manifold M with structure group an even dimensional unitary, special orthogonal or compact symplectic Lie group [Poddar, Takhur].

Blow-ups

Theorem (Blanchard)

The complex blow-up of a Kähler manifold (M, J, g) at a point p or along a compact complex submanifold Y is still Kähler.

Theorem (F, Tomassini)

The complex blow-up at a point or along a compact complex submanifold preserves the existence of pluriclosed metrics.

$\hookrightarrow$ resolutions of complex orbifolds with pluriclosed metrics.

Characterization in terms of currents

$\mathcal{D}^{p,q}(M) :=$ space of (p,q)-forms with cpt support on (M,J).

Definition

The space of currents of bi-dimension (p,q) or of bi-degree $(n-p,n-q)$ is the topological dual $\mathcal{D}'_{p,q}(M)$ of $\mathcal{D}^{p,q}(M)$.

A current of bi-dimension (p,q) on M can be locally identified with a $(n-p,n-q)$-form on M with coefficients distributions.

Definition

A current T of bi-dimension (p,p) is real if $T(\varphi) = T(\overline{\varphi})$, for any $\varphi \in \mathcal{D}^{p,p}(M)$.

Definition

A real $T \in \mathcal{D}'_{p,p}$ is positive if

$T(\frac{i^{p^2}}{2^p}\varphi^1 \wedge \ldots \wedge \varphi^p \wedge \overline{\varphi}^1 \wedge \ldots \wedge \overline{\varphi}^p) \geq 0$, for any $\varphi^j \in \mathcal{D}^{1,0}$.

T is strictly positive if

$\varphi^1 \wedge \ldots \wedge \varphi^p \neq 0 \Rightarrow T(\frac{i^{p^2}}{2^p}\varphi^1 \wedge \ldots \wedge \varphi^p \wedge \overline{\varphi}^1 \wedge \ldots \wedge \overline{\varphi}^p) > 0$, for any $\varphi^j \in \mathcal{D}^{1,0}$.

If $T \in \mathcal{D}'_{p,p}(M)$ is real, then $T = \frac{i^{(n-p)^2}}{2^{(n-p)}} \sum_{I,\bar{J}} T_{I\bar{J}} dz_I \wedge d\bar{z}_J$, where $T_{I\bar{J}}$ are distributions such that $T_{J\bar{I}} = \overline{T}_{I\bar{J}}$ and I, J are multi-indices of length $n-p$.

Theorem (Harvey, Lawson)

A compact (M, J) does not admit a Kähler metric $\iff$ (M, J) has a non-zero, positive current of bi-dimension $(1, 1)$ which is the $(1, 1)$- part of an exact current.

Theorem (Alessandrini, Bassanelli)

- *A compact (M, J) admits no symplectic forms taming J $\iff$ (M, J) has a positive, exact, non-zero current of bi-dimension $(1, 1)$.*
- *A compact (M, J) admits no pluriclosed metrics $\iff$ M admits a positive, non-zero, current of bi-dimension $(1, 1)$ which is $i\partial\overline{\partial}$-exact.*

An extension result

Theorem (Miyaoka)

If $M^{2n} \setminus \{p\}$ admits a Kähler metric, then there exists a Kähler metric on the complex manifold M^{2n}.

Theorem (F, Tomassini)

Let (M^{2n}, J), $n \geq 2$. If $M^{2n} \setminus \{p\}$ admits a pluriclosed metric, then there exists a pluriclosed metric on M^{2n}.

Remark

If ω is the fundamental form of a pluriclosed g on (M^{2n}, J), then ω corresponds to a real strictly positive current of bi-degree $(1,1)$ which is $\partial\overline{\partial}$-closed.

Sketch of the proof

It is sufficient to show that if ω is the fundamental 2-form of a pluriclosed metric on $\mathbb{B}^n(r) \setminus \{0\}$, $n \geq 2$, then $\exists\, 0 < R < r$ and $\hat{\omega} \in \Lambda^{1,1}(\mathbb{B}^n(R))$ such that
i) $\hat{\omega}$ is the fundamental 2-form of a pluriclosed metric on $\mathbb{B}^n(R)$;
ii) $\hat{\omega} = \omega$ on $\mathbb{B}^n(R) \setminus \mathbb{B}^n(\frac{2}{3}R)$.

