Lecture 2: Uniform estimates through qpsh envelopes

Joint work with H.C.Lu

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Complex Analysis and Geometry - XXV,
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- how "nice" μ should be;
- a recent and elementary proof of this fundamental estimate.

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If $X \subset \mathbb{CP}^N$ has degree d and $\omega = \omega_{FS|X}$ then $\alpha = \frac{1}{nd}$ works.

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$\hookrightarrow$ when $\varphi \sim \log||z||$ this corresponds to "radial singularities"...

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- $PSH(X, \omega) \subset L^{n+\varepsilon}(\mu) \Rightarrow \chi'(t) = O(|t|^{-1-\varepsilon/n}) \Rightarrow \chi(-\infty) > -\infty$.
- For $\chi(t) = -\log \circ \log(-t)$ and $n \geq 2$, gets $PSH(X, \omega) \subset L^n(\mu)$.

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(For non smooth functions, this requires some extra work). $\square$

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$\hookrightarrow$ New and simplified approach using quasi-psh envelopes.

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Previous Thm follows from Hölder inequality (with $m = n+1$).

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- If $H \subset X$ real analytic hypersurface and $\mu = (2n-1)$-Hausdorff measure on H, then $PSH(X,\omega) \subset L^m(\mu)$ for any $m > 1$.

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$$\int_0^{T_{max}} g'(t)\mu(\varphi < -t)dt \leq \int_0^{+\infty} \frac{dt}{(1+t)^2} = 1.$$

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We choose χ so that $\int_X (\chi' \circ \varphi)^{n+2\varepsilon} d\mu = B \leq 2$. More precisely:

- Recall that if $g : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing with $g(0) = 1$ then

$$\int_X g \circ (-\varphi)\, d\mu = \mu(X) + \int_0^{T_{max}} g'(t)\mu(\varphi < -t)dt.$$

- We set $g(t) = [\chi'(-t)]^{n+2\varepsilon}$ with $\chi(0) = 0, \chi'(0) = g(0) = 1$ and

$$g'(t) = \frac{1}{(1+t)^2 \mu(\varphi < -t)} \quad \text{when} \quad t < T_{max}.$$

- Thus χ has all the required properties, e.g.

$$\int_0^{T_{max}} g'(t)\mu(\varphi < -t)dt \leq \int_0^{+\infty} \frac{dt}{(1+t)^2} = 1.$$

- We note for later use that $-\chi(-1) = \int_{-1}^0 \chi'(t)dt \geq 1$.

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- Integrate between 1 and T_{max} and use $h(1) \geq 1$ to conclude $\square$.

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