L^2 extension theorems and applications to algebraic geometry

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes & Académie des Sciences de Paris

Complex Analysis and Geometry – XXV
CIRM – ICTP virtual meeting, smr 3601
June 7–11, 2021

Second lecture

Second lecture: extension with optimal L^2 estimates

Setup. Let $L \to X$ be a holomorphic line bundle, equipped with a singular hermitian metric $h = h_0 e^{-\varphi}$, φ quasi-psh. Let $\psi \in L^1_{\rm loc}$ such that $\varphi + \psi$ is quasi-psh, and $Y \subset X$ the subvariety defined by the conductor ideal $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

Second lecture: extension with optimal L^2 estimates

Setup. Let $L \to X$ be a holomorphic line bundle, equipped with a singular hermitian metric $h = h_0 e^{-\varphi}$, φ quasi-psh. Let $\psi \in L^1_{\rm loc}$ such that $\varphi + \psi$ is quasi-psh, and $Y \subset X$ the subvariety defined by the conductor ideal $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

For a section $f \in H^0(Y, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, the goal is to get an "extension" $F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$,

$$\textit{via} \quad \mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi}), \quad F \mapsto f,$$

with an explicit L^2 estimate of F on X in terms of a suitable L^2 integral of f on the subvariety Y.

Second lecture: extension with optimal L^2 estimates

Setup. Let $L \to X$ be a holomorphic line bundle, equipped with a singular hermitian metric $h = h_0 e^{-\varphi}$, φ quasi-psh. Let $\psi \in L^1_{\rm loc}$ such that $\varphi + \psi$ is quasi-psh, and $Y \subset X$ the subvariety defined by the conductor ideal $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

For a section $f \in H^0(Y, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, the goal is to get an "extension" $F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$,

$$\textit{via}\ \ \mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi}),\quad F \mapsto f,$$

with an explicit L^2 estimate of F on X in terms of a suitable L^2 integral of f on the subvariety Y.

Additionally, it will be convenient to assume that X is weakly pseudoconvex (this is weaker than being holomorphically convex). This means that there exists a smooth psh exhaustion γ on X.

Second lecture: extension with optimal L^2 estimates

Setup. Let $L \to X$ be a holomorphic line bundle, equipped with a singular hermitian metric $h = h_0 e^{-\varphi}$, φ quasi-psh. Let $\psi \in L^1_{\rm loc}$ such that $\varphi + \psi$ is quasi-psh, and $Y \subset X$ the subvariety defined by the conductor ideal $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

For a section $f \in H^0(Y, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, the goal is to get an "extension" $F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$,

$$\textit{via} \quad \mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi}), \quad F \mapsto f,$$

with an explicit L^2 estimate of F on X in terms of a suitable L^2 integral of f on the subvariety Y.

Additionally, it will be convenient to assume that X is weakly pseudoconvex (this is weaker than being holomorphically convex). This means that there exists a smooth psh exhaustion γ on X.

We first define the Ohsawa residual measure associated with f. As for f, this will be a measure supported on Y.

The Ohsawa residual measure

Given $f \in H^0(U, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, there exists a Stein covering (U_i) of X and liftings $\widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$ of f on U_i via $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$. We obtain in this way a C^∞ extension $\widetilde{f} = \sum \xi_i \widetilde{f}_i$ where (ξ_i) is a partition of unity.

The Ohsawa residual measure

Given $f \in H^0(U, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, there exists a Stein covering (U_i) of X and liftings $\widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$ of f on U_i via $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$. We obtain in this way a C^∞ extension $\widetilde{f} = \sum \xi_i \widetilde{f}_i$ where (ξ_i) is a partition of unity.

Definition of the Ohsawa residual measure

For $g \in C_c(Y)$, $g \geq 0$, and $0 \leq \widetilde{g} \in C_c(X)$ extending g, we set

$$\int_Y g\, dV_Y[f^2, h, \psi] := \inf_{\widetilde{g}} \limsup_{t \to -\infty} \int_{\{t<\psi<t+1\}} \widetilde{g}\, |\widetilde{f}|^2_{\omega,h} e^{-\psi} dV_{X,\omega}.$$

The Ohsawa residual measure

Given $f \in H^0(U, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, there exists a Stein covering (U_i) of X and liftings $\widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$ of f on U_i via $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$. We obtain in this way a C^∞ extension $\widetilde{f} = \sum \xi_i \widetilde{f}_i$ where (ξ_i) is a partition of unity.

Definition of the Ohsawa residual measure

For $g \in C_c(Y)$, $g \geq 0$, and $0 \leq \widetilde{g} \in C_c(X)$ extending g, we set

$$\int_Y g\, dV_Y[f^2, h, \psi] := \inf_{\widetilde{g}} \limsup_{t \to -\infty} \int_{\{t < \psi < t+1\}} \widetilde{g}\, |\widetilde{f}|^2_{\omega,h} e^{-\psi} dV_{X,\omega}.$$

Proposition

$dV_Y[f^2, h, \psi]$ is independent of the choice of $\widetilde{f}$ as well as of ω, and defines a positive measure on Y (but not necessarily locally finite).

The Ohsawa residual measure

Given $f \in H^0(U, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, there exists a Stein covering (U_i) of X and liftings $\widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$ of f on U_i via $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$. We obtain in this way a C^∞ extension $\widetilde{f} = \sum \xi_i \widetilde{f}_i$ where (ξ_i) is a partition of unity.

Definition of the Ohsawa residual measure

For $g \in C_c(Y)$, $g \geq 0$, and $0 \leq \widetilde{g} \in C_c(X)$ extending g, we set

$$\int_Y g\, dV_Y[f^2, h, \psi] := \inf_{\widetilde{g}} \limsup_{t \to -\infty} \int_{\{t<\psi<t+1\}} \widetilde{g}\, |\widetilde{f}|^2_{\omega,h} e^{-\psi} dV_{X,\omega}.$$

Proposition

$dV_Y[f^2, h, \psi]$ is independent of the choice of $\widetilde{f}$ as well as of ω, and defines a positive measure on Y (but not necessarily locally finite).

Proof. When $\delta\widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi}))$, then $|\delta\widetilde{f}_i|^2_{\omega,h} e^{-\psi} \in L^1_{\mathrm{loc}}(X)$ and the lim sup $\to 0$ for $\mathrm{Supp}(\widetilde{g}) \subset U$.

