Interplays of Complex and Symplectic Geometry
Lecture 2: Symplectic Calabi-Yau Problem

Anna Fino

Dipartimento di Matematica
Universitá di Torino

CIRM-ICTP Complex Analysis and Geometry - XXV
7-11 June 2021

Calibrated almost complex structures

Definition

An almost cx structure J on a symplectic manifold (M^{2n}, Ω) is calibrated by Ω (or Ω is compatible with J) if J is tamed and $\Omega(JX, JY) = \Omega(X, Y)$, $\forall X, Y$.

• If J is calibrated by $\Omega \Longrightarrow (\Omega, J)$ is an almost-Kähler (AK) structure $\Rightarrow g(X, Y) = \Omega(X, JY)$ is a J-Hermitian metric.

• If J is integrable, then the AK structure (Ω, J, g) is Kähler.

Given a Kähler structure (Ω, J) one can define

$$\mathcal{C}_\Omega := \{\omega \in [\Omega] \mid \omega \,\text{is compatible with}\, J, \;\; \omega > 0\}.$$

By dd^c-Lemma $\Rightarrow$

$$\mathcal{C}_\Omega := \{\Omega + dd^c u > 0 \mid u \in \mathcal{C}^\infty(M, \mathbb{R})\}.$$

The Ricci tensor of the metric g induced by (Ω, J) satisfies

$$Ric(JX, JY) = Ric(X, Y), \quad \forall X, Y.$$

The Calabi-Yau theorem

$\rho(X, Y) := Ric(JX, Y)$ is the Ricci form of (Ω, J).
$\Rightarrow d\rho = 0$ and $[\rho] = 2\pi c_1(M^{2n}, J)$

Theorem (Yau 1978)

Let (M^{2n}, J, Ω) be a compact Kähler manifold and let $\tilde{\rho}$ be a closed $(1,1)$-form such that $[\tilde{\rho}] = 2\pi c_1(M^{2n}, J)$. Then there exists a unique Kähler form $\tilde{\Omega} \in \mathcal{C}_\Omega$ such that $\tilde{\rho}$ is the Ricci form of $(\tilde{\Omega}, J)$.

There has been great interest in extending Yau's Theorem to non-Kähler settings!

Symplectic version of the Calabi-Yau theorem

Theorem (Yau, Symplectic version)

(M^{2n}, J, Ω) *compact Kähler manifold and* σ *a volume form such that* $\int_{M^{2n}} \Omega^n = \int_{M^{2n}} \sigma$.

Then there exists a unique Kähler form $\tilde{\Omega}$ *with* $[\tilde{\Omega}] = [\Omega]$ *such that*

$$\tilde{\Omega}^n = \sigma \longleftrightarrow \text{CY Equation} \leftrightarrow (*) \begin{cases} (\Omega + d\alpha)^n = e^f \Omega^n \\ d\alpha \text{ is of type } (1,1) \end{cases}$$

$\hookrightarrow$ complex Monge-Ampere equation $(\Omega + dd^c h)^n = e^f \Omega^n$

$\hookrightarrow$ Yau's theorem: $(*)$ has always a unique solution h.

Symplectic Calabi-Yau problem

Let (M^{2n}, J, Ω, g) be a compact AK manifold with a volume form $\sigma = \mathrm{e}^f \Omega^n$ satisfying $\int_{M^{2n}} \mathrm{e}^f \Omega^n = \int_{M^{2n}} \Omega^n$. Then

$$\text{CY equation} \longleftrightarrow \begin{cases} (\Omega + d\alpha)^n = e^f \Omega^n \\ J(d\alpha) = d\alpha \\ d^*\alpha = 0. \end{cases} \qquad (*)$$

- $(*)$ is elliptic for $n = 2$ and the solutions are unique [Donaldson];
- $(*)$ is overdetermined for $n > 2$.

Problem

For which almost Kähler 4-manifolds does (*) have solutions?

Uniqueness of solutions

Proposition (Donaldson)

If $n = 2$ the symplectic CY problem has a unique solution.

Proof.

