Inverse problems, data assimilation and methods in models of geodynamics

Mathematical Model of Geophysical Problem

Many geophysical problems can be described by mathematical models, i.e., by a set of partial differential equations and boundary and/or initial conditions defined in a specific domain.

A mathematical model links the *causal characteristics* of a geophysical process with its *effects*.

The causal characteristics of the process include, for example, parameters of the initial and boundary conditions, coefficients of the differential equations, and geometrical parameters of a model domain.

What are Direct Problems?

The principal aim of the *direct mathematical problem* is to determine the relationship between the *causes* and *effects* of the geophysical process or phenomenon and hence to find a solution to the mathematical problem for a given set of parameters and coefficients.

What are Inverse Problems?

Solution of an *inverse problem* entails determination of unknown *causes* of geophysical processes or phenomena based on observation of their *effects*.

An *inverse problem* is the opposite of a direct problem. An inverse problem is considered when there is a lack of information on the causal characteristics (but information on the effects of the geophysical process exists).

Inverse Problems

The inference of the values of some parameters based on observations is, of course, as old as quantitative science, but it was only in the middle of XVIII century that the first formalizations arose. The two basic problems of that time were the use of geodetic data to estimate the shape of the Earth, and the use of astronomical data to infer the orbits of planets and comets.

Marquis P-S. de Laplace
(1749-1827)

J. C. F. Gauss
(1777-1855)

Inverse Problems

Jacques Hadamard
(1865 – 1963)

Andrei Tikhonov
(1906 – 1993)

Jacques-Louis Lions
(1928 – 2001)

Inverse (Time-Reverse) Problems

The mantle is heated from the core and from inside due to decay of radioactive elements. Since mantle convection is described by heat advection and diffusion, one can ask: *Is it possible to tell, from the present temperature estimations of the Earth, something about the Earth's temperature in the geological past?*

Even though heat diffusion is irreversible in the physical sense, it is possible to predict initial conditions for mantle temperature and flow in the geological past using data assimilation techniques without contradicting the basic thermodynamic laws.

In other words, the present observations (mantle temperature, velocity, etc.) can be *assimilated* into the past to constrain the initial conditions for the mantle temperature and velocity.

Well- & Ill-Posed Problems

Inverse problems are often ill-posed. Jacques Hadamard introduced the idea of *well- (and ill-) posed* problems in the theory of partial differential equations (Hadamard 1902).

A mathematical model for a geophysical problem has to be *well-posed* in the sense that it has to have the properties of existence, uniqueness, and stability of a solution to the problem. Problems for which at least one of these properties does not hold are called
ill-posed.

The requirement of stability is the most important one. If a problem lacks the property of stability then its solution is almost impossible to compute because computations are polluted by unavoidable errors. If the solution of a problem does not depend continuously on the initial data, then, in general, the computed solution may have nothing to do with the true solution.

DATA ASSIMILATION IN GEODYNAMICS

How to assimilate the data in geodynamic models?

1: Collect the geophysical, geological, and geodetic data

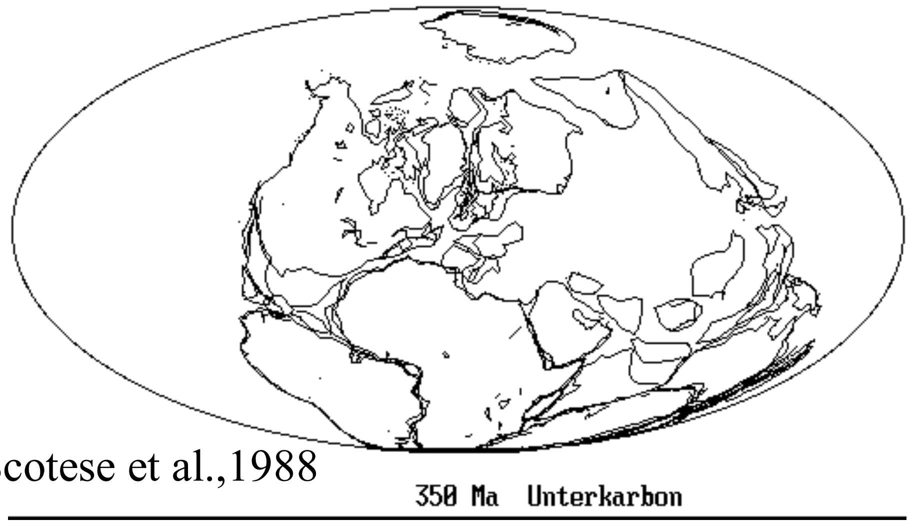
2: Learn mathematical and computational approaches to data assimilation

3: Assimilate present data to restore the dynamics of the Earth's mantle in the geological past.

DATA COLLECTION FOR ASSIMILATION

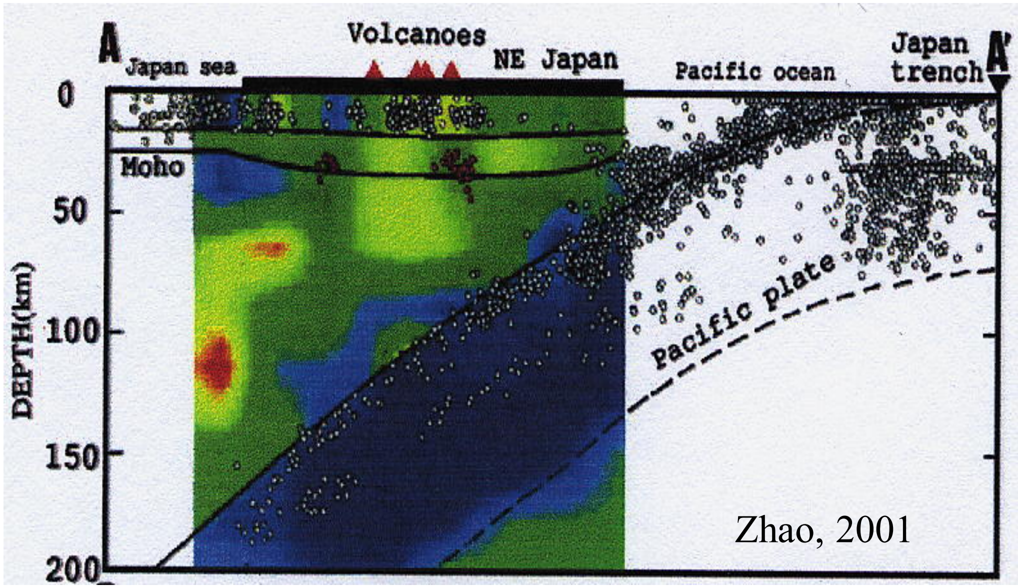
- Seismic images (tomography, reflection, other)
- Borehole data
- Heat flow data
- Composition of the crust and upper mantle
- Data from mineral physics
- Geodetic measurements
- Data on paleogeographic reconstructions
- Data on subsidence and uplift
- etc

WHY DO WE ASSIMILATE THE DATA?



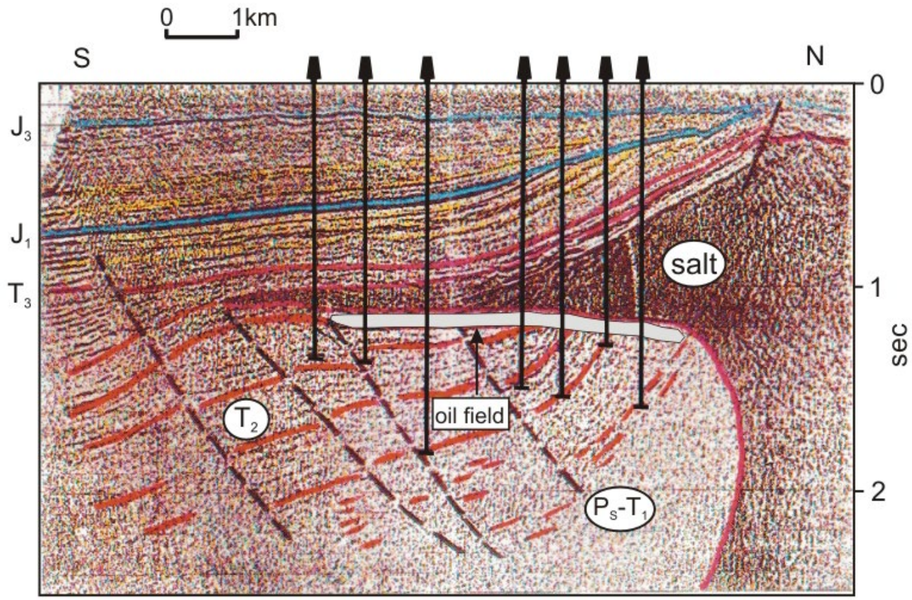
Scotese et al.,1988

To restore the temperature, composition, and movements in the mantle and the history of continental motion



Zhao, 2001

To understand the origin and evolution of the present high and low velocity anomalies in a particular region



Volozh, Talbot & Ismail-Zadeh, 2003

To reconstruct structural and geothermal evolutions of sedimentary basins (particularly, complicated by salt tectonics)