The Ohsawa residual measure (2)

Example 1. Take $\psi(z) = r \log |s(z)|^2_{h_E}$, where $s \in H^0(X, E)$ and $r = \mathrm{rank}(E)$. Assume that $Y = s^{-1}(0)$ is of codimension r, that s is generically transverse to 0 on Y and $h \in C^\infty$. Then

$$dV_Y[f^2, h, \psi] = c_{n,r} \frac{|f|^2_{\omega,h}\, dV_{Y,\omega}}{|\Lambda^r(ds)|^2_{\omega,h_E}} \quad \text{on } Y \smallsetminus \{\Lambda^r(ds) = 0\}.$$

The Ohsawa residual measure (2)

Example 1. Take $\psi(z) = r \log |s(z)|^2_{h_E}$, where $s \in H^0(X, E)$ and $r = \mathrm{rank}(E)$. Assume that $Y = s^{-1}(0)$ is of codimension r, that s is generically transverse to 0 on Y and $h \in C^\infty$. Then

$$dV_Y[f^2, h, \psi] = c_{n,r} \frac{|f|^2_{\omega,h}\, dV_{Y,\omega}}{|\Lambda^r(ds)|^2_{\omega,h_E}} \quad \text{on } Y \smallsetminus \{\Lambda^r(ds) = 0\}.$$

Proof. Near a regular point z_0 be can pick a holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$ of E and coordinates $(z_1, \ldots, z_n)$ such that (e_λ) is h-orthornormal and $(\partial/\partial z_j)$ is ω-orthonormal at z_0, and $s(z) = \sum_{1\le j\le r} \lambda_j z_j e_j$, $\lambda_j \neq 0$. Then $\omega \sim i \sum dz_j \wedge d\overline{z}_j$ and $\psi(z) \sim r \log(|\lambda_1|^2|z_1|^2 + \ldots + |\lambda_r|^2|z_r|^2)$. This is an easy calculation of integrals on ellipsoids.

The Ohsawa residual measure (2)

Example 1. Take $\psi(z) = r \log |s(z)|^2_{h_E}$, where $s \in H^0(X, E)$ and $r = \mathrm{rank}(E)$. Assume that $Y = s^{-1}(0)$ is of codimension r, that s is generically transverse to 0 on Y and $h \in C^\infty$. Then

$$dV_Y[f^2, h, \psi] = c_{n,r} \frac{|f|^2_{\omega,h}\, dV_{Y,\omega}}{|\Lambda^r(ds)|^2_{\omega,h_E}} \quad \text{on } Y \smallsetminus \{\Lambda^r(ds) = 0\}.$$

Proof. Near a regular point z_0 be can pick a holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$ of E and coordinates $(z_1, \ldots, z_n)$ such that (e_λ) is h-orthornormal and $(\partial/\partial z_j)$ is ω-orthonormal at z_0, and $s(z) = \sum_{1\le j\le r} \lambda_j z_j e_j$, $\lambda_j \ne 0$. Then $\omega \sim i \sum dz_j \wedge d\overline{z}_j$ and $\psi(z) \sim r \log(|\lambda_1|^2|z_1|^2 + \ldots + |\lambda_r|^2|z_r|^2)$. This is an easy calculation of integrals on ellipsoids.

Example 2. Take now $\psi(z) = \sum c_j \log |s_{D_j}|^2_{h_j}$ where $D = \sum c_j D_j$ is a simple normal crossing divisor, $c_j > 0$, and h_j is a C^∞ metric on $\mathcal{O}_X(D_j)$. Also assume $h \in C^\infty$.

Ohsawa residual measure for s.n.c. singularities

By a change of coordinates, we are reduced to computing $dV_Y[f^2, h, \psi]$ for $\psi(z) = \sum c_j \log |z_j|^2 + u(z)$, $u \in C^\infty$. However

$$dV_Y[f^2, h, \psi + u] = e^{-u}\, dV_Y[f^2, h, \psi],$$

thus we may assume $u = 0$. At a regular point of $D_j \smallsetminus \bigcup_{k \neq j} D_k$, (and $j = 1$, say) we apply the Fubini theorem with $z = (z_1, z')$, $z' = (z_2, \ldots, z_n)$. We have to compute limits of the form

$$\lim_{t \to -\infty} \int_{e^t < |z_1|^{2c_1} < e^{t+1}} \frac{\widetilde{g}(z) |\widetilde{f}(z)|^2}{|z_1|^{2c_1}}\, i dz_1 \wedge d\overline{z}_1 = \frac{2\pi}{m_1} g(0, z') |\widetilde{h}(0, z')|^2$$

when $c_1 = m_1 \in \mathbb{N}^*$ and $\widetilde{f}(z) = z_1^{m_1 - 1} \widetilde{h}(z)$. However, if $c_j < 1$, we get 0, and in general, if $c_j \notin \mathbb{N}^*$ and $c_j > 1$, we can get only 0 or ∞ values, according to the divisibility of f by $z_j^{m_j - 1}$, $m_j = \lfloor c_j \rfloor \in \mathbb{N}^*$.

Ohsawa residual measure for s.n.c. singularities

By a change of coordinates, we are reduced to computing $dV_Y[f^2, h, \psi]$ for $\psi(z) = \sum c_j \log |z_j|^2 + u(z)$, $u \in C^\infty$. However

$$dV_Y[f^2, h, \psi + u] = e^{-u}\, dV_Y[f^2, h, \psi],$$

thus we may assume $u = 0$. At a regular point of $D_j \smallsetminus \bigcup_{k \neq j} D_k$, (and $j = 1$, say) we apply the Fubini theorem with $z = (z_1, z')$, $z' = (z_2, \ldots, z_n)$. We have to compute limits of the form

$$\lim_{t \to -\infty} \int_{e^t < |z_1|^{2c_1} < e^{t+1}} \frac{\widetilde{g}(z)|\widetilde{f}(z)|^2}{|z_1|^{2c_1}}\, idz_1 \wedge d\overline{z}_1 = \frac{2\pi}{m_1}\, g(0, z')|\widetilde{h}(0, z')|^2$$

when $c_1 = m_1 \in \mathbb{N}^*$ and $\widetilde{f}(z) = z_1^{m_1 - 1}\widetilde{h}(z)$. However, if $c_j < 1$, we get 0, and in general, if $c_j \notin \mathbb{N}^*$ and $c_j > 1$, we can get only 0 or ∞ values, according to the divisibility of f by $z_j^{m_j - 1}$, $m_j = \lfloor c_j \rfloor \in \mathbb{N}^*$.

As a consequence, we can capture an interesting (i.e. locally finite, non zero) residual measure $dV_Y[f^2, h, \psi]$ only in the case where one of the coefficients c_j is an integer.

Ohsawa residual measure for analytic singularities

One general case of interests is when ψ has analytic singularities, i.e. locally $\psi(z) = c \log \sum |g_j(z)|^2 + u(z)$, $g_j \in \mathcal{O}_X(V)$, $u \in C^\infty(V)$.

Ohsawa residual measure for analytic singularities

One general case of interests is when ψ has analytic singularities, i.e. locally $\psi(z) = c \log \sum |g_j(z)|^2 + u(z)$, $g_j \in \mathcal{O}_X(V)$, $u \in C^\infty(V)$.

Then, it is interesting to look at the family of multiplier ideal sheaves $\mathcal{I}(e^{-s\psi})$ when $s \in \mathbb{R}_+$, which decrease as s increases. Assume without loss of generality that $c = 1$.

Ohsawa residual measure for analytic singularities

One general case of interests is when ψ has analytic singularities, i.e. locally $\psi(z) = c\log\sum|g_j(z)|^2 + u(z)$, $g_j \in \mathcal{O}_X(V)$, $u \in C^\infty(V)$.

