Interplays of Complex and Symplectic Geometry Lecture 3: Balanced Metrics and the Hull-Strominger System

Anna Fino

Dipartimento di Matematica
Universitá di Torino

CIRM-ICTP Complex Analysis and Geometry - XXV
7-11 June 2021

Balanced metrics

Definition (Michelsohn)

A balanced metric on a complex n-manifold is an Hermitian metric ω such that $d(\omega^{n-1}) = 0$.

• A metric is balanced if and only if $\Delta_{\partial} f = \Delta_{\overline{\partial}} f = 2\Delta_d f$ for every $f \in \mathcal{C}^\infty(M, \mathbb{C})$ (Gauduchon).

• A compact complex manifold M admits a balanced metric if and only if M carries no positive currents of degree $(1, 1)$ which are components of a boundary (Michelsohn).

In particular, Calabi-Eckmann manifolds have no balanced metrics!

Behaviour under modifications

Definition

Let M and N complex manifolds. A modification $f : M \to N$ is a holomorphic map such that $\exists$ a cpx submanifold $Y \subset N$ of codim ≥ 2 and a biholomorphism $f : X \setminus f^{-1}(Y) \to N \setminus Y$ given by restriction.

Theorem (Alessandrini, Bassanelli)

Let $f : M \to N$ be a modification with M and N compact complex manifolds. Then M is balanced if and only if N is balanced.

Every compact complex manifold bimeromorphic to a compact Kähler manifold is balanced $\Rightarrow$

Moishezon manifolds and complex manifolds in the Fujiki class $\mathcal{C}$ are balanced.

Balanced metrics and conifolds transitions

(Non-Kähler) balanced 3-folds can be constructed via the Clemens-Friedman construction [Fu, Li, Yau]:

Start with a Calabi-Yau 3-fold M with k mutually disjoint smooth rational curves $\mathcal{C}_j$ with normal bundles $\cong \mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$.

Contracting the k rational curves $\hookrightarrow$ a singular Calabi-Yau 3-fold M_0 with k ordinary double-point singularities $p_1, \ldots, p_k$:

- $M \setminus \cup_k \mathcal{C}_k \cong M_0 \setminus \{p_1, \ldots, p_k\}$
- a neighbourhood of p_j in $M_0 \cong$ a neighbourhood of 0 in $\{z_1^2 + z_2^2 + z_3^2 + z_4^4 = 0\} \subset \mathbb{C}^4$.

If $[\mathcal{C}_j] \in H^{2,2}(M, \mathbb{Q})$ satisfy $\sum_j n_j[\mathcal{C}_j] = 0$, with $n_j \neq 0, \forall j$

$\hookrightarrow \exists$ a family M_t over a disk $\Delta \subset \mathbb{C}$ [Friedman, Tian, Kawamata]:

$M \to M_0 \dashrightarrow M_t$ (conifold transition) such that

- M_t is a smooth 3-fold for $t \neq 0$ and for t small the local model is $\cong \{z_1^2 + z_2^2 + z_3^2 + z_4^4 = t\} \subset \mathbb{C}^4$;
- the central fibre is isomorphic to M_0.

Remark (Friedman)

$\#_k(S^3 \times S^3)$ for any $k \geq 2$ has in this way a cpx structure [contracting enough rational curves s.t. $H^2(M_t, \mathbb{R}) = 0$, for $t \neq 0$].

Theorem (Fu, Li, Yau)

For sufficiently small $t \neq 0$, M_t has a balanced metric.

$\hookrightarrow \#_k(S^3 \times S^3)$ is balanced!

More examples of balanced manifolds

- The twistor space of a 4-dim oriented anti-self-dual Riemannian manifold always has a balanced metric (Michelsohn; Gauduchon).

- Any left-invariant Hermitian metric on a unimodular complex Lie group is balanced [Abbena, Grassi] $\hookrightarrow$ complex parallelizable manifolds.

  If the complex Lie group is semisimple, ω^{n-1} is exact [Yachou].