Statement of the Problem

Model domain $\Omega = (0, x_1 = l_1) \times (0, x_2 = l_2) \times (0, x_3 = h), \quad t \in (\theta_1, \theta_2)$

Governing equations

Momentum (Stokes) equation

$$-\nabla P + \nabla \cdot (\mu[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]) + Ra T \mathbf{e} = 0 \qquad Ra = \frac{\alpha g \rho \Delta T h^3}{\tilde{\mu} \kappa}$$

Continuity equation $\nabla \cdot \mathbf{u} = 0$

Heat equation $\partial T / \partial t + \mathbf{u} \cdot \nabla T = \nabla^2 T$

Rheology equation

$$\mu(P, T) = \mu_* \exp\left[\frac{E_a + V_a P}{RT}\right]$$

Statement of the Problem

Boundary and Initial Conditions

At all model boundaries, impenetrability condition with perfect slip

$$\mathbf{n} \cdot \nabla \mathbf{u}_{tg} = 0, \quad \mathbf{n} \cdot \mathbf{u} = 0$$

Zero heat flux through the vertical boundaries

$$\mathbf{n} \cdot \nabla T = 0$$

Isothermal upper and lower boundaries

$$T = T_u \quad \text{at} \quad x_3 = h, \qquad T = T_l \quad \text{at} \quad x_3 = 0$$

→ At initial time $T(x_1, x_2, x_3, t = \theta_1) = T^*(x_1, x_2, x_3)$

← At final time $T(x_1, x_2, x_3, t = \theta_2) = T^{**}(x_1, x_2, x_3)$

Backward in Time

How to solve the inverse problem of thermal convection?

$-\nabla$ … $\mathbf{e} = 0$

∇ …

∂ …

Principal difficulty in backward modeling is neither the Stokes nor advection equations, but the heat equation

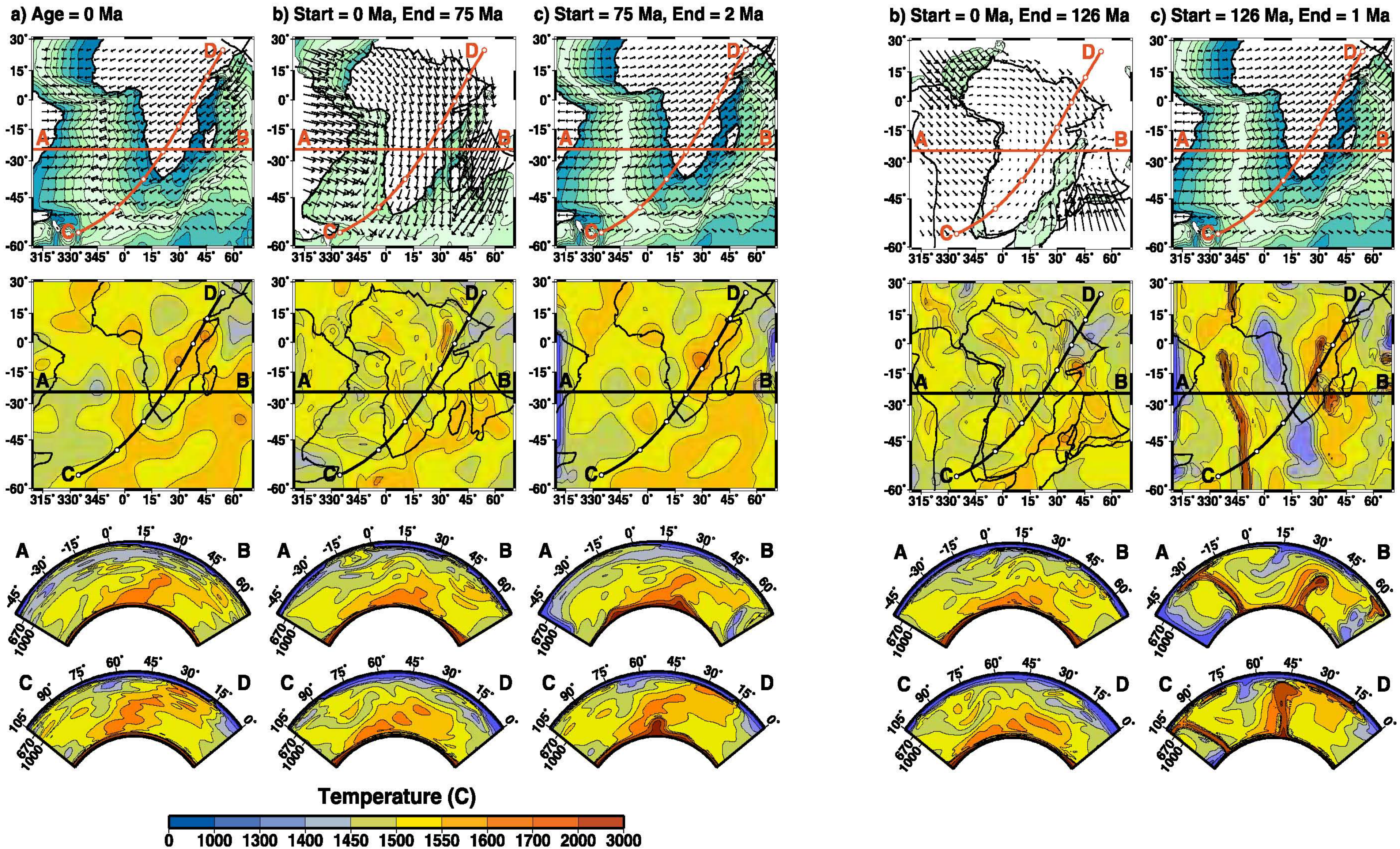
The inverse heat problem is ill-posed (Hadamard, 1923).

NO DIFFUSION

BAD Application to Plate Motion Reconstruction

Geochemistry Geophysics Geosystems G3

CONRAD AND GURNIS: SEISMIC TOMOGRAPHY 10.1029/2001GC000299



Data Assimilation Methods

- Backward advection method (BAD)

  Ismail-Zadeh et al., 1998, 2001, 2004
  Steinberger and O'Connell, 1998 (Harvard)
  Kaus and Podladchikov, 2001 (ETHZ)
  Conrad and Gurnis, 2003 (CalTech)

- Variational method (VAR)

  Ismail-Zadeh et al., 2003; 2004; 2006
  Bunge et al., 2003 (Princeton, now Munich)
  Hier-Majumder et. al, 2005; 2006 (Minnesota)
  Liu and Gurnis, 2008 (CalTech)

Variational Method (VAR)

The variational method finds the best fit between the forecast model state and the observations by minimizing an objective functional over space and time.

To minimize the objective functional over time, an assimilation time interval is defined and an adjoint model is typically used to find the derivatives of the objective functional with respect to the model states.

The variational method can be formulated with a weak constraint where errors in the model formulation are taken into account as process noise [e.g., *Bunge et al.*, 2003] or with a strong constraint where the model is assumed to be perfect [e.g., *Ismail-Zadeh et al.*, 2003]. The strong constraint makes the problem computationally tractable.

Variational Method (VAR)

Consider objective functional

$$J(\varphi) = \left\| T(\theta_2, \cdot; \varphi) - \chi(\cdot) \right\|^2 = \int_{\Omega} \left| T(\theta_2, x; \varphi) - \chi(x) \right|^2 dx$$

where

$T(\theta_2, x; \varphi)$ solution of the forward heat equation with appropriate boundary conditions at final time θ_2, which corresponds to unknown as yet the initial temperature distribution $\varphi = \varphi(x)$;

$\chi(x) = T(\theta_2, x; T_0)$ known temperature distribution at the final time for the initial temperature $T_0 = T_0(x)$.

The objective functional has its unique minimum at $\varphi = T_0$.

We seek **a minimum of the objective functional** with respect to initial temperature

$$\nabla J(\varphi) = 0$$

Variational Method (VAR)

It can be shown that $\nabla J(\varphi) = \Psi(\theta_1, x)$, where

$$\partial\Psi / \partial t + \mathbf{u} \cdot \nabla\Psi = -\nabla^2\Psi, \quad x \in \Omega, \; t \in (\theta_1, \theta_2),$$

$$\sigma_1\Psi + \sigma_2 \partial\Psi / \partial\mathbf{n} = 0, \qquad x \in \Gamma, \; t \in (\theta_1, \theta_2),$$

$$\Psi(\theta_2, x) = 2\left[T(\theta_2, x; \varphi) - \chi(x)\right], \quad x \in \Omega.$$

The boundary problem is referred to as the problem ***adjoint*** to the heat problem.

Note that the adjoint problem is *well-posed*.