Then, it is interesting to look at the family of multiplier ideal sheaves $\mathcal{I}(e^{-s\psi})$ when $s \in \mathbb{R}_+$, which decrease as s increases. Assume without loss of generality that $c = 1$.

By Hironaka, we know that there exists a composition of blow-ups $\mu : \widetilde{X} \to X$ such that the pull-back ideal $\mu^*(g_j) = (g_j \circ \mu)$ is an invertible ideal sheaf $\mathcal{O}_{\widetilde{X}}(-\sum m_j D_j)$ associated with a simple normal crossing divisor.

Ohsawa residual measure for analytic singularities

One general case of interests is when ψ has analytic singularities, i.e. locally $\psi(z) = c\log\sum|g_j(z)|^2 + u(z)$, $g_j \in \mathcal{O}_X(V)$, $u \in C^\infty(V)$.

Then, it is interesting to look at the family of multiplier ideal sheaves $\mathcal{I}(e^{-s\psi})$ when $s \in \mathbb{R}_+$, which decrease as s increases. Assume without loss of generality that $c = 1$.

By Hironaka, we know that there exists a composition of blow-ups $\mu : \widetilde{X} \to X$ such that the pull-back ideal $\mu^*(g_j) = (g_j \circ \mu)$ is an invertible ideal sheaf $\mathcal{O}_{\widetilde{X}}(-\sum m_j D_j)$ associated with a simple normal crossing divisor. The direct image formula implies

$$\mathcal{I}(e^{-s\psi}) = \mu_*(K_{\widetilde{X}/X} \otimes \mathcal{I}(e^{-s\,\psi\circ\mu})) = \mu_*\mathcal{O}_{\widetilde{X}}\Big(\sum(a_j - \lfloor sm_j \rfloor)D_j\Big)$$

where $K_{\widetilde{X}/X} = \mathcal{O}_{\widetilde{X}}(\sum a_j D_j)$.

Ohsawa residual measure for analytic singularities

One general case of interests is when ψ has analytic singularities, i.e. locally $\psi(z) = c \log \sum |g_j(z)|^2 + u(z)$, $g_j \in \mathcal{O}_X(V)$, $u \in C^\infty(V)$.

Then, it is interesting to look at the family of multiplier ideal sheaves $\mathcal{I}(e^{-s\psi})$ when $s \in \mathbb{R}_+$, which decrease as s increases. Assume without loss of generality that $c = 1$.

By Hironaka, we know that there exists a composition of blow-ups $\mu : \widetilde{X} \to X$ such that the pull-back ideal $\mu^*(g_j) = (g_j \circ \mu)$ is an invertible ideal sheaf $\mathcal{O}_{\widetilde{X}}(-\sum m_j D_j)$ associated with a simple normal crossing divisor. The direct image formula implies

$$\mathcal{I}(e^{-s\psi}) = \mu_*(K_{\widetilde{X}/X} \otimes \mathcal{I}(e^{-s\,\psi\circ\mu})) = \mu_* \mathcal{O}_{\widetilde{X}}\Big(\sum (a_j - \lfloor s m_j \rfloor) D_j\Big)$$

where $K_{\widetilde{X}/X} = \mathcal{O}_{\widetilde{X}}(\sum a_j D_j)$. This implies that $\mathcal{I}(e^{-s\psi})$ "jumps" precisely for a discrete sequence of rational numbers $0 = s_0 < s_1 < \ldots < s_k < \ldots$ such that $s_k m_j \in \mathbb{N}$ for some j.

Ohsawa residual measure for analytic singularities

One general case of interests is when ψ has analytic singularities, i.e. locally $\psi(z) = c \log \sum |g_j(z)|^2 + u(z)$, $g_j \in \mathcal{O}_X(V)$, $u \in C^\infty(V)$.

Then, it is interesting to look at the family of multiplier ideal sheaves $\mathcal{I}(e^{-s\psi})$ when $s \in \mathbb{R}_+$, which decrease as s increases. Assume without loss of generality that $c = 1$.

By Hironaka, we know that there exists a composition of blow-ups $\mu : \widetilde{X} \to X$ such that the pull-back ideal $\mu^*(g_j) = (g_j \circ \mu)$ is an invertible ideal sheaf $\mathcal{O}_{\widetilde{X}}(-\sum m_j D_j)$ associated with a simple normal crossing divisor. The direct image formula implies

$$\mathcal{I}(e^{-s\psi}) = \mu_*(K_{\widetilde{X}/X} \otimes \mathcal{I}(e^{-s\,\psi\circ\mu})) = \mu_*\mathcal{O}_{\widetilde{X}}\Big(\sum (a_j - \lfloor s m_j \rfloor) D_j\Big)$$

where $K_{\widetilde{X}/X} = \mathcal{O}_{\widetilde{X}}(\sum a_j D_j)$. This implies that $\mathcal{I}(e^{-s\psi})$ "jumps" precisely for a discrete sequence of rational numbers $0 = s_0 < s_1 < \ldots < s_k < \ldots$ such that $s_k m_j \in \mathbb{N}$ for some j.

For $f \in \mathcal{I}(e^{-s_{k-1}\psi})$, the measure $dV_Y[f^2, h, s_k\psi]$ will be interesting.

Restricted multiplier ideals

We first have to introduce a suitable sheaf of integrable functions on the subvariety Y associated with $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

Restricted multiplier ideals

We first have to introduce a suitable sheaf of integrable functions on the subvariety Y associated with $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

Definition of the restricted multiplier ideal

For $x \in Y$, we define $\mathcal{I}'_\psi(h)_x \subset \mathcal{I}(h)_x$ to be the ideal of germs of functions $\widetilde{f} \in \mathcal{I}(h)_x$ associated with $f = \widetilde{f} \mod \mathcal{I}(he^{-\psi})_x$ in $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})_x$, for which $dV[f^2, h, \psi]$ is locally finite near x on Y.

Restricted multiplier ideals

We first have to introduce a suitable sheaf of integrable functions on the subvariety Y associated with $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

Definition of the restricted multiplier ideal

For $x \in Y$, we define $\mathcal{I}'_\psi(h)_x \subset \mathcal{I}(h)_x$ to be the ideal of germs of functions $\widetilde{f} \in \mathcal{I}(h)_x$ associated with $f = \widetilde{f} \mod \mathcal{I}(he^{-\psi})_x$ in $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})_x$, for which $dV[f^2, h, \psi]$ is locally finite near x on Y.

Clearly, $\mathcal{I}(he^{-\psi}) \subset \mathcal{I}'_\psi(h) \subset \mathcal{I}(h)$.