- A characterization of compact complex homogeneous spaces with invariant volume admitting a balanced metric (in particular $c_1 \neq 0$) [F, Grantcharov, Vezzoni].

- Special classes of compact locally homogeneous spaces $\Gamma \backslash G$ (nilmanifolds or solvmanifolds) with an invariant complex structure.

Interplay with other types of Hermitian metrics

A Hermitian metric which is balanced and puriclosed is Kähler [Alexandrov, Ivanov; Popovici].

Conjecture

Every compact complex manifold admitting a balanced and a pluriclosed metric is Kähler.

The conjecture is true for

- the twistor space of a compact anti-self-dual 4-dim Riemannian manifold [Verbitsky]
- compact complex manifolds in the Fujiki class $\mathcal{C}$ [Chiose]

• The non-Kähler balanced manifolds constructed by Li, Fu and Yau by using conifold transitions. In particular, $\#_k(S^3 \times S^3)$ $k \geq 2$, since they have no pluriclosed metrics.

• 2-step nilmanifolds with invariant complex structures [F, Vezzoni].

• 6-dim solvmanifolds with invariant complex structures and holomorphically trivial canonical bundle [F, Vezzoni].

• Almost abelian solvmanifolds with invariant complex structures [F, Paradiso].

• Oeljeklaus-Toma (OT) manifolds [Otiman].

Theorem (F, Grantcharov)

A compact Hermitian manifold (M, J, g) with holomorphically trivial canonically bundle whose Bismut Ricci tensor vanishes, must be globally conformally balanced.

$\hookrightarrow$ counter-example to a conjecture by Gutowski, Ivanov, Papadopolous.

Theorem (F, Grantcharov, Vezzoni)

There exists a compact complex non-Kähler manifold admitting a balanced and an astheno-Kähler metric.

$\hookrightarrow$ negative answer to a question posed by Székeleyhidi, Tosatti, Weinkove.

Balanced flow

Let (M^{2n}, J, ω_0) be a complex manifold with a balanced metric ω_0.

Definition (Bedulli, Vezzoni)

A parabolic flow preserving the balanced condition is given by:

$$\partial_t \varphi(t) = i\partial\overline{\partial} *_t (\rho^C_{\omega(t)} \wedge *_t \varphi(t)) + \Delta_{BC}\varphi(t), \quad \varphi(0) = *_0\omega_0,$$

where $\rho^C_{\omega(t)}$ is the Ricci form of the Chern connection and

$$\Delta_{BC} = \partial\overline{\partial}\overline{\partial}^*\partial^* + \overline{\partial}^*\partial^*\partial\overline{\partial} + \overline{\partial}^*\partial\partial^*\overline{\partial} + \partial^*\overline{\partial}\overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial} + \partial^*\partial$$

is the Bott-Chern Laplacian.

Short-time existence and uniqueness for compact manifolds [Bedulli, Vezzoni].

Remark

If ω_0 is Kähler, then the flow coincides with the Calabi flow:

$$\begin{cases} \partial_t \omega(t) = i\partial\overline{\partial} s_{\omega(t)}, & \omega(t) \in \{\omega_0 + i\partial\overline{\partial} u > 0\} \subset [\omega_0] \\ \omega(0) = \omega_0, \end{cases}$$

where $s_{\omega(t)}$ is the scalar curvature of $\omega(t)$.

Theorem (F, Paradiso)

Let (G, J, ω_0) be a 6-dim balanced almost abelian Lie group. Then

- *the solution $\omega(t)$ to the balanced flow is defined for all positive times (eternal solution);*
- *Cheeger-Gromov convergence to a Kähler almost abelian Lie group.*

The physical motivation of the Hull-Strominger system

• The Hull-Strominger system describes the geometry of compactification of heterotic superstrings with torsion to 4-dimensional Minkowski spacetime.