Variational Method (VAR)

as applied to the heat problem

- Solve the forward heat problem with initial temperature $T=\varphi_k$ and find $T(\theta_2, x; \varphi_k)$ at the final time:

$$\partial T / \partial t + \mathbf{u} \cdot \nabla T = \nabla^2 T, \qquad x \in \Omega,\ t \in (\theta_1, \theta_2)$$

$$\sigma_1 T + \sigma_2 \partial T / \partial \mathbf{n} = 0, \qquad x \in \Gamma,\ t \in (\theta_1, \theta_2),$$

$$T(\theta_1, x) = \varphi_k(x), \quad x \in \Omega.$$

- Solve the adjoint problem backwards in time, find $\Psi(\theta_1, x)$ and hence $\nabla J(\varphi) = \Psi(\theta_1, x)$:

$$\partial \Psi / \partial t + \mathbf{u} \cdot \nabla \Psi = -\nabla^2 \Psi, \quad x \in \Omega,\ t \in (\theta_2, \theta_1),$$

$$\sigma_1 \Psi + \sigma_2 \partial \Psi / \partial \mathbf{n} = 0, \qquad x \in \Gamma,\ t \in (\theta_2, \theta_1),$$

$$\Psi(\theta_2, x) = 2\left[T(\theta_2, x; \varphi) - \chi(x)\right], \quad x \in \Omega.$$

- Determine α_k and then update the initial temperature, that is, find φ_{k+1}:

$$\alpha_k = \min\left[1/(k+1); J(\varphi_k) / \left\|\nabla J(\varphi_k)\right\|\right],$$

$$\varphi_{k+1} = \varphi_k - \alpha_k \nabla J(\varphi_k), \quad \varphi_0 = T_*, \quad k = 0,1,2,\ldots$$

- Compute the iterations until

$$\delta\varphi_n = J(\varphi_n) + \left\|\nabla J(\varphi_n)\right\|^2 < \varepsilon$$

Numerical Approach

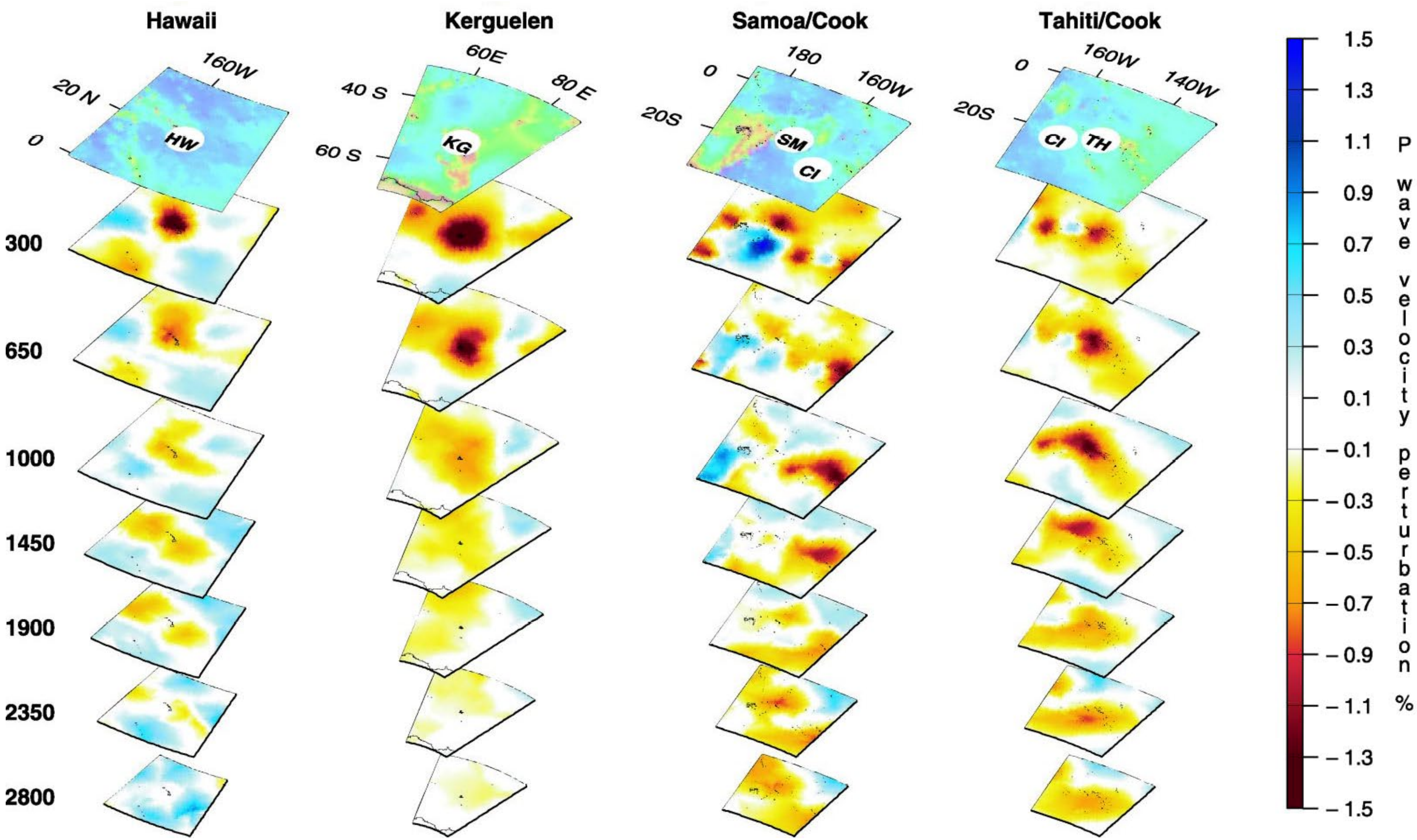
To restore thermoconvective flow and temperature, time slot $[\theta_1, \theta_2]$ is divided into m subslots by $t_i = \theta_2 - \tau i$.

At each time subslot:

- the Stokes and continuity equations are solved to find the velocity field; and
- the heat problem is solved by the variational method.

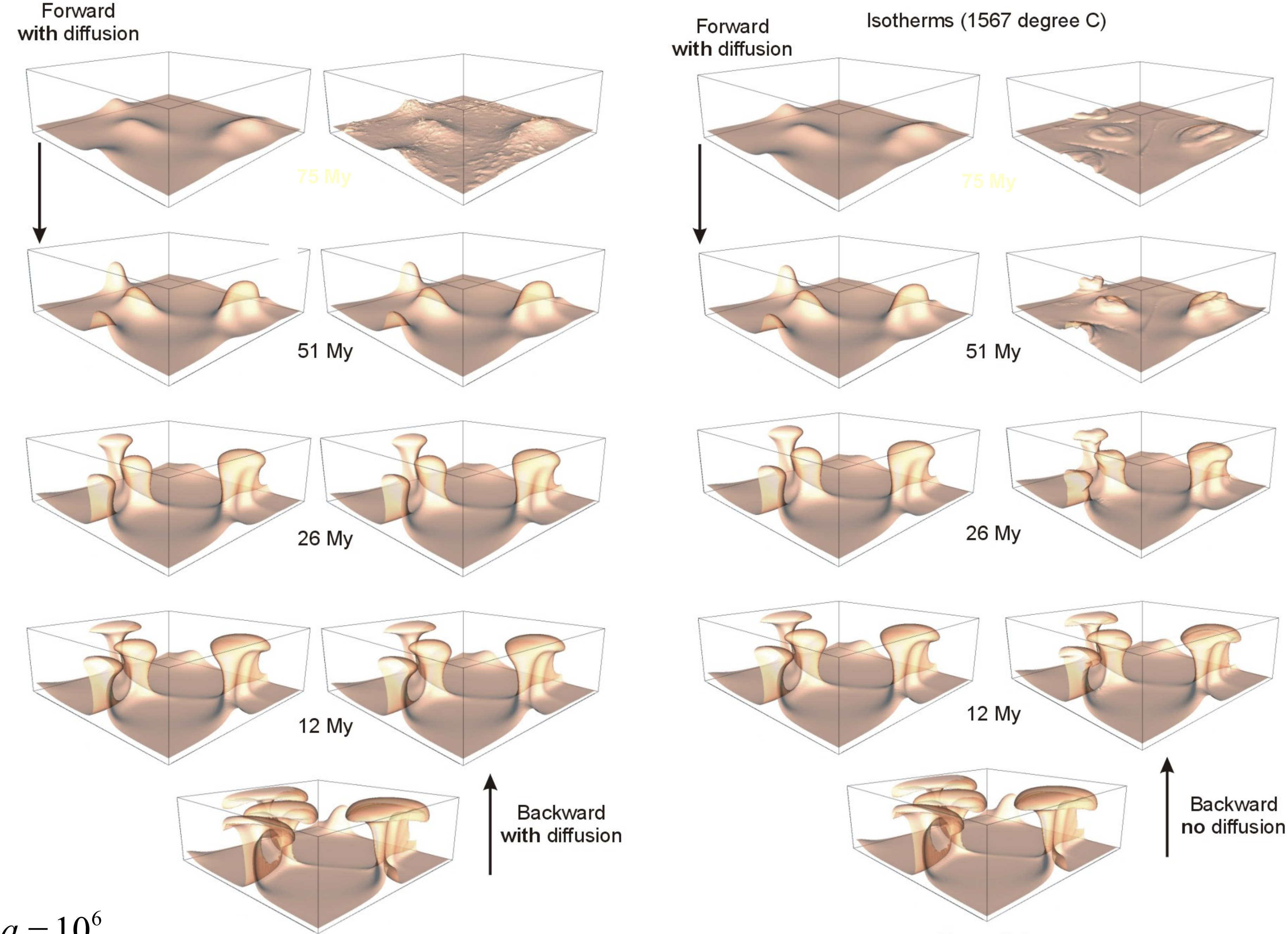
Inverse (Time-Reverse) Modelling of Mantle Plume Evolution

