Alik Ismail-Zadeh

Data assimilation in geodynamical models

Part 2

DATA ASSIMILATION METHODS

Last Lecture

- Backward advection method (BAD)

  Ismail-Zadeh et al., 1998, 2001, 2004
  Steinberger and O'Connell, 1998 (Harvard)
  Kaus and Podladchikov, 2001 (ETHZ)
  Conrad and Gurnis, 2003 (CalTech)

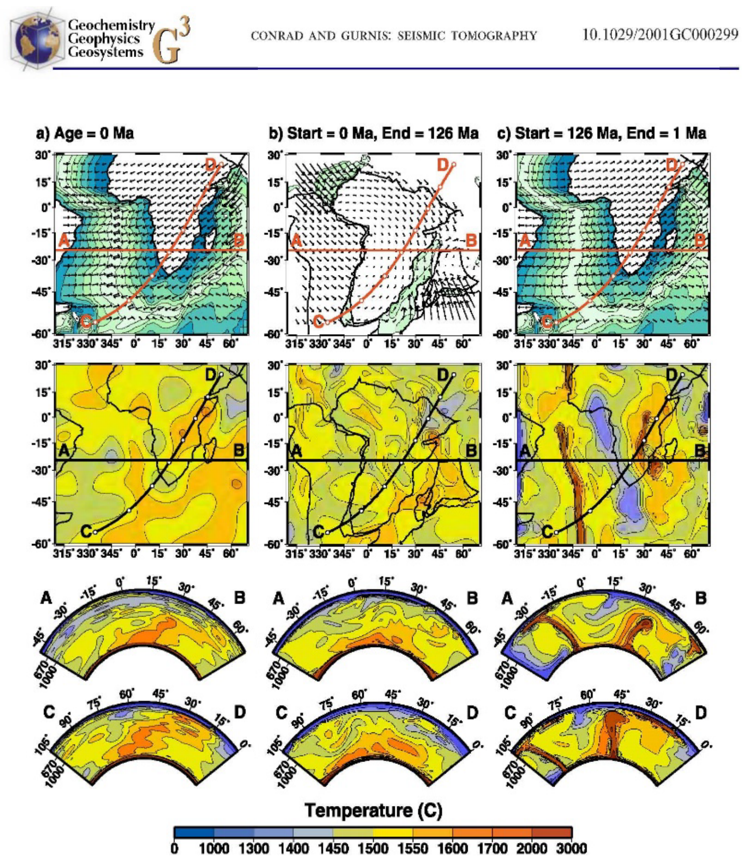
Last Lecture

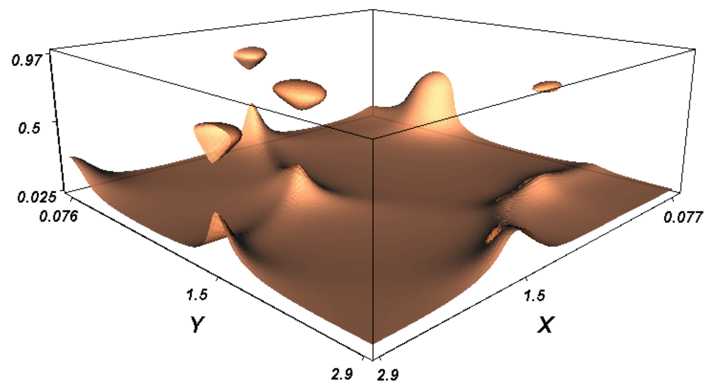
- Variational method (VAR)

  Ismail-Zadeh et al., 2003; 2004; 2006
  Bunge et al., 2003 (Princeton)
  Hier-Majumder et. al, 2005; 2006 (Minnesota)
  Liu and Gurnis, 2008 (CalTech)

- **Quasi-reversibility method (QRV)**

  Ismail-Zadeh et al., 2007, 2008
  Glisovic et al., 2009 (U. Toronto)





Quasi-Reversibility Method

Jacques-Louis Lions
(1928 – 2001)

The principal idea of the quasi-reversibility method is based on the transformation of an ill-posed problem into a well-posed problem (Lattes & Lions 1969).