Restricted multiplier ideals

We first have to introduce a suitable sheaf of integrable functions on the subvariety Y associated with $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

Definition of the restricted multiplier ideal

For $x \in Y$, we define $\mathcal{I}'_\psi(h)_x \subset \mathcal{I}(h)_x$ to be the ideal of germs of functions $\widetilde{f} \in \mathcal{I}(h)_x$ associated with $f = \widetilde{f} \mod \mathcal{I}(he^{-\psi})_x$ in $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})_x$, for which $dV[f^2, h, \psi]$ is locally finite near x on Y.
Clearly, $\mathcal{I}(he^{-\psi}) \subset \mathcal{I}'_\psi(h) \subset \mathcal{I}(h)$.

Typical case of application. Assume that $h = e^{-\varphi}$ and ψ have analytic singularities, and that $s_k = 1$ is one of jumping values for $s \mapsto \mathcal{I}(e^{-s\psi})$ (case of log canonical singularities: $s_1 = 1$).

Restricted multiplier ideals

We first have to introduce a suitable sheaf of integrable functions on the subvariety Y associated with $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

Definition of the restricted multiplier ideal

For $x \in Y$, we define $\mathcal{I}'_\psi(h)_x \subset \mathcal{I}(h)_x$ to be the ideal of germs of functions $\widetilde{f} \in \mathcal{I}(h)_x$ associated with $f = \widetilde{f} \mod \mathcal{I}(he^{-\psi})_x$ in $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})_x$, for which $dV[f^2, h, \psi]$ is locally finite near x on Y. Clearly, $\mathcal{I}(he^{-\psi}) \subset \mathcal{I}'_\psi(h) \subset \mathcal{I}(h)$.

Typical case of application. Assume that $h = e^{-\varphi}$ and ψ have analytic singularities, and that $s_k = 1$ is one of jumping values for $s \mapsto \mathcal{I}(e^{-s\psi})$ (case of log canonical singularities: $s_1 = 1$).

Then $\mathcal{I}'_\psi(h) \subset \mathcal{I}(he^{-s_{k-1}\psi})$ on X, and $\mathcal{I}'_\psi(h) = \mathcal{I}(he^{-s_{k-1}\psi})$ on a Zariski open subset $X_0 = X \smallsetminus Z$, $Z \subsetneq Y$ (however, the ideals may differ on Z).

Use of more "flexible" weights

The next issue is that we need special and rather flexible weights. Let $\alpha \in]0,1[$ and $A = \sup_X \psi \in]-\infty,+\infty]$. We consider functions $\rho : [-\infty, A] \to \mathbb{R}_+^*$, such as

$$\rho(u) = 1 - (A + 1 + \alpha^{-1/2} - u)^{-1},$$

that are continuous strictly decreasing, with the property that ρ is concave near $-\infty$.

Use of more “flexible” weights

The next issue is that we need special and rather flexible weights. Let $\alpha \in \,]0,1[$ and $A = \sup_X \psi \in \,]-\infty,+\infty]$. We consider functions $\rho : [-\infty, A] \to \mathbb{R}_+^*$, such as

$$\rho(u) = 1 - (A + 1 + \alpha^{-1/2} - u)^{-1},$$

that are continuous strictly decreasing, with the property that ρ is concave near $-\infty$.

We assume moreover that

$$\int_t^A \rho(u)\,du + \frac{\rho(A)}{\alpha} \leq \frac{\rho(t)^2}{|\rho'(t)|} \quad \text{for all } t \in \,]-\infty, A].$$

Use of more "flexible" weights

The next issue is that we need special and rather flexible weights. Let $\alpha \in]0,1[$ and $A = \sup_X \psi \in]-\infty,+\infty]$. We consider functions $\rho : [-\infty, A] \to \mathbb{R}_+^*$, such as

$$\rho(u) = 1 - (A + 1 + \alpha^{-1/2} - u)^{-1},$$

that are continuous strictly decreasing, with the property that ρ is concave near $-\infty$.

We assume moreover that

$$\int_t^A \rho(u)\,du + \frac{\rho(A)}{\alpha} \le \frac{\rho(t)^2}{|\rho'(t)|} \quad \text{for all } t \in]-\infty, A].$$

The L^2 estimates will involve integrals of the form $\int_X |F|^2_{\omega,h} e^{-\psi}|\rho'(\psi)|\,dV_{X,\omega}$, where $|\rho'(\psi)| = (C-\psi)^{-2}$ in the above example, so that $e^{-\psi}|\rho'(\psi)|$ is locally sommable when ψ has log canonical singularities.

General L^2 extension theorem

Theorem (X. Zhou-L. Zhu 2019)

Let (X, ω) be a weakly pseudoconvex Kähler manifold, L a holomorphic line bundle with a hermitian metric $h = h_0 e^{-\varphi}$, $h_0 \in C^\infty$, φ quasi-psh on X, and $\psi \in L^1_{\mathrm{loc}}(X)$.

General L^2 extension theorem

Theorem (X. Zhou-L. Zhu 2019)

Let (X, ω) be a weakly pseudoconvex Kähler manifold, L a holomorphic line bundle with a hermitian metric $h = h_0 e^{-\varphi}$, $h_0 \in C^\infty$, φ quasi-psh on X, and $\psi \in L^1_{\mathrm{loc}}(X)$. Assume $\exists \alpha > 0$ constant such that

$$\Theta_{L,h} + (1 + \nu\alpha) i\partial\overline{\partial}\psi \geq 0 \quad \text{on } X, \quad \nu = 0, 1.$$

General L^2 extension theorem

Theorem (X. Zhou-L. Zhu 2019)

Let (X, ω) be a weakly pseudoconvex Kähler manifold, L a holomorphic line bundle with a hermitian metric $h = h_0 e^{-\varphi}$, $h_0 \in C^\infty$, φ quasi-psh on X, and $\psi \in L^1_{\rm loc}(X)$. Assume $\exists \alpha > 0$ constant such that

$$\Theta_{L,h} + (1 + \nu\alpha) i\partial\overline{\partial}\psi \geq 0 \quad \text{on } X, \quad \nu = 0, 1.$$

Then, for every $f \in H^0(Y, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}'_\psi(h)/\mathcal{I}(he^{-\psi}))$ s.t.

$$\int_Y dV_Y[f^2, h, \psi] < +\infty,$$

there exists $F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}'_\psi(h)$ that is mapped to f by the morphism $\mathcal{I}'_\psi(h) \to \mathcal{I}'_\psi(h)/\mathcal{I}(he^{-\psi})$, such that

$$\int_X |F|^2_{\omega,h} e^{-\psi} |\rho'(\psi)|\, dV_{X,\omega} \leq \rho(-\infty) \int_Y dV_Y[f^2, h, \psi].$$

(1) Construction of a smooth extension

Every section $f \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$ admits a C^∞ lifting

$$\widetilde{f} = \sum \xi_i \widetilde{f}_i, \quad \widetilde{f}_i \in H^0\big(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

by means of a Stein covering (U_i) of X and a partition of unity (ξ_i) subordinate to (U_i).

(1) Construction of a smooth extension

Every section $f \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$ admits a C^∞ lifting

$$\widetilde{f} = \sum \xi_i \widetilde{f}_i, \quad \widetilde{f}_i \in H^0\big(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

by means of a Stein covering (U_i) of X and a partition of unity (ξ_i) subordinate to (U_i).