Variety of mantle plumes



Montelli et al., 2004

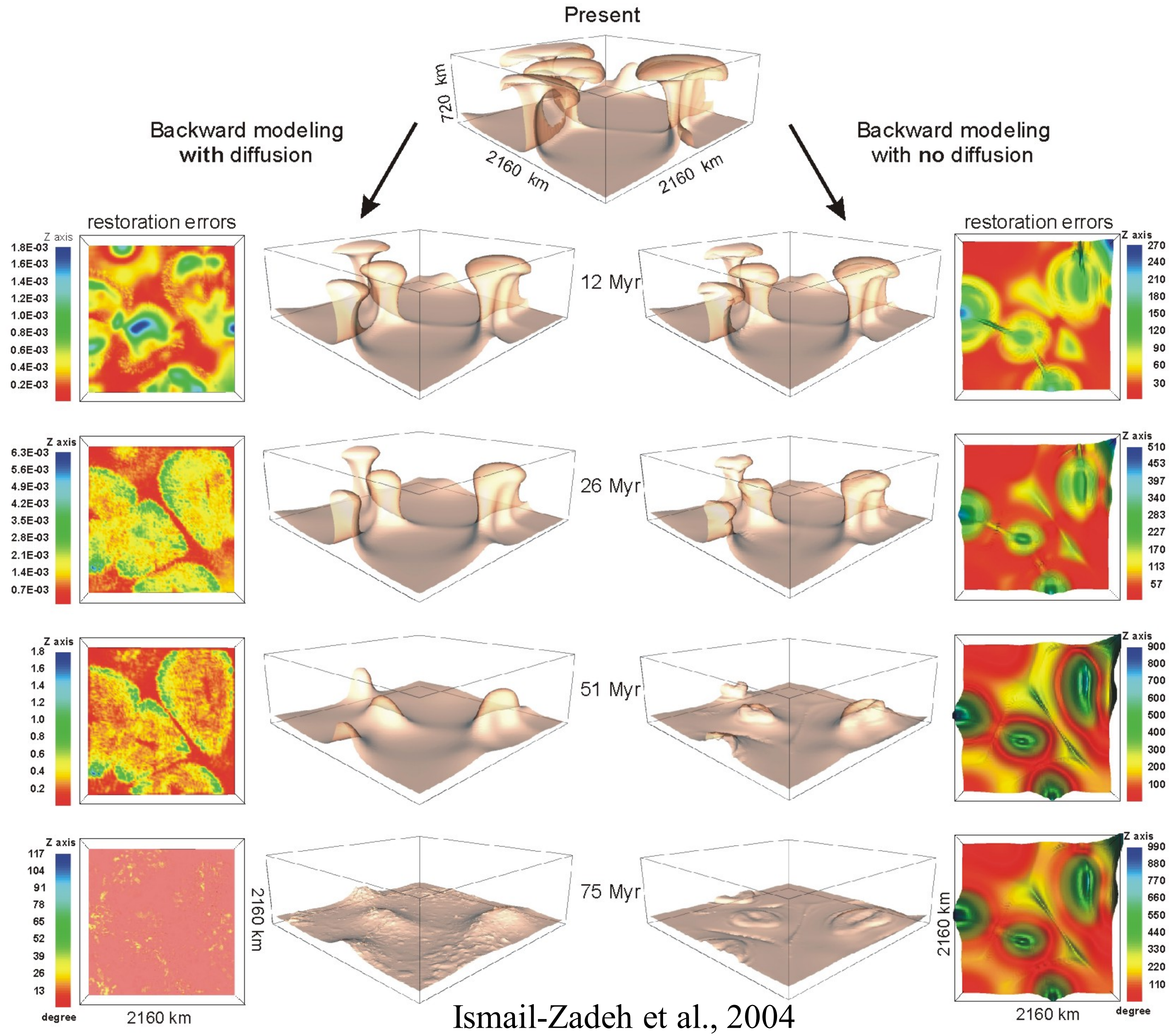
VAR versus BAD



$Ra = 10^6$

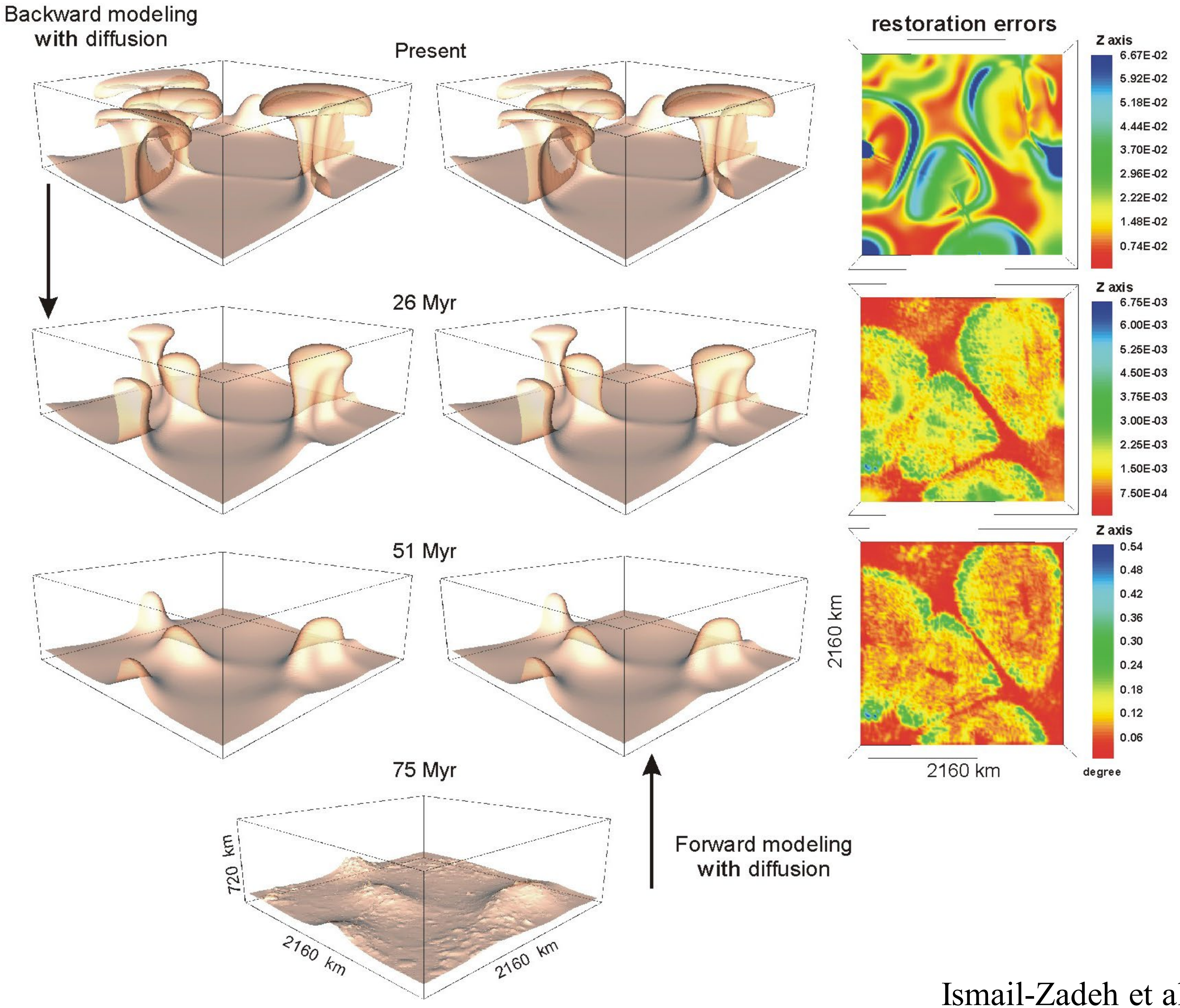
Ismail-Zadeh et al., 2004

VAR versus BAD



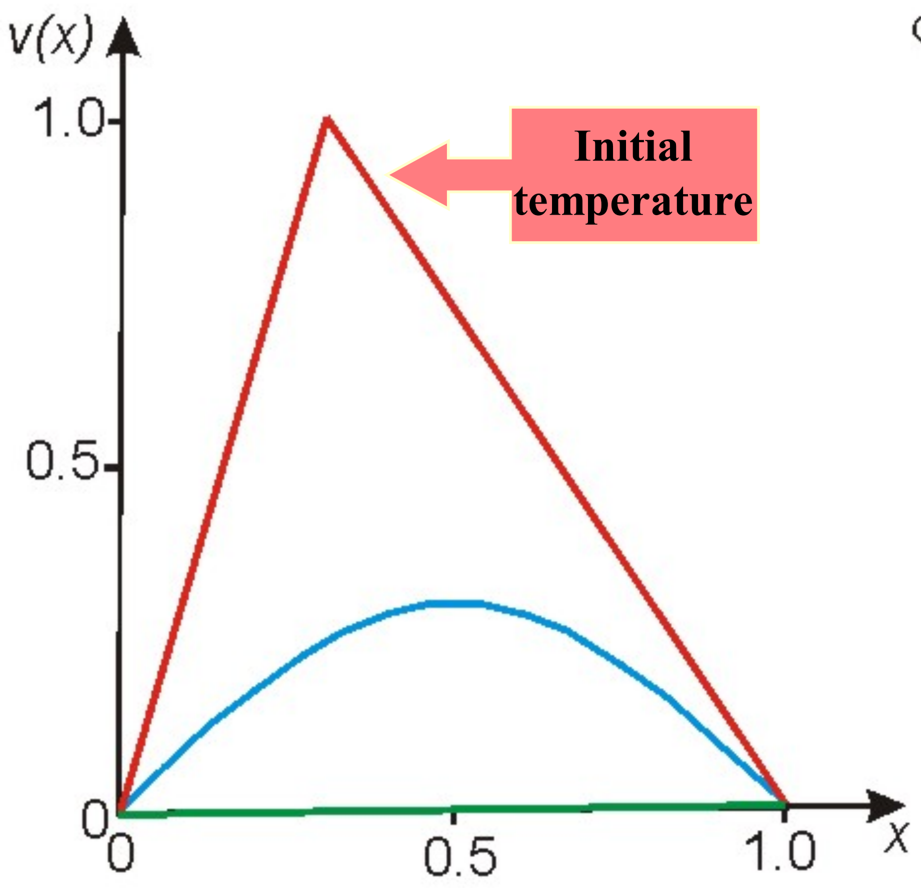