Let Ω_1 and Ω_2 be two solutions to the symplectic CY problem. Then

$$\begin{cases} \Omega_1^2 = \Omega_2^2\,, \\ \Omega_2 = \Omega_1 + d\alpha \end{cases} \quad \Longrightarrow d\alpha^2 + 2\Omega_1 \wedge d\alpha = 0\,.$$

Consider $\bar{\Omega} = \Omega_1 + \Omega_2 \hookrightarrow \bar{\Omega}$ is a symplectic form of type $(1,1)$ and defines a Riemannian metric $\tilde{g}$.

$$\bar{\Omega} \wedge d\alpha = 0 \Longrightarrow d\alpha \in \Lambda^2_+ \Longrightarrow d\alpha = 0\,.$$

$\square$

Donaldson's Conjecture

Conjecture

Let (M^4, Ω, J, σ) be a compact sympletic 4-manifold with an almost cpx structure J tamed by Ω and a normalized volume form σ. If $\tilde{\Omega} \in [\Omega]$ is a symplectic form which is calibrated by J and solves the CY equation $\tilde{\Omega}^2 = \sigma$, then there are $\mathcal{C}^\infty$ a priori bounds on $\tilde{\Omega}$ depending only on Ω, J and σ.

Applications:

• the Symplectic CY theorem holds for compact AK 4-manifolds M with $b_+(M^4) = 1$.

• If $b_+(M^4) = 1$ and $\exists\, \Omega$ taming J, then $\exists\, \tilde{\Omega}$ calibrated by J.

Question (Donaldson)

Suppose J is an almost complex structure on a cpt 4-manifold M. If J is tamed by a symplectic form Ω, is there a symplectic form compatible with J?

Remark

The question is true locally for all almost complex 4-manifolds, but this is no longer the case in higher dimensions!
(examples by Tomassini; Lejmi; Vezzoni).

Donaldson's question is confirmed when

- $M = \mathbb{CP}^2$ [Gromov; Taubes].
- J is integrable [Li, Zhang; Streets, Tian].

Taubes's result

Theorem (Taubes)

Let (M, Ω) be a cpt symplectic 4-manifold with $b_+(M^4) = 1$. A generic Ω-tamed almost cpx structure on M is compatible with a symplectic form $\tilde{\Omega}$ on M. Moreover, the class $[\tilde{\Omega}]$ in $H^2(M, \mathbb{R})$ can be taken to be that of Ω if the latter's class comes from $H^2(M, \mathbb{Q})$.

Remark

• "Generic" means in an open and dense subset of the $\mathcal{C}^\infty$-Frechét space of Ω-tamed almost cpx structures.

• The new symplectic form $\tilde{\Omega}$ is constructed by integrating over a space of currents that are defined by pseudo-holomorphic curves.

Sufficient conditions

Theorem (Weinkove)

Let (M^4, J, Ω, g) be a compact AK manifold with $b_+(M^4) = 1$. Then the symplectic CY problem has a solution if $\|N\|_{L^1(g)} < \epsilon$ for ϵ depending only on g and $\|f\|_{\mathcal{C}^2(g)}$.

On a AK (M^4, Ω, J, g) $\exists!$ connection ∇^C (the canonical or Chern connection) such that $\nabla^C J = \nabla^C \Omega = 0$, $\mathrm{Tor}^{1,1}(\nabla^C) = 0$.

Theorem (Tosatti,Weinkove,Yau)

If $\mathcal{R}(g, J) \geq 0$, then the Calabi-Yau problem can be solved for every normalized volume form on (M^4, Ω, J, g), where $\mathcal{R}(g, J)$ is defined by $\mathcal{R}_{i\bar{j}k\bar{l}} = (R^C)^j_{ik\bar{l}} + 4 N^r_{\bar{l}\bar{j}} \overline{N^i_{\bar{r}k}}$.

The CY equation on the Kodaira-Thurston manifold

Remark

The existence result by Tosatti-Weinkove-Yau cannot be applied to the KT manifold $M^4 = (\Gamma \backslash Nil^3) \times S^1$.

$$Nil^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \right\}$$

- M^4 has a global invariant coframe $\{e^i\}$ such that

$$e^1 = dy, \qquad e^2 = dx, \qquad e^3 = dt, \qquad e^4 = dz - x\,dy.$$

We will denote $Nil^3 \times \mathbb{R}$ simply by $(0, 0, 0, 12)$.

- M^4 is the total space of a T^2-bundle over $\mathbb{T}^2$:

$$T^2 = S^1 \times S^1 \quad \hookrightarrow \quad \begin{array}{c} \Gamma\backslash Nil^3 \\ \downarrow \pi_{xy} \\ \mathbb{T}^2_{xy} \end{array} \quad \times S^1$$

- M^4 has the Lagrangian (with respect to π_{xy}) AK structure

$$\Omega = e^1 \wedge e^4 + e^2 \wedge e^3, \quad g = \sum_{i=1}^{4} (e^i)^2,$$

i.e. Ω vanishes on the fibers.