Since $\sum \overline{\partial}\xi_i = 0$, we have $\overline{\partial}\widetilde{f} = \sum \overline{\partial}\xi_i(\widetilde{f}_i - \widetilde{f}_j)$ on U_j, and since $\widetilde{f}_i - \widetilde{f}_j$ has coefficients in $\mathcal{I}(he^{-\psi})$, we see that $\overline{\partial}\widetilde{f}$ is valued in

$$\mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi}) \otimes_{\mathcal{O}_X} C^\infty.$$

(1) Construction of a smooth extension

Every section $f \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$ admits a C^∞ lifting

$$\widetilde{f} = \sum \xi_i \widetilde{f}_i, \quad \widetilde{f}_i \in H^0\big(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

by means of a Stein covering (U_i) of X and a partition of unity (ξ_i) subordinate to (U_i).

Since $\sum \overline{\partial}\xi_i = 0$, we have $\overline{\partial}\widetilde{f} = \sum \overline{\partial}\xi_i(\widetilde{f}_i - \widetilde{f}_j)$ on U_j, and since $\widetilde{f}_i - \widetilde{f}_j$ has coefficients in $\mathcal{I}(he^{-\psi})$, we see that $\overline{\partial}\widetilde{f}$ is valued in

$$\mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi}) \otimes_{\mathcal{O}_X} C^\infty.$$

As X is assumed to be weakly pseudoconvex, we can consider $X_c = \{z \in X\,;\, \gamma(z) < c\} \Subset X$, $\forall c \in \mathbb{R}$, and get by compactness

$$\int_{X_c} |\overline{\partial}\widetilde{f}|^2_{\omega,h} e^{-\psi} dV_{X,\omega} < +\infty.$$

(1) Construction of a smooth extension

Every section $f \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$ admits a C^∞ lifting

$$\widetilde{f} = \sum \xi_i \widetilde{f}_i, \quad \widetilde{f}_i \in H^0\big(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

by means of a Stein covering (U_i) of X and a partition of unity (ξ_i) subordinate to (U_i).

Since $\sum \overline{\partial}\xi_i = 0$, we have $\overline{\partial}\widetilde{f} = \sum \overline{\partial}\xi_i(\widetilde{f}_i - \widetilde{f}_j)$ on U_j, and since $\widetilde{f}_i - \widetilde{f}_j$ has coefficients in $\mathcal{I}(he^{-\psi})$, we see that $\overline{\partial}\widetilde{f}$ is valued in

$$\mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi}) \otimes_{\mathcal{O}_X} C^\infty.$$

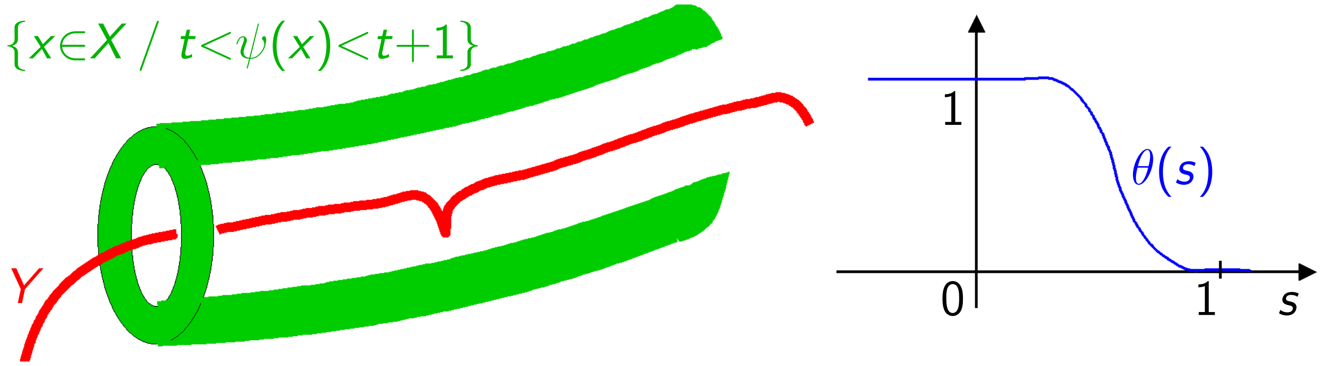
As X is assumed to be weakly pseudoconvex, we can consider $X_c = \{z \in X\,;\, \gamma(z) < c\} \Subset X$, $\forall c \in \mathbb{R}$, and get by compactness

$$\int_{X_c} |\overline{\partial}\widetilde{f}|^2_{\omega,h} e^{-\psi} dV_{X,\omega} < +\infty.$$

It will be enough to get estimates on X_c, and then let $c \to +\infty$.

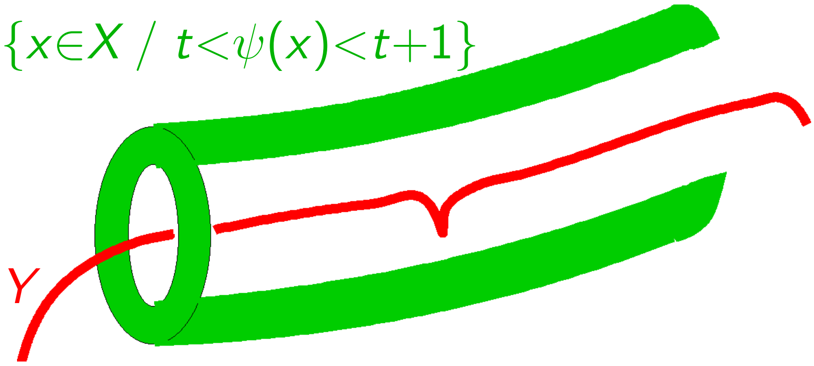
(2) Solving the $\overline{\partial}$ equation

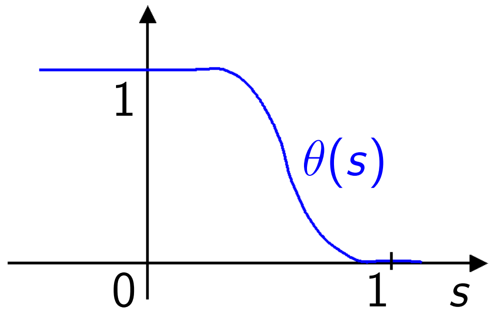
The next idea is to truncate $\widetilde{f}$ by multiplying $\widetilde{f}$ with a cut-off function $\theta(\psi - t)$ equal to 1 near $Y \subset \psi^{-1}(-\infty)$.



(2) Solving the $\overline{\partial}$ equation

The next idea is to truncate $\widetilde{f}$ by multiplying $\widetilde{f}$ with a cut-off function $\theta(\psi - t)$ equal to 1 near $Y \subset \psi^{-1}(-\infty)$.