Ismail-Zadeh et al., 2004

Model Verification

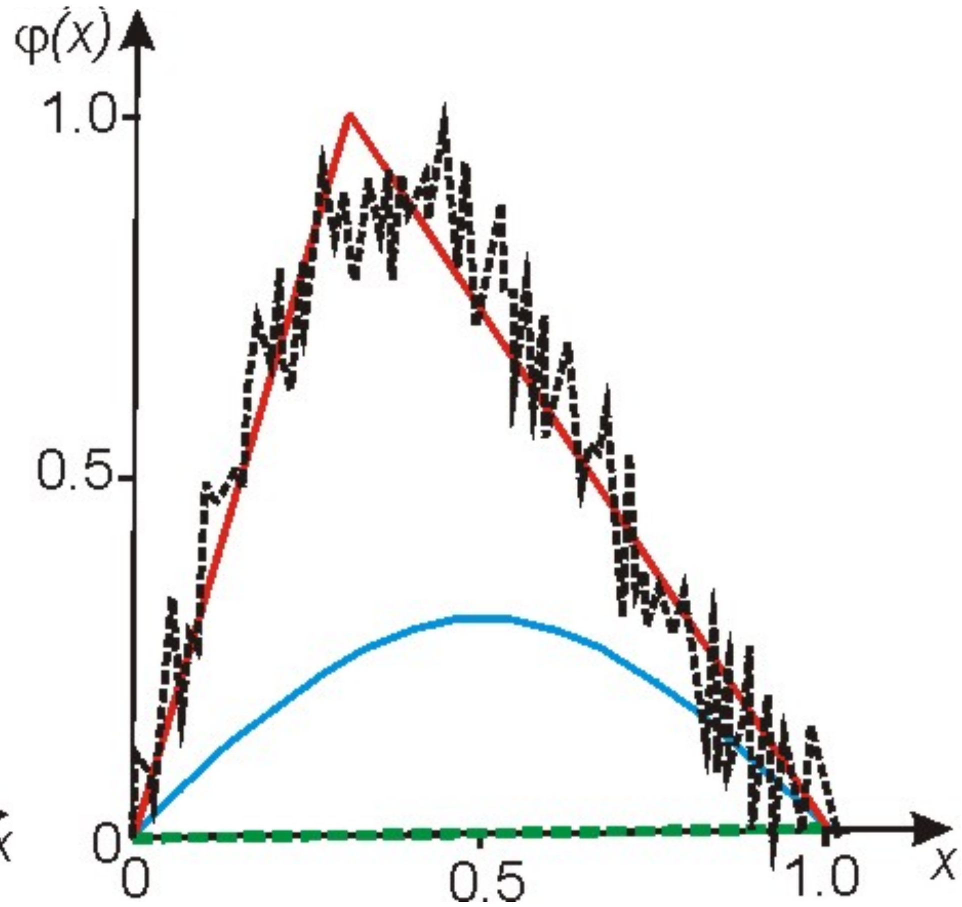


Ismail-Zadeh et al., 2004

1-D Heat Diffusion Problem

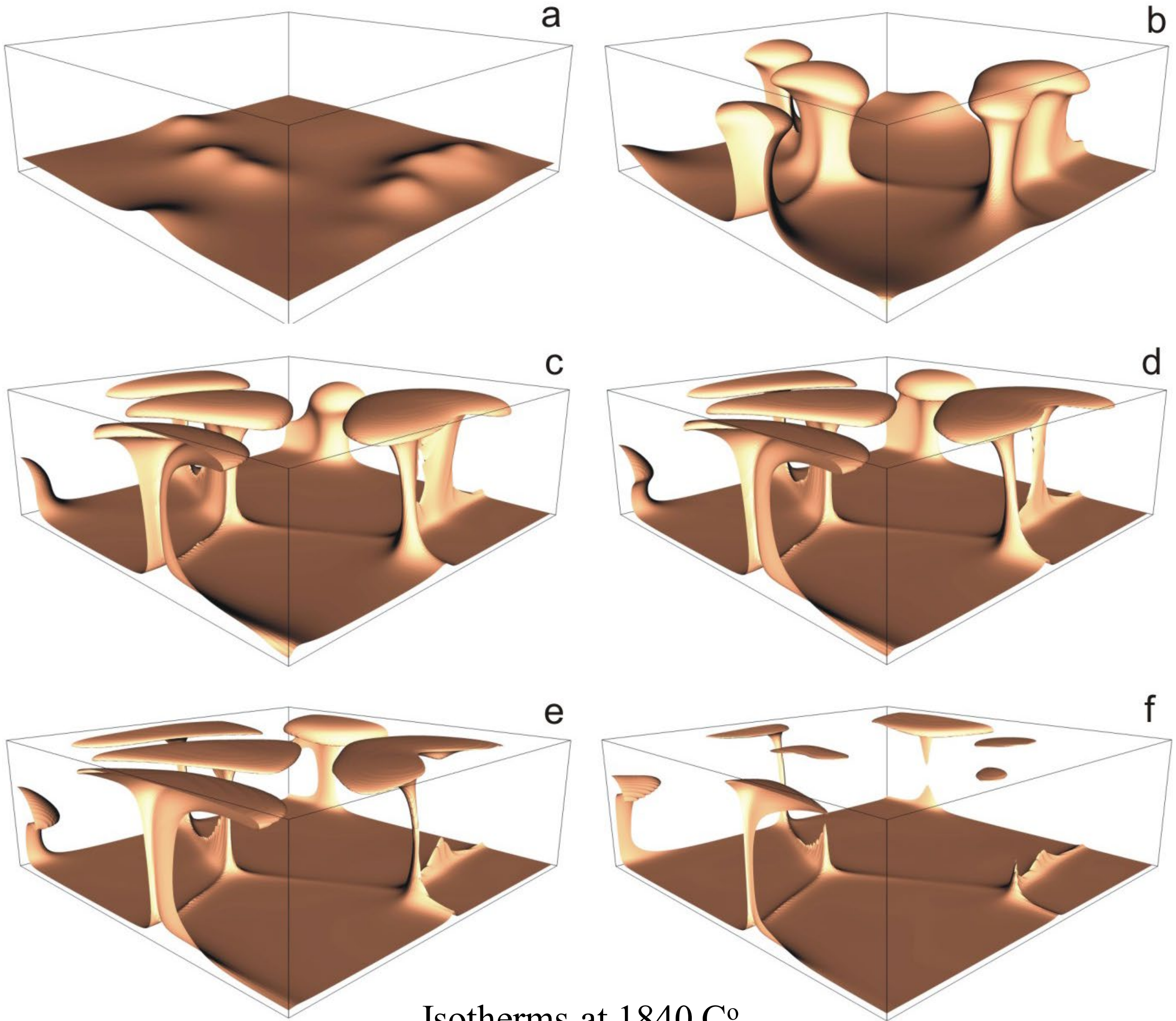


Forwards in time



Backwards in time