Tosatti and Weinkove result

Theorem (Tosatti, Weinkove)

The CY equation on the KT manifold (M^4, J, Ω, g) can be solved for every T^2-invariant volume form σ.

Argument of the proof:

1. Writing $\sigma = e^f\,\Omega^2$, for a $\mathcal{C}^\infty$ T^2-invariant function f, then by the normalization of σ one has $\int_{M^4} e^f \Omega^2 = \int_{M^4} \Omega^2$.
   Every solution $\tilde{\Omega} = \Omega + d\alpha$ of the CY problem satisfies the following a priori bound on the metric $\tilde{g}$ associated to $(\tilde{\Omega}, J)$:
   $$tr_g\,\tilde{g} \leq Min_{M^4} \Delta\, f\,.$$
2. The continuity method gives the result.

CY equation on the KT manifold II

Consider the Calabi-Yau equation $(\Omega + d\alpha)^2 = e^f\,\Omega^2$. Let $\alpha = d^c v - v\,e^1 = v\,e^1 + v_x\,e^3 + v_y\,e^4\,, \quad v \in C^\infty(\mathbb{T}^2)\,.$

Then $d\alpha = v_{xx}\,e^{23} + v_{xy}(e^{13} + e^{24}) + v_{yy}\,e^{14}$ and the CY equation $(\Omega + d\alpha)^2 = e^f\,\Omega^2$ becomes the Monge-Ampère equation

$$(1 + v_{xx})(1 + v_{yy}) - v_{xy}^2 = e^f$$

Theorem (Li)

The Monge-Ampère equation on the standard torus $\mathbb{T}^n$ has always a solution.

Goal: To generalize this argument to other AK structures on T^2-bundles over $\mathbb{T}^2$.

Definition (Thurston)

A geometric 4-manifold is a pair (X, G) where X is a complete, simply-connected Riemannian 4-manifold, G is a group of isometries acting transitively on X that contains a discrete subgroup Γ such that $\Gamma \backslash X$ has finite volume.

Let $Nil^4 = (0, 13, 0, 12)$, $\quad Sol^3 \times \mathbb{R} = (0, 0, 13, 41)$.

Theorem (Ue)

Every orientable T^2-bundle over a $\mathbb{T}^2$ is a geometric 4-manifold, where (X, G) is one of the following

$$(\mathbb{R}^4, SO(4) \ltimes \mathbb{R}^4), \quad (Nil^3 \times \mathbb{R}, Nil^3 \times S^1),$$
$$(Nil^4, Nil^4), \quad (Sol^3 \times \mathbb{R}, Sol^3 \times \mathbb{R})$$

and it is infra-solvmanifold, i.e. a smooth quotient $\Gamma \backslash X$ covered by a solvmanifold or equivalently a quotient $\Gamma \backslash X$, where the discrete group Γ contains a lattice $\tilde{\Gamma}$ of X such that $\tilde{\Gamma} \backslash \Gamma$ is finite.

Definition

An AK structure (J, Ω, g) on an infra-solvmanifold $M^4 = \Gamma \backslash X$ is called invariant if it is induced by a left-invariant one on X and it is Γ-invariant.

Proposition (F, Li, Salamon, Vezzoni)

On a 4-dimensional infra-solvmanifold $(M^4 = \Gamma \backslash X, J, \Omega, g)$ with an invariant AK structure, the Tosatti-Weinkove-Yau condition $\mathcal{R}(g, J) \geq 0$ is satisfied if and only if (Ω, J) is Kähler.

Results on T^2-bundles over $\mathbb{T}^2$

Theorem (F, Li, Salamon, Vezzoni / Buzano, F, Vezzoni)

Let $M^4 = \Gamma\backslash X$ be a T^2-bundle over a $\mathbb{T}^2$ endowed with an invariant AK structure (J, Ω, g). Then for every normalized T^2-invariant volume form $\sigma = e^F \Omega^2$, $F \in C^\infty(\mathbb{T}^2)$ the associated CY problem has a unique solution.

Layout of the proof:

- Use the classification of T^2-bundles over $\mathbb{T}^2$;
- Classify in each case invariant Lagrangian AK structures and invariant non-Lagrangian AK structures;
- Rewrite the problem in terms of a Monge-Ampère equation;
- Show that such an equation has a solution.

Classification of T^2-bundles over $\mathbb{T}^2$

By Sakamoto and Fukuhara the diffeomorphism classes of T^2-bundles over $\mathbb{T}^2$ are classified in 8 families:

	G	Structure equations of X
$i), ii)$	$SO(4) \ltimes \mathbb{R}^4$	$(0,0,0,0)$
$iii), iv), v)$	$Nil^3 \times S^1$	$(0,0,0,12)$
$vi)$	Nil^4	$(0,13,0,12)$
$vii), viii)$	$Sol^3 \times \mathbb{R}$	$(0,0,13,41)$

The Lie group G is the geometry type of $\Gamma\backslash X$.