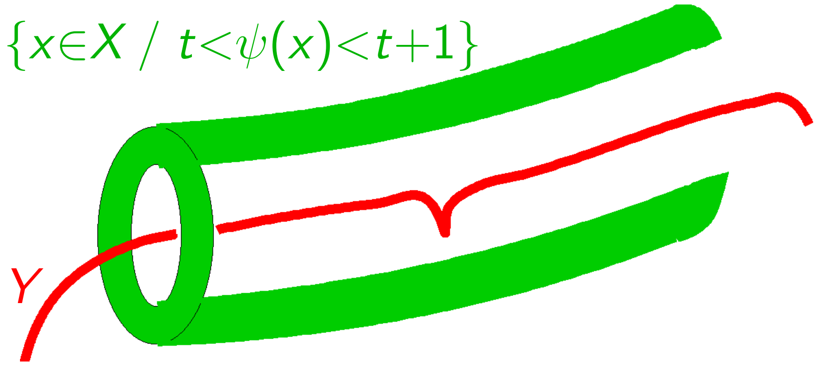
We next solve the approximate $\overline{\partial}$-equation

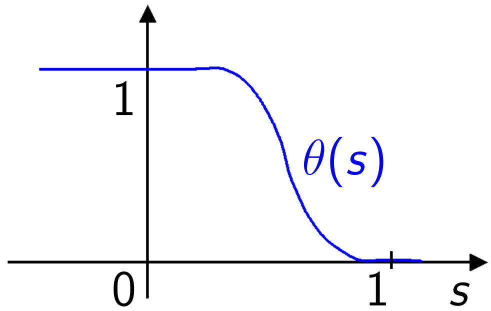
$$(*) \qquad \overline{\partial} u_{t,\varepsilon} = v_t + w_{t,\varepsilon}$$

with $v_t := \overline{\partial}(\theta(\psi - t) \cdot \widetilde{f}) = \theta(\psi - t) \cdot \overline{\partial}\widetilde{f} + \theta'(\psi - t)\overline{\partial}\psi \wedge \widetilde{f}$.

(2) Solving the $\overline{\partial}$ equation

The next idea is to truncate $\widetilde{f}$ by multiplying $\widetilde{f}$ with a cut-off function $\theta(\psi - t)$ equal to 1 near $Y \subset \psi^{-1}(-\infty)$.





We next solve the approximate $\overline{\partial}$-equation

$(*)$ $\qquad \overline{\partial} u_{t,\varepsilon} = v_t + w_{t,\varepsilon}$

with $v_t := \overline{\partial}(\theta(\psi - t) \cdot \widetilde{f}) = \theta(\psi - t) \cdot \overline{\partial}\widetilde{f} + \theta'(\psi - t)\overline{\partial}\psi \wedge \widetilde{f}$.

It the weights ψ and φ of $h = h_0 e^{-\varphi}$ are not smooth, we use regularizations $\varphi_\delta \downarrow \varphi$, $\psi_\delta \downarrow \psi$ and complete Kähler metrics $\omega_\delta \downarrow \omega$ on $X \smallsetminus Z_\delta$. (We omit details here).

(3) L^2 estimates for solution and error term

The existence theorem with twisting factors $\eta_{t,\varepsilon}$, $\lambda_{t,\varepsilon}$ yields

$$\int_{X_c}(\eta_{t,\varepsilon}+\lambda_{t,\varepsilon})^{-1}|u_{t,\varepsilon}|^2_{\omega,h_0}e^{-\varphi-\psi}dV_{X,\omega}+\frac{1}{\varepsilon}\int_{X_c}|w_{t,\varepsilon}|^2_{\omega,h_0}\,e^{-\varphi-\psi}dV_{X,\omega}$$
$$\le 4\int_{X_c\cap\{\psi<t+1\}}|\overline{\partial}\widetilde{f}|^2_{\omega,h_0}\,e^{-\varphi-\psi}dV_\omega$$
$$+\,4\int_{X_c\cap\{t<\psi<t+1\}}\langle(B_t+\varepsilon\,\mathrm{Id})^{-1}\overline{\partial}\psi\wedge\widetilde{f},\overline{\partial}\psi\wedge\widetilde{f}\rangle_{\omega,h_0}\,e^{-\varphi-\psi}.$$

(3) L^2 estimates for solution and error term

The existence theorem with twisting factors $\eta_{t,\varepsilon}$, $\lambda_{t,\varepsilon}$ yields

$$\begin{aligned}\int_{X_c} (\eta_{t,\varepsilon} + \lambda_{t,\varepsilon})^{-1} |u_{t,\varepsilon}|^2_{\omega,h_0} e^{-\varphi-\psi} dV_{X,\omega} + \frac{1}{\varepsilon}\int_{X_c} |w_{t,\varepsilon}|^2_{\omega,h_0}\, e^{-\varphi-\psi} dV_{X,\omega} \\ \leq 4 \int_{X_c\cap\{\psi<t+1\}} |\overline{\partial}\widetilde{f}|^2_{\omega,h_0}\, e^{-\varphi-\psi} dV_\omega \\ + 4 \int_{X_c\cap\{t<\psi<t+1\}} \langle (B_t + \varepsilon\,\mathrm{Id})^{-1}\overline{\partial}\psi \wedge \widetilde{f}, \overline{\partial}\psi \wedge \widetilde{f}\rangle_{\omega,h_0}\, e^{-\varphi-\psi}.\end{aligned}$$

The first integral in the right hand side tends to 0 as $t \to -\infty$.

(3) L^2 estimates for solution and error term

The existence theorem with twisting factors $\eta_{t,\varepsilon}$, $\lambda_{t,\varepsilon}$ yields

$$\int_{X_c}(\eta_{t,\varepsilon}+\lambda_{t,\varepsilon})^{-1}|u_{t,\varepsilon}|^2_{\omega,h_0}e^{-\varphi-\psi}dV_{X,\omega}+\frac{1}{\varepsilon}\int_{X_c}|w_{t,\varepsilon}|^2_{\omega,h_0}\,e^{-\varphi-\psi}dV_{X,\omega}$$
$$\le 4\int_{X_c\cap\{\psi<t+1\}}|\overline\partial\widetilde f|^2_{\omega,h_0}\,e^{-\varphi-\psi}dV_\omega$$
$$+4\int_{X_c\cap\{t<\psi<t+1\}}\langle(B_t+\varepsilon\,\mathrm{Id})^{-1}\overline\partial\psi\wedge\widetilde f,\overline\partial\psi\wedge\widetilde f\rangle_{\omega,h_0}\,e^{-\varphi-\psi}.$$

The first integral in the right hand side tends to 0 as $t\to-\infty$.

Again, the main point is to choose ad hoc factors η_t, λ_t, and we want here the last integral to converge to a finite limit.