Set $T = -\omega$ with $\omega =$ fundamental form of a pluriclosed metric on $\mathbb{B}^n(r) \setminus \{0\}$. We apply

Theorem (Alessandrini, Bassanelli)

Y analytic subset in $\Omega \subset \mathbb{C}^n$. If T is a plurisubharmonic, negative current of bi-dim (p,p) on $\Omega \setminus Y$ and $\dim_{\mathbb{C}} Y < p$, then $\exists$ the simple (or trivial) extension T^0 of T across Y and T^0 is plurisubharmonic.

$\hookrightarrow$ $T = -\omega$ can be extended as a current to $\mathbb{B}^n(r)$.

Set $\omega^0 = -T^0$.

Theorem (Siu; Bassanelli)

Let T be a current of bi-degree (h, k) on Ω. If T is of order 0 and $i\,\partial\overline{\partial} T = 0$, then, locally,

$$T = \partial G + \overline{\partial} H\,,$$

with G and H with locally integrable functions as coefficients.

Then

$$\omega^0 = \partial G + \overline{\partial G},$$

on $\mathbb{B}^n(R)$ for some $0 < R < r$, where G is a current of bi-degree $(0, 1)$. In fact, G is smooth on $\mathbb{B}^n(R) \setminus \{0\}$.

Finally, we can regularize G to obtain a $\partial\overline{\partial}$-closed and positive $(1, 1)$-form on $\mathbb{B}^n(R)$.

Some negative results

Theorem (Enrietti, F, Vezzoni)

A compact nilmanifold $M = \Gamma \backslash G$ with J invariant has a symplectic form taming $J \iff M$ is a torus.

Sketch of the proof:

• If (M, J) admits a pluriclosed metric, then J has to preserve the center ξ of $\mathfrak{g}$.

• We use that $\xi \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$ for a nilpotent Lie algebra.

Remark

For a solvable Lie algebra $(\mathfrak{g}, J)$ admitting a pluriclosed metric it is not true in general that J preserves the center ξ of $\mathfrak{g}$.

Compact nilmanifolds are in particular compact solvmanifolds of completely solvable type.

Definition

A compact solvmanifold $\Gamma\backslash G$ is completely solvable if the adjoint representation of the Lie algebra $\mathfrak{g}$ of G has only real eigenvalues.

Theorem (Baues, Cortes; Hasegawa)

A compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus.

Theorem (F, Kasuya)

A compact solvmanifold $(M = \Gamma\backslash G, J)$ of completely solvable type endowed with an invariant complex structure J admits a symplectic form Ω taming J if and only if M is a complex torus.

Sketch of the Proof:

• $\mathfrak{g}$ contains a nontrivial isotropic ideal $\mathfrak{h}$, i.e. $\Omega|_{\mathfrak{h}\times\mathfrak{h}} = 0$ [Baues, Cortes].

• If $\dim \mathfrak{h} = 1$, then the Lie algebra $\mathfrak{h}^{\perp_\Omega}/\mathfrak{h}$ admits a symplectic form $\tilde{\Omega}$ taming a complex structure $\tilde{J}$ and $\mathfrak{h}^{\perp_\Omega}/\mathfrak{h}$ is unimodular.

• By induction $\mathfrak{h}^{\perp_\Omega}/\mathfrak{h}$ is Kähler and so $\mathfrak{h}^{\perp_\Omega}/\mathfrak{h}$ is abelian.

Twistor space

M compact, anti-selfdual Riemannian 4-manifold ($W_+ = 0$)
$Tw(M) := S(\Lambda^+ M)$ the set of unit vectors in $\Lambda^+ M$, where

$$\Lambda^+ M := \{\alpha \in \Lambda^2 M = \mathfrak{so}(TM) \mid *\alpha = \alpha\}.$$

$\hookrightarrow$ the unit vectors $\alpha \in \Lambda^+ M$ correspond to oriented, orthogonal complex structures on $T_m M$.
At each point $(m, s) \in Tw(M)$, consider the decomposition

$$T_{(m,s)} Tw(M) = T_m M \oplus T_s S(\Lambda^+_m M),$$

induced by the Levi-Civita connection.