The geometric objects are a 10-dim Lorentzian manifold M^{10} (product of $\mathbb{R}^{1,3}$ and a compact 6-manifold M^6) and a vector bundle E over M^6 $\hookrightarrow$ reduce all the equations required by superstring theory to geometry of M^6 (and E).

• (Candelas, Horowitz, Strominger, Witten'85) fluxfree compactification: $M^{10} = \mathbb{R}^{1,3} \times M^6$ equipped with a product metric, "embed the gauge into spin connection"($E = TM^6$) $\Rightarrow$ M^6 must be a Calabi-Yau 3-fold with Kähler Ricci-flat metric (solved by Yau'77)

• (Hull'86, Strominger'86) compactification with flux: $M^{10} = \mathbb{R}^{1,3} \times M^6$ equipped with a warped product metric $\Rightarrow$ Hull-Strominger system, in particular M^6 is a Calabi-Yau 3-fold ($K_{M^6} \cong \mathcal{O}$, not necessarily Kähler).

Hull-Strominger System

• M compact 3-dim complex manifold with a nowhere vanishing holomorphic $(3,0)$-form Ω.

• E complex vector bundle over M with a Hermitian metric H along its fibers and $\alpha' \in \mathbb{R}$ constant (slope parameter).

The Hull-Strominger system, for the Hermitian metric ω on M, is:

(1) $F_H^{2,0} = F_H^{0,2} = 0$, $F_H \wedge \omega^2 = 0$ (Hermitian-Yang-Mills),

(2) $d(\|\Omega\|_\omega \, \omega^2) = 0$ (ω is conformally balanced),

(3) $i\partial\overline{\partial}\omega = \frac{\alpha'}{4}(Tr(R_\nabla \wedge R_\nabla) - Tr(F_H \wedge F_H))$ (Bianchi identity)

where F_H, R_∇ are the curvatures of H and of a metric connection ∇ on TM.

Remark

The Hull-Strominger system is a generalization of Ricci-flat metrics on non-Kähler Calabi-Yau 3-folds coupled with Hermitian-Yang-Mills equation!

- $F_H^{2,0} = F_H^{0,2} = 0$, $F_H \wedge \omega^2 = 0$ is the Hermitian-Yang-Mills equation which is equivalent to E being a stable bundle.
- Calabi-Yau manifolds can be viewed as special solutions: take $E = T^{1,0}M$, and $H = \omega$, then the Hull-Strominger system reduces to $i\partial\overline{\partial}\omega = 0, d(\|\Omega\|_\omega \omega^2) = 0$, which imply that ω is Kähler and Ricci-flat.

Link with balanced metrics

The 2nd equation $d(\|\Omega\|_\omega \omega^2) = 0$ says that ω is conformally balanced.

Remark

It was originally written as $d^*\omega = i(\bar{\partial} - \partial) \ln(\|\Omega\|_\omega)$ (the equivalence was proved by Li and Yau).

The Hull-Strominger system can be interpreted as a notion of "canonical metric" for conformally balanced manifolds.

The anomaly cancellation equation

The third equation $i\partial\overline{\partial}\omega = \frac{\alpha'}{4}(Tr(R_\nabla \wedge R_\nabla) - Tr(F_H \wedge F_H))$ is the anomaly cancellation equation (or Bianchi identity) and couples the two metrics ω and H.

Remark

- It is the main equation accounting for both the novelty and the difficulty in solving the Hull-Strominger system.
- It originates from the famous Green-Schwarz anomaly cancellation mechanism required for the consistency of superstring theory.

Remark

- Since ω may not be Kähler, there is a one-parameter line of natural unitary connections on $T^{1,0}M$ defined by ω, passing through the Chern connection and the Bismut connection.
- From physical perspective one has $\alpha' \geq 0$ with $\alpha' = 0$ corresponding to the Kähler case, but in mathematical literature the case $\alpha' < 0$ is also considered [Phong, Picard, Zhang].

We will consider the case when ∇ is the Chern connection of ω.

Remark

Finding a solution of the Hull-strominger system is a priori not enough to find a supersymmetric classical solution of the theory.