(3) L^2 estimates for solution and error term

The existence theorem with twisting factors $\eta_{t,\varepsilon}$, $\lambda_{t,\varepsilon}$ yields

$$\int_{X_c}(\eta_{t,\varepsilon}+\lambda_{t,\varepsilon})^{-1}|u_{t,\varepsilon}|^2_{\omega,h_0}e^{-\varphi-\psi}dV_{X,\omega}+\frac{1}{\varepsilon}\int_{X_c}|w_{t,\varepsilon}|^2_{\omega,h_0}\,e^{-\varphi-\psi}dV_{X,\omega}$$

$$\leq 4\int_{X_c\cap\{\psi<t+1\}}|\overline{\partial}\widetilde{f}|^2_{\omega,h_0}\,e^{-\varphi-\psi}dV_\omega$$

$$+\,4\int_{X_c\cap\{t<\psi<t+1\}}\langle(B_t+\varepsilon\,\mathrm{Id})^{-1}\overline{\partial}\psi\wedge\widetilde{f},\overline{\partial}\psi\wedge\widetilde{f}\rangle_{\omega,h_0}\,e^{-\varphi-\psi}.$$

The first integral in the right hand side tends to 0 as $t\to-\infty$.

Again, the main point is to choose ad hoc factors η_t, λ_t, and we want here the last integral to converge to a finite limit. One can check that this works with

$$\zeta(u)=\log\frac{\rho(-\infty)}{\rho(u)},\quad \chi(u)=\frac{\int_u^A\rho(v)dv+\frac{1}{\alpha\rho(A)}}{\rho(u)},\quad \beta=\frac{(\chi')^2}{\chi\zeta''-\chi''},$$

$$\sigma_{t,\varepsilon}(u)=\max\nolimits_\varepsilon(u,t),\quad \eta_{t,\varepsilon}=\chi(\sigma_{t,\varepsilon}(\psi)),\quad \lambda_{t,\varepsilon}=\beta(\sigma_{t,\varepsilon}(\psi)).$$

Extension from hypersurface (Stein case)

In the hypersurface case, one gets the following simpler statement.

Theorem

Let X be a Stein manifold of dimension n. Let φ and ψ be plurisubharmonic functions on X. Assume that w is a holomorphic function on X such that $\sup_X(\psi + 2\log|w|) \le 0$ and dw does not vanish identically on any branch of $w^{-1}(0)$.

Extension from hypersurface (Stein case)

In the hypersurface case, one gets the following simpler statement.

Theorem

Let X be a Stein manifold of dimension n. Let φ and ψ be plurisubharmonic functions on X. Assume that w is a holomorphic function on X such that $\sup_X(\psi + 2\log|w|) \leq 0$ and dw does not vanish identically on any branch of $w^{-1}(0)$.

Denote $Y = w^{-1}(0)$ and $Y_0 = \{x \in Y : dw(x) \neq 0\}$.

Extension from hypersurface (Stein case)

In the hypersurface case, one gets the following simpler statement.

Theorem

Let X be a Stein manifold of dimension n. Let φ and ψ be plurisubharmonic functions on X. Assume that w is a holomorphic function on X such that $\sup_X(\psi + 2\log|w|) \le 0$ and dw does not vanish identically on any branch of $w^{-1}(0)$.

Denote $Y = w^{-1}(0)$ and $Y_0 = \{x \in Y : dw(x) \neq 0\}$.

Then for any holomorphic $(n-1)$-form f on Y_0 satisfying

$$\int_{Y_0} \mathrm{e}^{-\varphi-\psi} i^{(n-1)^2} f \wedge \bar{f} < +\infty,$$

there exists a holomorphic n-form F on X satisfying $F_{|Y_0} = dw \wedge f$ and an optimal estimate

$$\int_X \mathrm{e}^{-\varphi} i^{n^2} F \wedge \bar{F} \le 2\pi \int_{Y_0} \mathrm{e}^{-\varphi-\psi} i^{(n-1)^2} f \wedge \bar{f}.$$

The Suita conjecture

The Suita conjecture was posed originally on open Riemann surfaces in 1972. The motivation was to answer a question posed by Sario and Oikawa about the relation between the Bergman kernel B_Ω for holomorphic $(1,0)$ forms on an open Riemann surface Ω which admits a Green function G_Ω.

The Suita conjecture

The Suita conjecture was posed originally on open Riemann surfaces in 1972. The motivation was to answer a question posed by Sario and Oikawa about the relation between the Bergman kernel B_Ω for holomorphic $(1,0)$ forms on an open Riemann surface Ω which admits a Green function G_Ω.

Recall that the logarithmic capacity $c_\beta(z)$ is locally defined by

$$c_\beta(z) = \exp \lim_{\xi \to z} \big(G_\Omega(\xi, z) - \log|\xi - z| \big) \quad \text{on } \Omega.$$

Suita conjecture

$(c_\beta(z))^2 |dz|^2 \leq \pi B_\Omega(z)$, for every $z \in \Omega$.

The Suita conjecture

The Suita conjecture was posed originally on open Riemann surfaces in 1972. The motivation was to answer a question posed by Sario and Oikawa about the relation between the Bergman kernel B_Ω for holomorphic $(1,0)$ forms on an open Riemann surface Ω which admits a Green function G_Ω.

Recall that the logarithmic capacity $c_\beta(z)$ is locally defined by

$$c_\beta(z) = \exp \lim_{\xi \to z} \big(G_\Omega(\xi, z) - \log|\xi - z| \big) \ \text{ on } \Omega.$$

Suita conjecture

$(c_\beta(z))^2 |dz|^2 \leq \pi B_\Omega(z)$, for every $z \in \Omega$.

Theorem

The Suita conjecture holds true (planar case: Błocki 2013; general case: Guan-Zhou 2014).

The Suita conjecture

The Suita conjecture was posed originally on open Riemann surfaces in 1972. The motivation was to answer a question posed by Sario and Oikawa about the relation between the Bergman kernel B_Ω for holomorphic $(1,0)$ forms on an open Riemann surface Ω which admits a Green function G_Ω.

Recall that the logarithmic capacity $c_\beta(z)$ is locally defined by

$$c_\beta(z) = \exp \lim_{\xi \to z} \big(G_\Omega(\xi, z) - \log |\xi - z| \big) \quad \text{on } \Omega.$$

Suita conjecture

$(c_\beta(z))^2 |dz|^2 \leq \pi B_\Omega(z)$, for every $z \in \Omega$.

Theorem

The Suita conjecture holds true (planar case: Błocki 2013; general case: Guan-Zhou 2014). Moreover (Guan-Zhou 2014), equality holds iff Ω biholomorphic to disc minus a closed polar set.

Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a Kähler current T is a closed (1,1)-current T such that $T \geq \delta\omega$ for a smooth $(1,1)$ form $\omega > 0$ and $\delta \ll 1$.

Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a Kähler current T is a closed (1,1)-current T such that $T \geq \delta\omega$ for a smooth $(1,1)$ form $\omega > 0$ and $\delta \ll 1$.

Easy observation

$\alpha \in \mathcal{E}^\circ$ (interior of $\mathcal{E}$) $\iff$ $\alpha = \{T\}$, $T =$ a Kähler current.
We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a Kähler current T is a closed (1,1)-current T such that $T \geq \delta\omega$ for a smooth $(1,1)$ form $\omega > 0$ and $\delta \ll 1$.