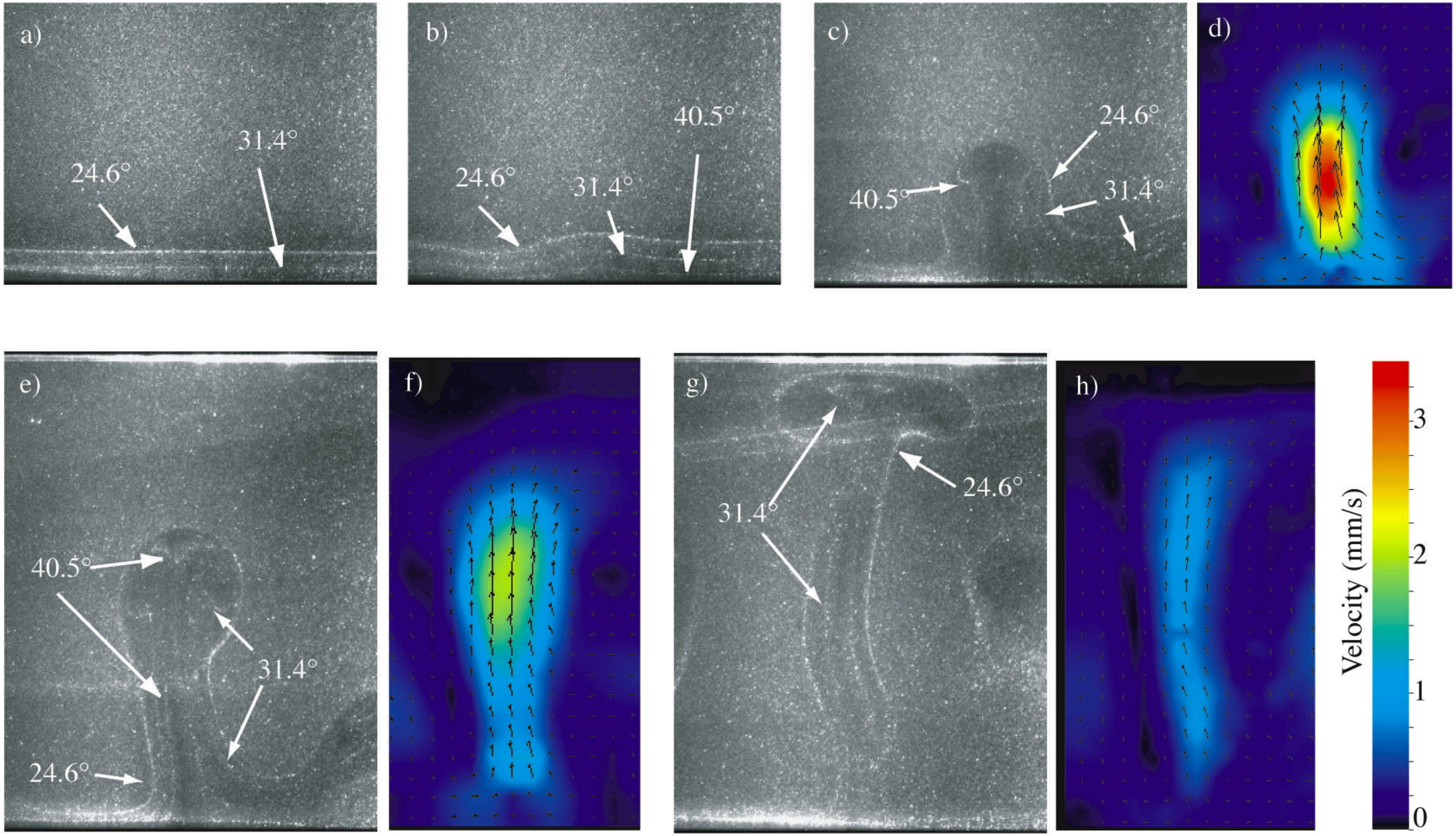
Effect of Heat Diffusion
(numerical experiment)



Isotherms at 1840 C°

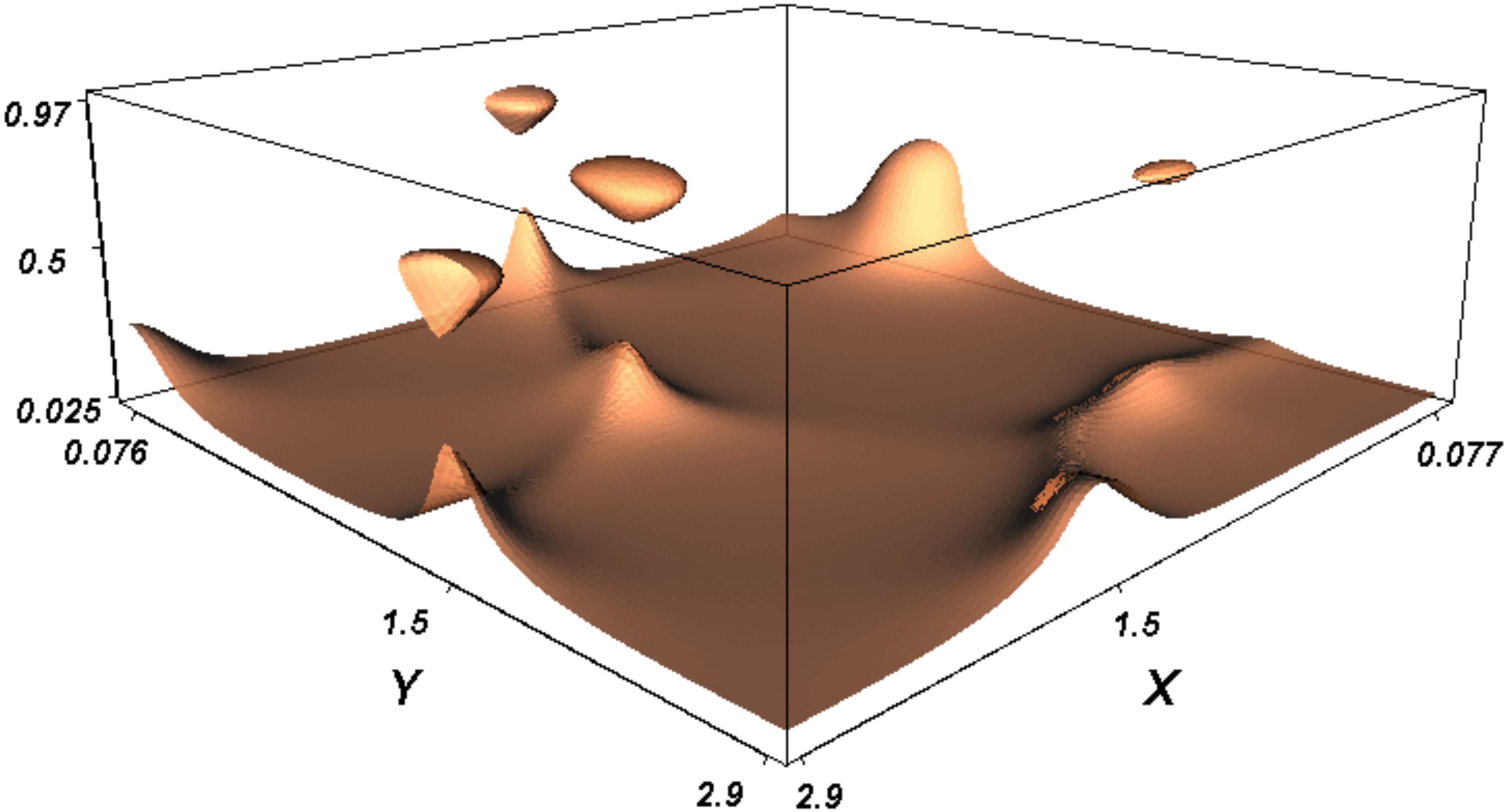
Ismail-Zadeh et al., 2006

Effect of Heat Diffusion
(laboratory experiment)