Theorem (Ivanov)

A solution of the Hull-Strominger system satisfies the heterotic equations of motion if and only if the connection ∇ in the anomaly cancellation equation is an instanton.

Known non-Kähler solutions

• The first Non-Kähler solutions have been found by Fu and Yau on a class of toric fibrations over K3 surfaces, constructed by Goldstein and Prokushkin.

• Non-Kähler solutions on Lie groups and their quotients by discrete subgroups [Fernández, Ivanov, Ugarte, Villacampa; Fei, Yau; Grantcharov...].

• New solutions on non-Kähler torus fibrations over K3 surfaces, leading to the first examples of T-dual solutions of the Hull-Strominger system [Garcia-Fernandez].

• Solutions on non-Kähler fibrations $p : M^6 \to \Sigma$ with fiber a compact HK manifold N^4, where Σ is a compact Riemann surface of genus $g \geq 3$ [Fei, Huang, Picard].

The construction of Goldstein and Prokushkin

Let (S, ω_S) be a K3 surface with Ricci flat Kähler metric ω_S.

• To any pair ω_1, ω_2 of anti-self-dual (1,1)-forms on S such that $[\omega_i] \in H^2(S, \mathbb{Z})$, Goldstein and Prokushkin associated a toric fibration

$$\pi : M \to S,$$

with a nowhere vanishing holomorphic 3-form $\Omega = \theta \wedge \pi^*(\Omega_S)$, for a (1,0)-form $\theta = \theta_1 + i\theta_2$, where θ_i are connection 1-forms on M such that $d\theta_i = \pi^* \omega_i$.

• The $(1,1)$-form

$$\omega_0 = \pi^*(\omega_S) + i\theta \wedge \overline{\theta}$$

is a balanced Hermitian metric on M, i.e. $d\omega_0^2 = 0$.

The Fu -Yau solution

Fu and Yau found a solution of the Hull-Strominger system with M given by the Goldstein-Prokushkin construction, and the following ansatz for the metric on M:

$$\omega_u = \pi^*(e^u \omega_S) + i\theta \wedge \overline{\theta},$$

where u is a function on S. This reduces the Hull-Strominger system to a 2-dim Monge-Ampère equation with gradient terms:

$$i\partial\overline{\partial}(e^u - fe^{-u}) \wedge \omega + \alpha' i\partial\overline{\partial}u \wedge i\partial\overline{\partial}u + \mu = 0,$$

under the ellipticity condition

$$(e^u + fe^{-u})\omega + 4\alpha' i\partial\overline{\partial}u > 0,$$

where $f \geq 0$ is a known function, and μ is a $(2,2)$-form with average 0.

The Fu-Yau equation

Theorem (Fu, Yau)

Consider the above complex Monge-Ampère equation with the above ellipticity condition. Then there exists a solution $u \in \mathcal{C}^\infty(S)$ satisfying the above ellipticity condition. In particular, there exists a solution of the Hull-Strominger system with M a toric fibration over the K3 surface S.

A generalization to n-dimensions of the Fu-Yau equation has been obtained by Phong, Picard and Zhang.

Extension to torus bundles aver K3 orbifolds

Theorem (F, Grantcharov, Vezzoni)

• S a *compact K3 orbifold* with a Ricci-flat Kähler form ω_S and orbifold Euler number $e(S)$.
• ω_i, $i = 1, 2$ *anti-self-dual (1, 1)-forms* on S s. t. $[\omega_i] \in H^2(S, \mathbb{Z})$ and the total space M of the *principal T^2 orbifold bundle* $\pi : M \to S$ determined by them is *smooth*.
• W a *stable vector bundle of degree 0* over (S, ω_S) such that

$$\alpha'(e(S) - (c_2(W) - \frac{1}{2}c_1^2(S))) = \frac{1}{4\pi^2}\int_S (\|\omega_1\|^2 + \|\omega_2\|^2)^2 \frac{\omega_S^2}{2}.$$

Then M has a Hermitian structure (M, ω_u) and $\exists$ a metric h along the fibers of W such that $(E = \pi^* W, H = \pi^*(h), M, \omega_u)$ solves the Hull-Strominger system.