Easy observation

$\alpha \in \mathcal{E}^\circ$ (interior of $\mathcal{E}$) $\iff$ $\alpha = \{T\}$, $T =$ a Kähler current.
We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

Theorem on approximate Zariski decomposition (D, 1992)

Any Kähler current can be written $T = \lim T_m$ where $T_m \in \{T\}$ has analytic singularities & logarithmic poles, i.e. $\exists$ modification $\mu_m : \widetilde{X}_m \to X$ such that $\mu_m^\star T_m = [E_m] + \beta_m$, where $E_m \geq 0$ is a $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coeff. in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on $\widetilde{X}_m$.

Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a Kähler current T is a closed (1,1)-current T such that $T \geq \delta\omega$ for a smooth $(1,1)$ form $\omega > 0$ and $\delta \ll 1$.

Easy observation

$\alpha \in \mathcal{E}^\circ$ (interior of $\mathcal{E}$) $\iff$ $\alpha = \{T\}$, $T =$ a Kähler current.

We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

Theorem on approximate Zariski decomposition (D, 1992)

Any Kähler current can be written $T = \lim T_m$ where $T_m \in \{T\}$ has analytic singularities & logarithmic poles, i.e. $\exists$ modification $\mu_m : \widetilde{X}_m \to X$ such that $\mu_m^\star T_m = [E_m] + \beta_m$, where $E_m \geq 0$ is a $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coeff. in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on $\widetilde{X}_m$.

Moreover (Boucksom), $\mathrm{Vol}(\beta_m) = \int_{\widetilde{X}_m} \beta_m^n \to \mathrm{Vol}(T)$ as $m \to +\infty$.

Proof of the analytic Zariski decomposition

- Write locally on any coordinate ball $\Omega \subset X$

$$T = i\partial\overline{\partial}\varphi$$

for some strictly plurisubharmonic psh potential φ on X.

Proof of the analytic Zariski decomposition

- Write locally on any coordinate ball $\Omega \subset X$

$$T = i\partial\overline{\partial}\varphi$$

for some strictly plurisubharmonic psh potential φ on X.

- Approximate T on Ω by

$$T_m = i\partial\overline{\partial}\varphi_m, \quad \text{where} \quad \varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \big\{f \in \mathcal{O}(\Omega)\,;\ \|f\|^2_{m\varphi} := \int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty\big\}.$$

Proof of the analytic Zariski decomposition

- Write locally on any coordinate ball $\Omega \subset X$
$$T = i\partial\overline{\partial}\varphi$$
for some strictly plurisubharmonic psh potential φ on X.
- Approximate T on Ω by
$$T_m = i\partial\overline{\partial}\varphi_m, \quad \text{where} \quad \varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$
where $(g_{\ell,m})$ is a Hilbert basis of the space
$$\mathcal{H}(\Omega, m\varphi) = \Big\{f \in \mathcal{O}(\Omega)\,;\ \|f\|^2_{m\varphi} := \int_\Omega |f|^2 e^{-2m\varphi}dV < +\infty\Big\}.$$
- We have $\varphi_m(z) = \dfrac{1}{2m}\sup\limits_{\|f\|_{m\varphi}\le 1}\log|f(z)|^2$.

Proof of the analytic Zariski decomposition

- Write locally on any coordinate ball $\Omega \subset X$

$$T = i\partial\overline{\partial}\varphi$$

for some strictly plurisubharmonic psh potential φ on X.

- Approximate T on Ω by

$$T_m = i\partial\overline{\partial}\varphi_m, \quad \text{where} \quad \varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \big\{f \in \mathcal{O}(\Omega)\,;\ \|f\|^2_{m\varphi} := \int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty\big\}.$$

- We have $\varphi_m(z) = \dfrac{1}{2m} \sup\limits_{\|f\|_{m\varphi}\le 1} \log|f(z)|^2$.

The mean value inequality implies

$$|f(z)|^2 \le \frac{1}{\pi^n r^{2n}/n!} \sup_{B(z,r)} e^{2m\varphi(z)} \Rightarrow \varphi_m(z) \le \sup_{B(z,r)} \varphi + \frac{n}{m}\log\frac{C}{r}.$$

Use of the pointwise Ohsawa-Takegoshi theorem

• The Ohsawa-Takegoshi L^2 extension theorem (extension from a single isolated point) implies that for every $z_0 \in \Omega$, there exists $f \in \mathcal{O}(\Omega)$ such that $f(z_0) = c\, e^{m\varphi(z_0)}$ ($c > 0$ small), such that

$$\|f\|^2_{m\varphi} = \int_\Omega |f|^2 e^{-2m\varphi} dV \le C \int_{\{z_0\}} |f|^2 e^{-2m\varphi} \delta_{z_0} = 1$$

for $c = C^{-1/2}$. As a consequence $\varphi_m(z) \ge \varphi(z) + \dfrac{1}{2m} \log c.$

Use of the pointwise Ohsawa-Takegoshi theorem

- The Ohsawa-Takegoshi L^2 extension theorem (extension from a single isolated point) implies that for every $z_0 \in \Omega$, there exists $f \in \mathcal{O}(\Omega)$ such that $f(z_0) = c\, e^{m\varphi(z_0)}$ ($c > 0$ small), such that

$$\|f\|^2_{m\varphi} = \int_\Omega |f|^2 e^{-2m\varphi} dV \le C \int_{\{z_0\}} |f|^2 e^{-2m\varphi} \delta_{z_0} = 1$$

for $c = C^{-1/2}$. As a consequence $\varphi_m(z) \ge \varphi(z) + \frac{1}{2m} \log c.$

- By the above inequalities one easily concludes that the Lelong number at any point $z_0 \in \Omega$ satisfies

$$\nu(\varphi, z_0) - \frac{n}{m} \le \nu(\varphi_m, z_0) \le \nu(\varphi, z_0).$$

Use of the pointwise Ohsawa-Takegoshi theorem

• The Ohsawa-Takegoshi L^2 extension theorem (extension from a single isolated point) implies that for every $z_0 \in \Omega$, there exists $f \in \mathcal{O}(\Omega)$ such that $f(z_0) = c\, e^{m\varphi(z_0)}$ ($c > 0$ small), such that

$$\|f\|^2_{m\varphi} = \int_\Omega |f|^2 e^{-2m\varphi} dV \le C \int_{\{z_0\}} |f|^2 e^{-2m\varphi} \delta_{z_0} = 1$$

for $c = C^{-1/2}$. As a consequence $\varphi_m(z) \ge \varphi(z) + \dfrac{1}{2m} \log c$.

• By the above inequalities one easily concludes that the Lelong number at any point $z_0 \in \Omega$ satisfies

$$\nu(\varphi, z_0) - \frac{n}{m} \le \nu(\varphi_m, z_0) \le \nu(\varphi, z_0).$$

This implies Siu's analyticity result for Lelong upper level sets $E_c(T)$.

• The case of a global current $T = \alpha + dd^c\varphi$ is obtained by using a covering of X by balls Ω_j, and gluing the local approximations $\varphi_{j,m}$ of φ into a global one φ_m by a partition of unity.