• In the cases different from $iii)$ the fibration of M^4 as torus bundle is unique.

• In the case $iii)$ one has two fibrations

$$\pi_{xy} : M^4 \longrightarrow \mathbb{T}^2_{xy}, \quad \pi_{yt} : M^4 \longrightarrow \mathbb{T}^2_{yt}.$$

Theorem (Geiges)

Let $M^4 = \Gamma \backslash X$ be an orientable T^2-bundle over a $\mathbb{T}^2$. Then

- *M^4 has a symplectic form and every class $a \in H^2(M^4, \mathbb{R})$ with $a^2 \neq 0$ can be represented by a symplectic form;*
- *M^4 has a Kähler structure if and only if $X = \mathbb{R}^4$;*
- *If $X = Nil^4$ then every invariant AK structure on M^4 is Lagrangian;*
- *If $X = Sol^3 \times \mathbb{R}$ every invariant AK structure on M^4 is non-Lagrangian.*

The Monge-Ampère equation

The following equation covers all cases

$$A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + E_2 e^f,$$

where

$$\begin{aligned} A_{11}[u] &= u_{xx} + B_{11}u_x + C_{11} + Du, \\ A_{12}[u] &= u_{xy} + B_{12}u_y + C_{12}, \\ A_{22}[u] &= u_{yy} + B_{22}u_y + C_{22} \end{aligned}$$

and B_{ij}, C_{ij}, D, E_i are constants.

In the Lagrangian case $D = 0$.

Solutions to the Monge-Ampère equation

Goal: Show that $A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + E_2 e^f$ has a solution on $\mathbb{T}^2$.

• The first step consists in showing that the solutions to the equation are unique up to a constant.

• We look for a solution u satisfying $\int_{\mathbb{T}^2} u = 0$.

• We apply the continuity method to

$$A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + (1-t)E_2 + tE_2 e^f, \quad t \in [0,1].$$

using a priori estimate

$$\|u\|_{\mathcal{C}^2} \leq 2(B_{11}+1)|B_{22}|e^{2C_{22}} + C_{11} + C_{22}.$$

KT manifold viewed as an S^1-bundle over a 3-torus

M^4 is the total space of an S^1-bundle over $\mathbb{T}^3$:

$$\begin{array}{ccc} T^2 = S^1_z & \hookrightarrow & \Gamma\backslash Nil^3 \quad \times S^1_t \\ & & \downarrow \pi \\ & & \mathbb{T}^2_{xy} \times S^1_t \end{array}$$

Consider the AK structure $(\Omega = e^{13} + e^{42}, g = \sum_{i=1}^4 (e^i)^2)$

Theorem (Buzano, F, Vezzoni)

The CY equation $(\Omega + d\alpha)^2 = e^f \Omega^2$ has a unique solution $\tilde{\Omega} = \Omega + d\alpha$ for every S^1-invariant volume form $\sigma = e^f \Omega^2$ such that $\int_{\mathbb{T}^3} e^f \, dV = 1$.

Sketch of the proof:

- Step 1 Setting $\alpha = d^c u - ue^1$, then

$$J(d\alpha) = d\alpha$$

and we reduce the CY problem to a fully nonlinear PDE on the 3-dimensional base torus $\mathbb{T}^3$

$$(u_{xx} + 1)(u_{yy} + u_{tt} + u_t + 1) - u_{xy}^2 - u_{xt}^2 = e^f$$

- Step 2 $\mathcal{C}^0$-a priori estimates

Let $u \in \mathcal{C}^2_0(\mathbb{T}^3)$ such that

$$(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1)-u_{xy}^2-u_{xt}^2=e^f$$

Then

$$|u_x|<1$$
$$\|\nabla|u|^{\frac{p}{2}}\|_{L^2}^2 \le \frac{p^2}{16}\|u\|_{L^p}^p+\frac{5p^3}{16}\|1+e^f\|_{\mathcal{C}^0}\|u\|_{L^p}^{p-1}$$
$$\|u\|_{L^2}\le\|1+e^f\|_{\mathcal{C}^0}$$

$\Rightarrow \quad \|u\|_{\mathcal{C}^0}\le C$, where $C=C(\|f\|_{\mathcal{C}^0})$