Sketch of the proof

• If θ_i are the connection 1-forms with $d\theta_i = \pi^*\omega_i$, then the smooth T^2-bundle $\pi : M \to S$, determined by ω_i, has a complex structure such that $\theta = \theta_1 + i\theta_2$ is a $(1,0)$-form and π is a holomorphic projection.

• The Hermitian metric $\omega = \pi^*(\omega_S) + \theta_1 \wedge \theta_2$ on M is balanced if and only if $tr_{\omega_S}\omega_1 = tr_{\omega_S}\omega_2 = 0$.

If we choose ω_1, ω_2 to be harmonic, then this is equivalent to the topological condition $[\omega_S] \cup [\omega_1] = [\omega_S] \cup [\omega_2] = 0$.

• If Ω_S is a holomorphic $(2,0)$-form on S with $||\Omega_S||_{\omega_S} = const$, then the form $\Omega = \Omega_S \wedge \theta$ is holomorphic with constant norm with respect to ω.

• For every smooth function u on S, the metric $\omega_u = e^u \pi^*(\omega_S) + \theta_1 \wedge \theta_2$ on M is conformally balanced with conformal factor $||\Omega||_{\omega_u}$.

• If W is a stable bundle on S with respect to ω_S of degree 0 and Hermitian-Yang-Mills metric h and curvature F_h, then $E = \pi^*(W)$ is a stable bundle of degree 0 on M with respect to ω_u with Hermitian-Yang-Mills metric $H = \pi^*(h)$ and curvature $F_H := \pi^*(F_h)$.

• We use that the argument by Fu and Yau depends only on the foliated structure of the manifold M.

• $(\theta, \omega_B = \pi^*(\omega_S), \Omega_B = \pi^*(\Omega_S))$ satisfy

$$d\omega_B = 0\,, \quad \omega_B \wedge d\theta = 0\,, \quad \iota_Z \Omega_B = 0,$$

where Z is the dual to θ with respect to ω.
Then (ω_B, Ω_B) induces a transverse Calabi-Yau structure on M.

• We reduce the Hull-Strominger system on M to a transversally elliptic equation, proving a generalization of the Fu-Yau theorem to Hermitian 3-folds with a transverse Calabi-Yau structure.

• We solve the transversally elliptic equation using a result of El Kacimi.

New simply connected examples

To construct explicit examples we consider T^2-bundles over an orbifold S which are given by the following sequence

$$\begin{array}{ccc} S^1 & \hookrightarrow & M \\ & & \downarrow \\ S^1 & \hookrightarrow & M_1 \\ & & \downarrow \\ & & S \end{array}$$

where $M_1 \to S$ is a Seifert S^1-bundle, M_1 is smooth and $M \to M_1$ is a regular principal S^1-bundle over M_1.

Roughly speaking, Seifert fibered manifolds are $(2n+1)$-manifolds L which admit a differentiable map $f : L \to X$ to a complex n-manifold X such that every fiber is a circle.

The natural setting is study Seifert bundles where the base X is a complex locally cylic orbifold, i.e. locally it looks like $\mathbb{C}^n/G$ where G is a cyclic group acting linearly.

The main idea is that there is a divisor $\cup_i D_i \subset X$ such that $L \to X$ is a circle bundle over $X \setminus \cup_i D_i$ and natural multiplicities m_i are assigned to the fibers over each D_i.

$\Delta := \sum_i (1 - \frac{1}{m_i}) D_i$ is a $\mathbb{Q}$ divisor and is called the branch divisor of X.

Theorem (Kollar)

If (X, Δ) has trivial $H^1_{orb}(X, \mathbb{Z})$, then a Seifert S^1-bundle L is uniquely determined by its first Chern class

$$c_1(L/X) := [B] + \sum_{i=1}^{n} \frac{b_i}{m_i}[D_i] \in H^2(X, \mathbb{Q})$$

where b_i are integers such that $0 \leq b_i < m_i$ and relatively prime to m_i and B is a Weil divisor over X.