One can define

$$\mathcal{I}_{m,s} := I_s \oplus I_{S(\Lambda^+_m M)},$$

where I_s is the cpx structure on $T_m M$ induced by s and $I_{S(\lambda^*_m M)}$ is the cpx structure on $S(\Lambda^+_m M) = S^2$ induced by the metric and orientation.

Theorem (Verbitsky)

If the twistor space $(Tw(M), \mathcal{I})$ of a compact, anti-selfdual Riemannian manifold admits a pluriclosed metric, then $Tw(M)$ is Kähler, hence isomorphic to $\mathbb{CP}^3$ or a flag space.

The result is obtained from rational connectedness of the twistor space, due to F. Campana. Indeed, using this one can show that $Tw(M)$ is Moishezon $\hookrightarrow$ satisfies $\partial\bar{\partial}$-Lemma $\hookrightarrow$ admits a symplectic form taming the complex structure $\hookrightarrow$ a contradiction!

The pluriclosed flow

On a compact Kähler manifold (M, J, g) the Ricci flow

$$\partial_t g(t) = -Ric(g(t)), \quad g(0) = g,$$

preserves the Kähler condition ($\hookrightarrow$ Kähler Ricci flow) and reduces to a parabolic Monge-Ampére equation (Cao, Tian....).

Remark

For a non-Kähler manifold (M, J, g)

• the Levi-Civita connection does not not preserve the complex structure and the Ricci flow does not preserve the Hermitian condition!

• One may consider other connections preserving both the complex structure and the metric (e.g. the Bismut connection).

Let $(M^{2n}, J, g_0, \omega_0)$ be an Hermitian manifold. Streets and Tian introduced the geometric flow

$$\partial_t \omega(t) = -(\rho^B)^{1,1}(\omega(t)), \quad \omega(0) = \omega_0.$$

$\omega \to -(\rho^B)^{1,1}(\omega)$ is a real quasi-linear second-order elliptic operator when restricted to pluriclosed J-Hermitian metrics $\hookrightarrow$

Theorem (Streets, Tian)

Let (M^{2n}, J) be a compact complex manifold. If ω_0 is pluriclosed, then $\exists \epsilon > 0$ and a unique solution $\omega(t)$ to the pluriclosed flow with initial condition ω_0.
If ω_0 is Kähler, then $\omega(t)$ is the unique solution to the Kähler-Ricci flow with initial data ω_0.

Remark

In local cpx coordinates the pluriclosed flow can be written as:

$$\partial_t \omega(t) = \partial \partial^* \omega(t) + \overline{\partial \partial^*} \omega(t) + i\partial\overline{\partial} \log \det g(t).$$

Proposition (Streets, Tian)

If a pluriclosed metric ω on (M^{2n}, J) satisfies $(\rho^B)^{1,1} = \lambda\omega$, for a constant $\lambda \neq 0$, then $\omega = \Omega^{1,1}$ with Ω a symplectic form Ω taming the complex structure J.

Problem

- *Describe the maximal smooth existence time T.*
- *Study the limiting behavior at the time T.*

Consider the real $(1,1)$ Aeppli cohomology:

$$H^{1,1}_{\mathcal{A},\mathbb{R}} := \frac{\{\mathrm{Ker}\, i\partial\overline{\partial} : \Lambda^{1,1} \to \Lambda^{2,2}\}}{\{\partial\overline{\eta} + \overline{\partial}\eta \mid \eta \in \Lambda^{1,0}\}}.$$

$\hookrightarrow$ the (1, 1) Aeppli positive cone

$$\mathcal{P} := \{[\psi] \in H^{1,1}_{\mathcal{A},\mathbb{R}} \mid \exists \omega \in [\psi],\, \omega > 0\}.$$

consists precisely of the (1, 1) Aeppli classes represented by pluriclosed metrics.