• Step 3 First order estimates

Let $u \in \mathcal{C}^4_0(\mathbb{T}^3)$ solving

$$(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1)-u_{xy}^2-u_{xt}^2=e^f,$$

then

$\|\Delta u\|_{\mathcal{C}^0} \leq C_1(1+\|u\|_{\mathcal{C}^1})$, where $C_1 = C_1(\|f\|_{\mathcal{C}^2})$

$\|u\|_{\mathcal{C}^1} \leq C_2$, where $C_2 = C_2((\|f\|_{\mathcal{C}^2})$

- Step 4 $\mathcal{C}^{2,\rho}$ estimates

Theorem (Tosatti-Wang-Weikove-Yang)

Let $\tilde{\Omega}$ be the solution of the CY equation on (M^{2n}, Ω, J, g). Assume that there exist constants $\tilde{C}_0 > 0$ and $0 < \rho_0 < 1$ such that $f \in \mathcal{C}^{\rho_0}(M^{2n})$ and $tr\tilde{g} \leq \tilde{C}_0$. Then, there exist two constants $\tilde{C} > 0$ and $0 < \rho < 1$, depending only on M^{2n}, Ω, J, C_0 and $\|f\|_{\mathcal{C}^\rho}$, such that $\|\tilde{g}\|_{\mathcal{C}^\rho} \leq \tilde{C}$.

Proposition

If $u \in \mathcal{C}^4_0(\mathbb{T}^3)$ is a solution of

$$(u_{xx} + 1)(u_{yy} + u_{tt} + u_t + 1) - u_{xy}^2 - u_{xt}^2 = e^f,$$

then $\exists$ constants $C_3 > 0$ and $\rho > 0$, both depending only on $\|f\|_{\mathcal{C}^2}$, such that $\|u\|_{\mathcal{C}^{2,\rho}} \leq C_3$.

- Step 5 Continuity method

Let S be the set of $\tau \in [0,1]$ such that

$$(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1)-u_{xy}^2-u_{xt}^2=1-\tau+\tau e^f$$

has a solution in $\mathcal{C}_0^\infty(\mathbb{T}^3)$.

S is non-empty, open and closed in $[0,1]$.

Then $1 \in S$ and the claim follows.

New proof of the result by Tosatti and Weinkove

Tosatti and Weinkove have found a simplified proof of the $\mathcal{C}^0$-priori estimate based on the Aleksandrov-Bakelman-Pucci estimate.

Proposition (Székelyihidi)

Let $0 < r \leq 1$ and $v : \overline{B_r}(0) \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth map satisfying $v(0) + \epsilon \leq inf_{\partial B_r(0)} v$, for some $\epsilon > 0$. Then

$$\epsilon^n \leq C_0 \int_P det(D^2 v)$$

where $C_0 = C_0(n)$ and

$$P = \{x \in B_r(0) : |Dv(x)| < \tfrac{\epsilon}{2}, v(y) \geq v(x) + Dv(x)(y - x), \forall y \in B_r(0)\}.$$

Let $u \in \mathcal{C}^\infty(\mathbb{T}^3)$ such that

$$(u_{xx}+1)(u_{yy}+u_{tt}+u_t+1)-u_{xy}^2-u_{xt}^2=e^f,\ u\leq 0,\ \min u<-1.$$

Let $x_0 \in \mathbb{T}^3$ be such that $\min_{\mathbb{T}^3} u = u(x_0)$ and regard u as a map $u : B_r(0) \subset \mathbb{R}^3 \to \mathbb{R}$ with $O \equiv x_0$.
Define $v = u + \frac{\epsilon}{r^2}(x^2+y^2+t^2)$. Then

$$\epsilon^3 \leq C_0 \int_P det(D^2 v) \quad \text{and} \quad det(D^2 v(x)) \leq C, \quad \forall x \in P$$

for a uniform C.

Therefore $\epsilon^3 \leq C|P|$ and

$$\|u\|_{C^0} \leq \frac{C^{1/p}}{\epsilon^{3/p}} \|u\|_{L^p} + 1.$$

On the other hand, $\Delta u + u_t > -2$ which implies that $\|u\|_{L^p}$ is uniformly bounded and so one gets an L^∞ bound of u.

Theorem (Tosatti-Weinkove)

Let (Ω, J) be an invariant AK structure on the KT manifold inducing the standard metric $g = \sum_{i=1}^4 (e^i)^2$. The the CY equation on (M, J, Ω) can be solved for every S^1-invariant normalised volume form σ.

It is possible to generalize the theorem if we assume span $< e_1, e_2, e_3 >$ orthogonal to e_4.