• We consider as CY orbifold surface (K3 orbifold) S an intersection of two degree 6 hypersurfaces in $\mathbb{P}(2,2,2,3,3)$ in generic position (S has 9 isolated A_1-singularities and $\pi_1^{orb}(S) = 1$).

• Blowing up S at $9 - k$ points, $1 \leq k \leq 8$ (i.e. using partial resolutions) we construct a smooth Seifert S^1-bundle $M_1 \to S$.

• By applying the main theorem to $M = M_1 \times S^1$ we obtain a solution of the Hull-Strominger system on M.

• Using Barden's results and a Kollar's result for simply connected 5-manifolds with a semi-free S^1-action we show that M is diffeomorphic $S^1 \times \sharp_k(S^2 \times S^3)$, where k is determined by the orbifold second Betti number of the surface.

To obtain simply connected examples the construction is similar:

• We consider the blow-up $\tilde{S}$ of S at $k \geq 2$ of the singular points.

• We construct two indipendent over $\mathbb{Q}$ divisors D_1 and D_2 such that the Seifert S^1-bundle $\tilde{M}_1 \to \tilde{S}$ corresponding to D_1 is simply connected and a smooth S^1-bundle $\pi_2 : \tilde{M} \to \tilde{M}_1$ determined by the pull-back of D_2 to $\tilde{M}_1$.

• By a Kollar's result $\tilde{M}_1$ is diffeomorphic to $\#_k(S^2 \times S^3)$.

• Since $\tilde{M}$ is a simply-connected 6-manifold with a free S^1-action and $w_2(\tilde{M}) = 0$, then $\tilde{M}$ has no torsion in the cohomology.

• $\tilde{M}$ is diffeomorphic to $\#_r(S^2 \times S^4)\#_{r+1}(S^3 \times S^3)$, where $r = rk(H^2(\tilde{M}_1, \mathbb{Q})) - 1 = rk(H^2(S, \mathbb{Q})) - 2$.

Theorem (F, Grantcharov, Vezzoni)

Let $13 \leq k \leq 22$ and $14 \leq r \leq 22$. Then on the smooth manifolds $S^1 \times \#_k(S^2 \times S^3)$ and $\#_r(S^2 \times S^4)\#_{r+1}(S^3 \times S^3)$ there are complex structures with trivial canonical bundle admitting a balanced metric and a solution to the Hull-Strominger system via the Fu-Yau ansatz.

Remark

- The cases $k = 22$ and $r = 22$ correspond to Fu-Yau solutions.
- They have the structure of a principal S^1-bundle over Seifert S^1-bundles.
- The simply-connected examples are obtained starting from a K3 orbifold with isolated A1 singular points and trivial orbifold fundamental group.

The Anomaly flow

The solutions of the Hull-Strominger system can be viewed as stationary points of the following flow of positive $(2,2)$-forms, called the "Anomaly flow"

$$\begin{cases} \partial_t(\|\Omega\|_{\omega(t)}\omega(t)^2) = i\partial\overline{\partial}\omega(t) + \alpha'(Tr(R_t \wedge R_t) - Tr(F_t \wedge F_t)) \\ H(t)^{-1}\partial_t H(t) = \dfrac{\omega(t)^2 \wedge F_t}{\omega(t)^3}, \quad \omega(0) = \omega_0, \ F(0) = F_0. \end{cases}$$

with ω_0 (conformally balanced) [Phong, Picard, Zhang].

In the compact case:

- Short-time existence and uniqueness [Phong, Picard, Zhang].
- For $t \to \infty$ the limit solves the Hull-Strominger system $\hookrightarrow$ new proof of Fu-Yau non-Kähler solutions [Phong, Picard, Zhang].

THANK YOU VERY MUCH FOR THE ATTENTION!!