Remark

For a general complex manifold (M^{2n}, J)

$$c_1(M^{2n}) \in H^{1,1}_{BC,\mathbb{R}} := \frac{\{\mathrm{Ker}\, d : \Lambda^{1,1} \to \Lambda^{2,2}\}}{\{i\partial\overline{\partial} f \mid f \in \mathcal{C}^\infty\}} \hookrightarrow H^{1,1}_{\mathcal{A},\mathbb{R}}.$$

As in the Kähler-Ricci flow case for the real $(1,1)$ Aeppli class: $[\omega(t)] = [\omega_0] - t\, c_1(M^{2n})$.

$\hookrightarrow$ The maximal smooth existence time T for the pluriclosed flow with initial condition g_0 satisfies:

$$T \leq \tau^*(\omega_0) := \sup\{t \geq 0 \mid [\omega_0] - t\, c_1(M^{2n}) \in \mathcal{P}\}.$$

Conjecture (Streets, Tian)

Let (M^{2n}, J, g_0) be a compact complex manifold with pluriclosed metric. The maximal smooth solution of pluriclosed flow with initial condition g_0 exists on $[0, \tau^*(\omega_0))$.

Nilpotent Lie groups case

For a Lie group G with left-invariant Hermitian structure (J, g), one may deform the Lie bracket instead of the Hermitian metric g

Theorem (Enrietti, F, Vezzoni)

The pluriclosed flow on a 2-step nilpotent simply-connected Lie group (G, J) starting from a left-invariant Hermitian metric g has a long-time solution.

The solutions converge in the Gromov-Hausdorff sense, after a suitable normalization, to self-similar solutions of the flow [Arroyo-Lafuente].

Bismut Kähler-like conditions

Remark

In general ∇^B does not satisfy the first Bianchi identity, since

$$\begin{aligned}\sigma_{X,Y,Z}\, R^B(X,Y,Z,U) &= dT^B(X,Y,Z,U) + (\nabla^B_U T^B)(X,Y,Z)\\ &\quad -\sigma_{X,Y,Z}\, g(T^B(X,Y), T^B(Z,U)).\end{aligned}$$

Definition

∇^B is Kähler-like if it satisfies the first Bianchi identity

$$\sigma_{X,Y,Z}\, R^B(X,Y,Z) = 0$$

and the type condition

$$R^B(X,Y,Z,W) = R^B(JX,JY,Z,W),\ \forall X,Y,Z,W.$$

Conjecture (Angella, Otal, Ugarte, Villacampa)

If for a Hermitian manifold (M^{2n}, J, g) the Bismut connection ∇^B is Kähler-like, then g is pluriclosed.

Theorem (Zhao, Zheng)

∇^B is Kähler-like $\iff$ g is pluriclosed and $\nabla^B T^B = 0$.

Problem

Study the behaviour of the Bismut Kähler-like condition along the pluriclosed flow.

Remark

If $n = 2$, then $T^B = - * \theta$.

Complex surfaces case

Definition

A Hermitian metric g on a complex manifold M^{2n} is a Vaisman metric if $d\omega = \theta \wedge \omega$, for some d-closed 1-form θ with $\nabla^{LC}\theta = 0$.

$\hookrightarrow$ Vaisman metrics are Gauduchon and $|\theta|$ is constant.

Theorem (F, Tardini)

Let (M^4, J) be a complex surface.
• A Hermitian metric g is Vaisman if and only if g is pluriclosed and ∇^B satisfies the first Bianchi identity.
• If (M^4, J) admits a Vaisman metric g_0 with constant scalar curvature, then pluriclosed flow starting with ω_0 preserves the Vaisman condition.

We use that, if (M^4, J, g) is a compact Vaisman surface, then $\rho^C = h\, dJ\theta$, for some $h \in \mathcal{C}^\infty(M^4)$. Moreover, Scal$(g)$ is constant if and only if h is constant and, in such a case $c_1(M^4) = 0$.

Nilpotent Lie group case

Remark

If a 6-dimensional nilpotent Lie group (G, J) admits a Bismut Kähler-like metric, then the left-invariant complex structure J has to be abelian.

Theorem (F, Tardini, Vezzoni)

Let (G, J, g_0) be a 2-step nilpotent Lie group with a left-invariant Bismut Kähler-like Hermitian structure and let $g(t)$ be the solution to the pluriclosed flow starting from g_0. Then $g(t)$ is Bismut Kähler-like for every t.