Davaille & Vatteville, 2005

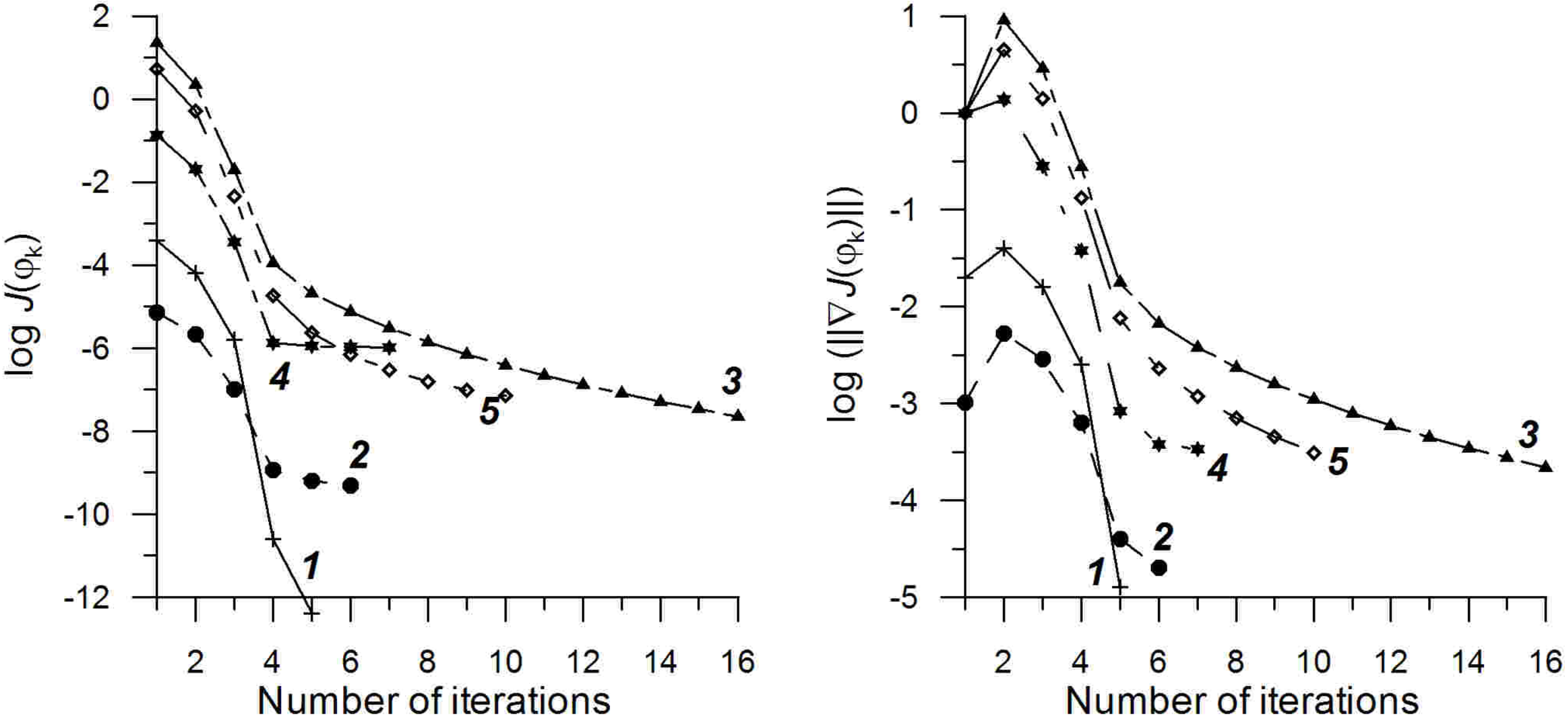
Reconstruction of Diffused Mantle Plumes



Isotherms at 1840 C°

Ismail-Zadeh et al., 2006

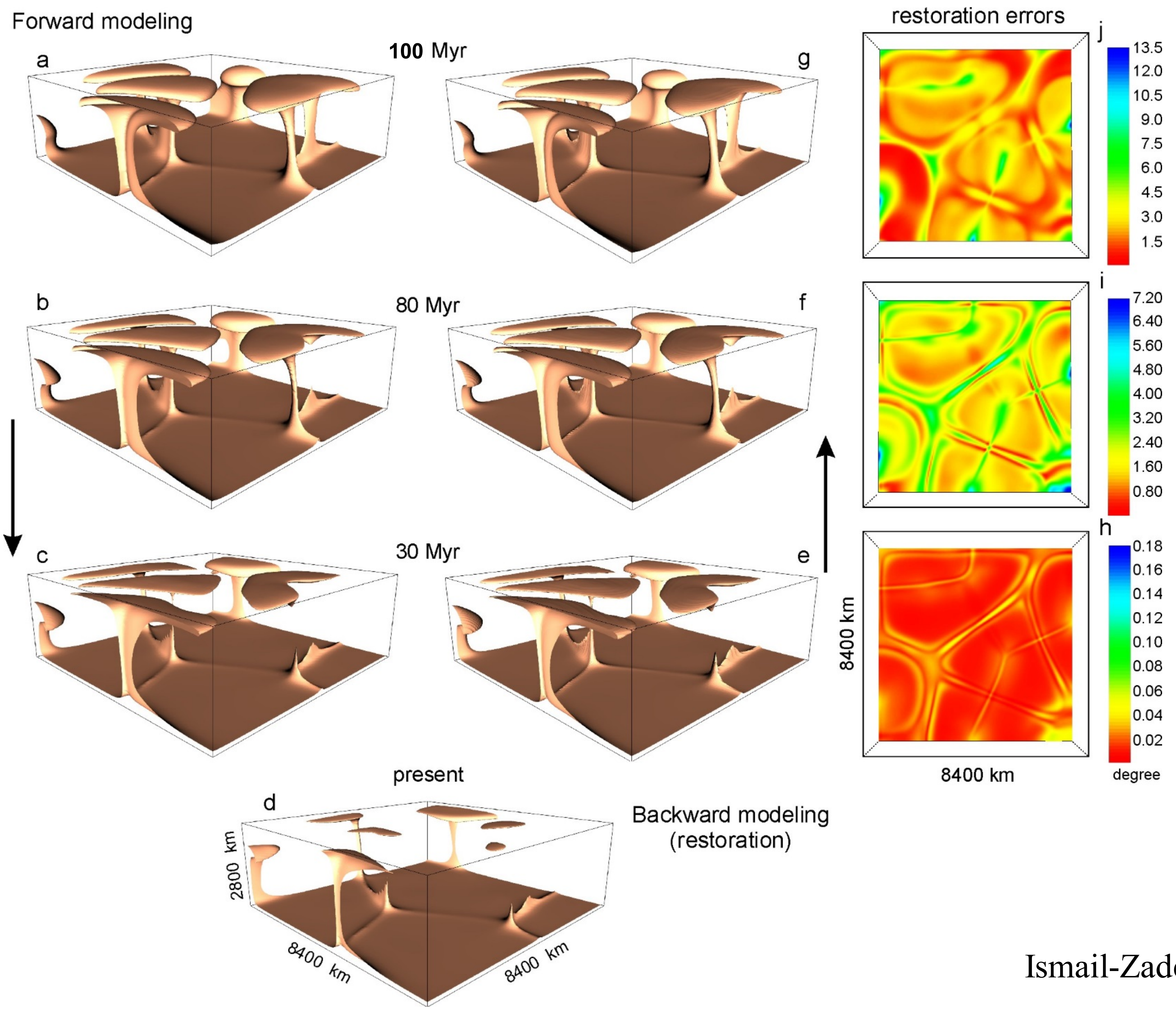
Conduction vs. Advection



Curves: r=20 and Ra=10^6 (***1***), Ra=10^4 (***2***), Ra=10^3 (***3***); r=200 and Ra=10^4 (***4***); Ra=10^3 (***5***).

Ismail-Zadeh et al., 2006

VAR Application to Restoration of Mantle Plumes



Ismail-Zadeh et al., 2006

VAR Application to Restoration of Slab Subduction

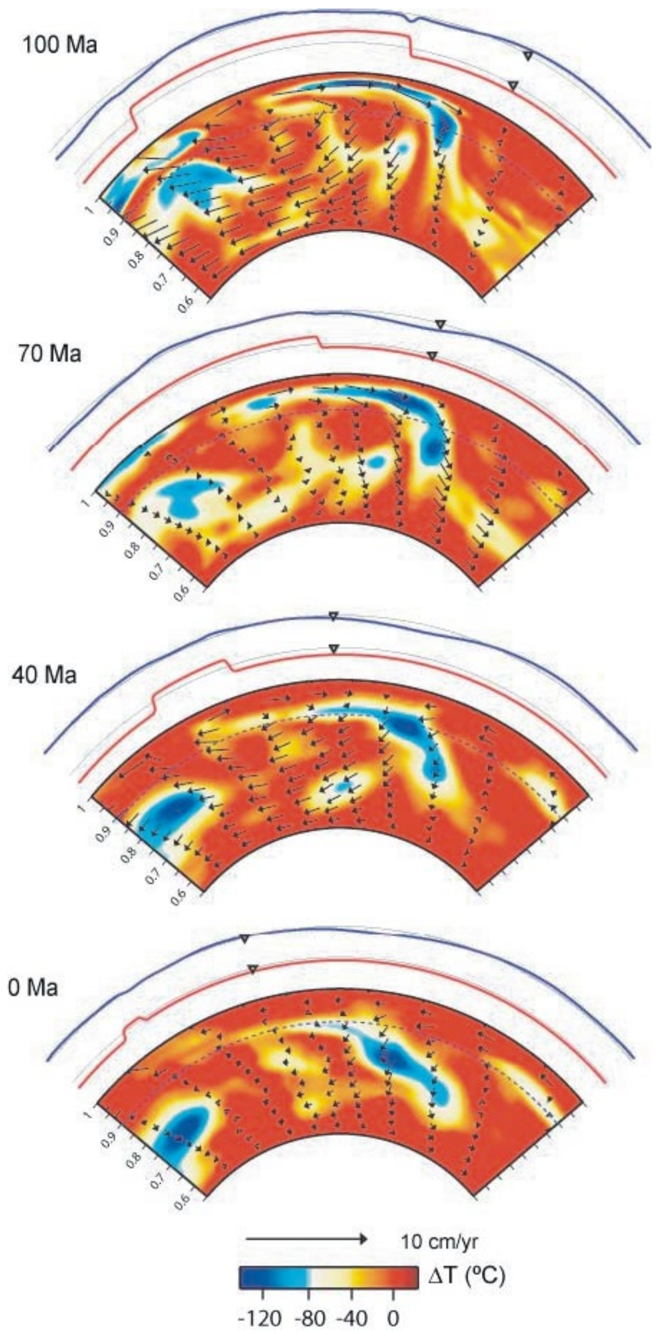
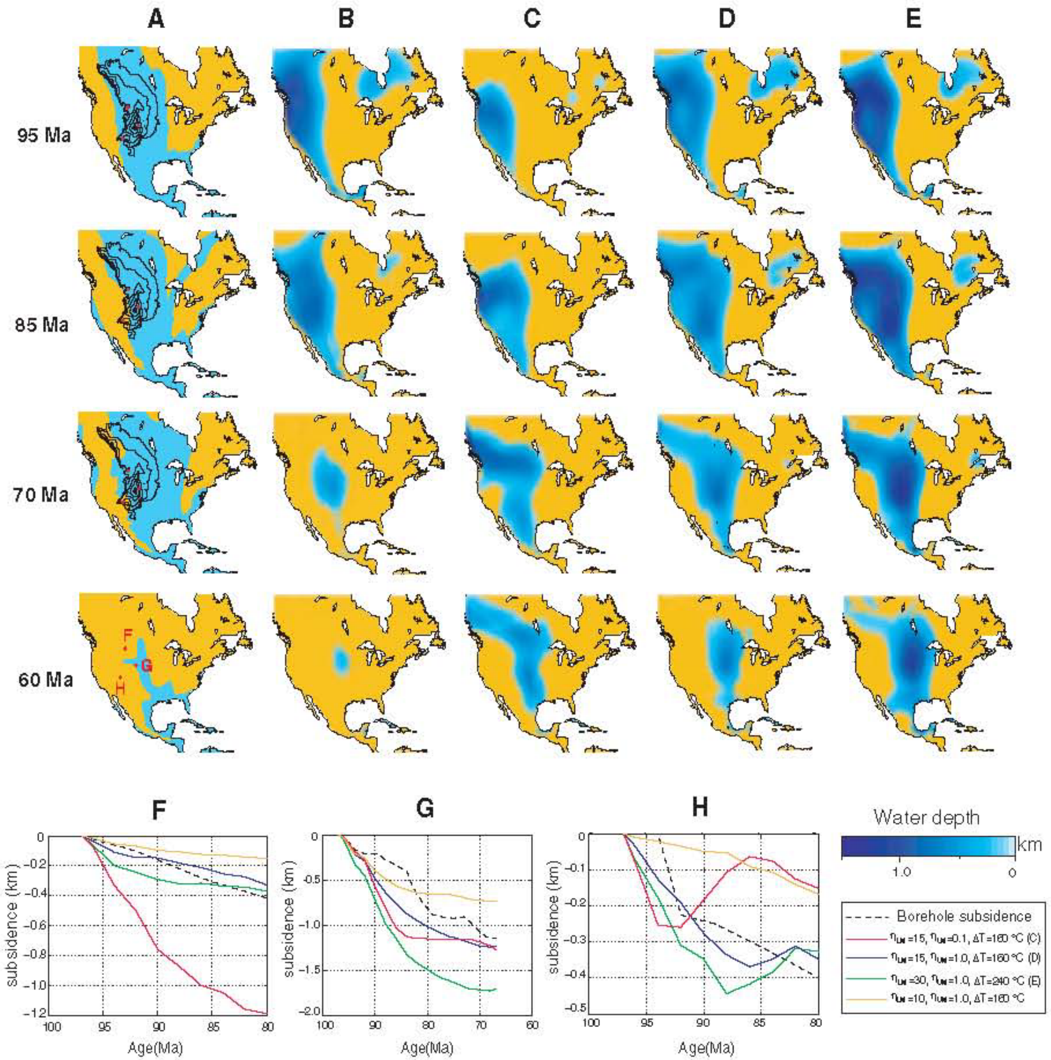


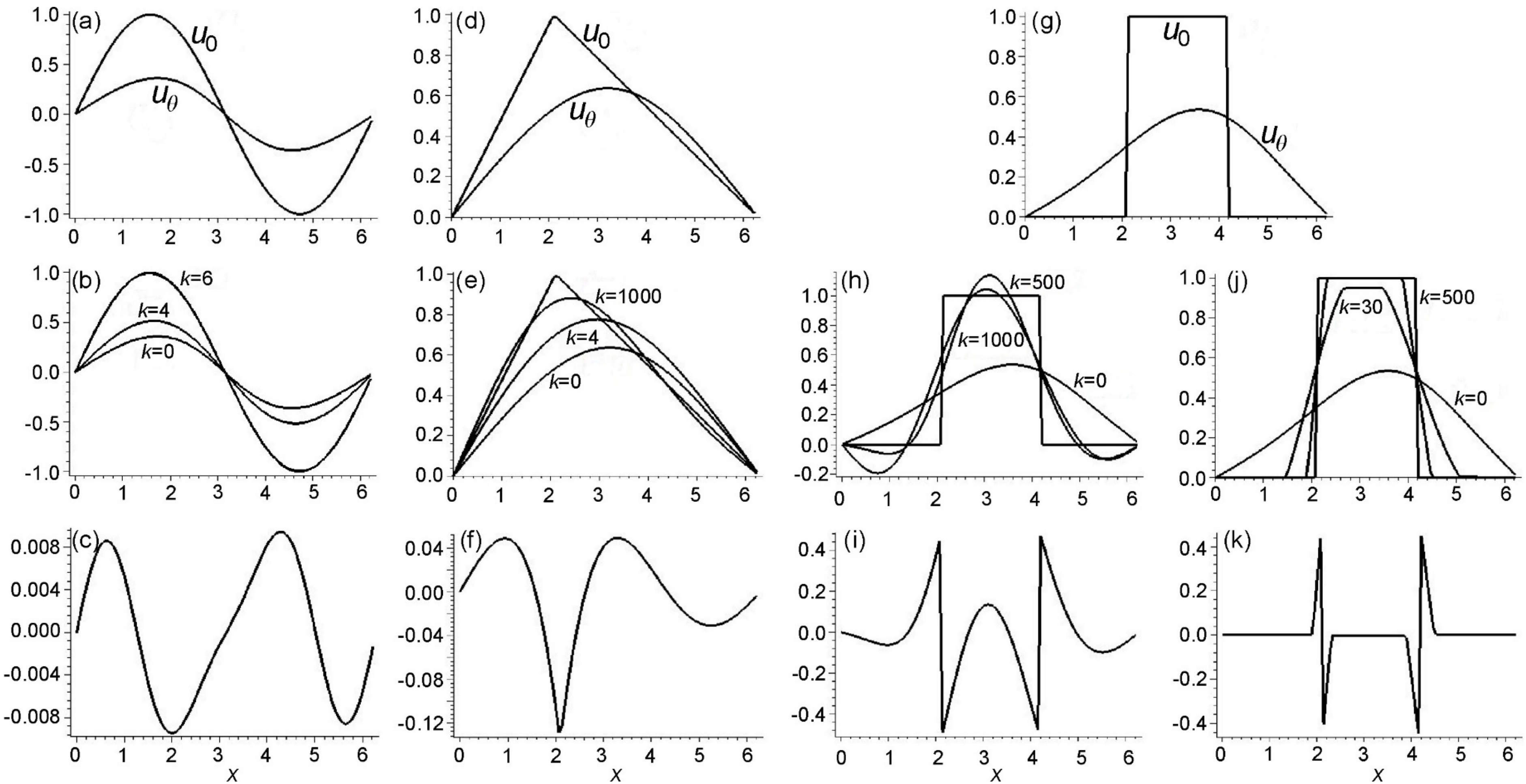
Fig. 2. Observed and predicted continental flooding and borehole subsidence for different temperature scaling and mantle viscosities. (**A**) Geologically inferred flooding (blue areas) bounded by paleoshorelines (*13*) with cumulative Cretaceous isopachs (*14*) overlain (2-km contour interval), with red dots indicating the three boreholes shown in (F) to (H). (**B** to **E**) Predicted flooding with lower-mantle viscosity η_{LM} = 30, 15, 30 and upper-mantle viscosity η_{UM} = 1, 0.1, 1, 1, respectively (relative to 10^{21} Pa s); cases (B) to (D) all have an effective temperature magnitude ΔT = 160°C, and (E) has ΔT = 240°C, where (D) is the best-fit model. (**F** to **H**) Observed and predicted borehole dynamic subsidence from four different models.



Liu et al., 2008

Numerical Noise and Smoothness of Data

$$u_t + uu_x = u_{xx},\ 0 \le t \le 1,\quad 0 \le x \le 2\pi$$
$$u(t,0) = 0\ ,\ u(t,2\pi) = 0\ ,\ 0 \le t \le 1$$
$$u_\theta = u(1, x; u_0),\ 0 \le x \le 2\pi\ \text{ at } t = 1$$



Ismail-Zadeh et al., 2006

DATA ASSIMILATION METHODS

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  Hier-Majumder et. al, 2005; 2006 (Minnesota)
  Liu and Gurnis, 2008 (CalTech)

- Quasi-reversibility method (QRV) ➡ **NEXT LECTURE**

  Ismail-Zadeh et al., 2007, 2008
  Glisovic et al., 2009 (U. Toronto)