In the case of the backward heat equation, this implies an introduction of an additional term into the equation, which involves the product of a small regularization parameter and higher order temperature derivative. The additional term should be sufficiently small compared to other terms of the heat equation and allow for simple additional boundary conditions.

The data assimilation in this case is based on a search of the best fit between the forecast model state and the observations by minimizing the regularization parameter. The QRV method is proven to be well suited for smooth and non-smooth input data (Lattes & Lions 1969; Samarskii & Vabishchevich 2004).

1-D heat conduction problem

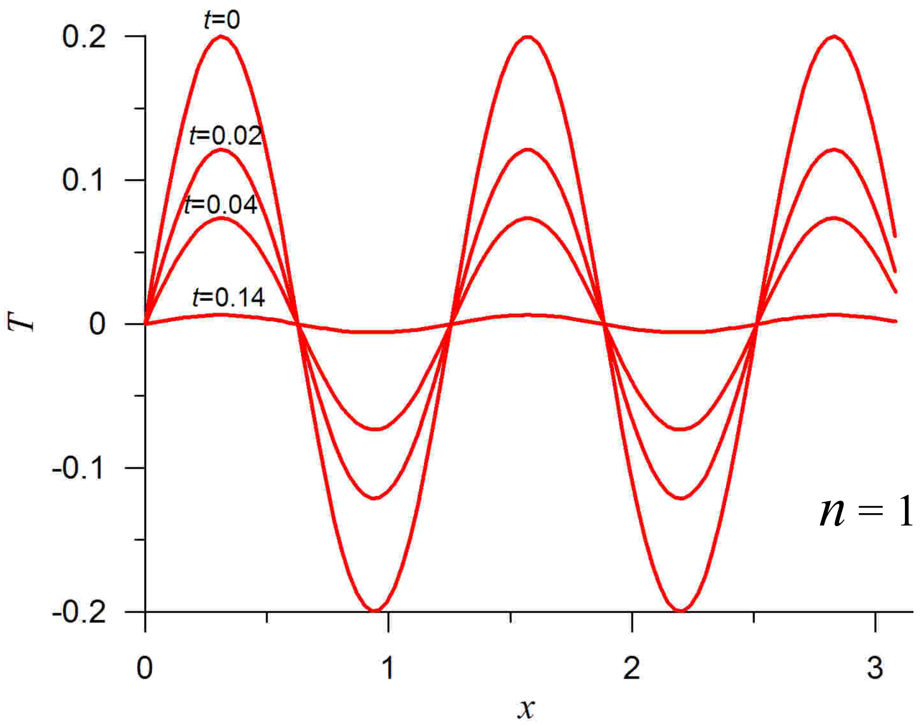
$$\frac{\partial T(t,x)}{\partial t} = \frac{\partial^2 T(t,x)}{\partial x^2}, \quad 0 \le x \le \pi, \quad 0 \le t \le t^*,$$

$$T(t, x=0) = T(t, x=\pi) = 0, \quad 0 \le t \le t^*,$$

$$T(t=0, x) = \frac{1}{4n+1}\sin((4n+1)x), \quad 0 \le x \le \pi.$$

Solution

$$T_n(t,x) = \frac{1}{4n+1} \exp(-(4n+1)^2 t) \sin((4n+1)x)$$



1-D regularized backward heat conduction problem

$$\frac{\partial T_\beta(t,x)}{\partial t} = \frac{\partial^2 T_\beta(t,x)}{\partial x^2} - \beta \frac{\partial^5 T_\beta(t,x)}{\partial t \partial x^4}, \quad 0 \le x \le \pi, \quad t \le t^*,$$

$$T_\beta(t, x=0) = T_\beta(t, x=\pi) = 0, \quad t \le t^*,$$

$$\frac{\partial^2 T_\beta(t, x=0)}{\partial x^2} = \frac{\partial^2 T_\beta(t, x=\pi)}{\partial x^2} = 0, \quad t \le t^*,$$

$$T_\beta(t=t^*, x) = \frac{1}{4n+1} \exp(-(4n+1)^2 t^*) \sin((4n+1)x), \quad 0 \le x \le \pi.$$

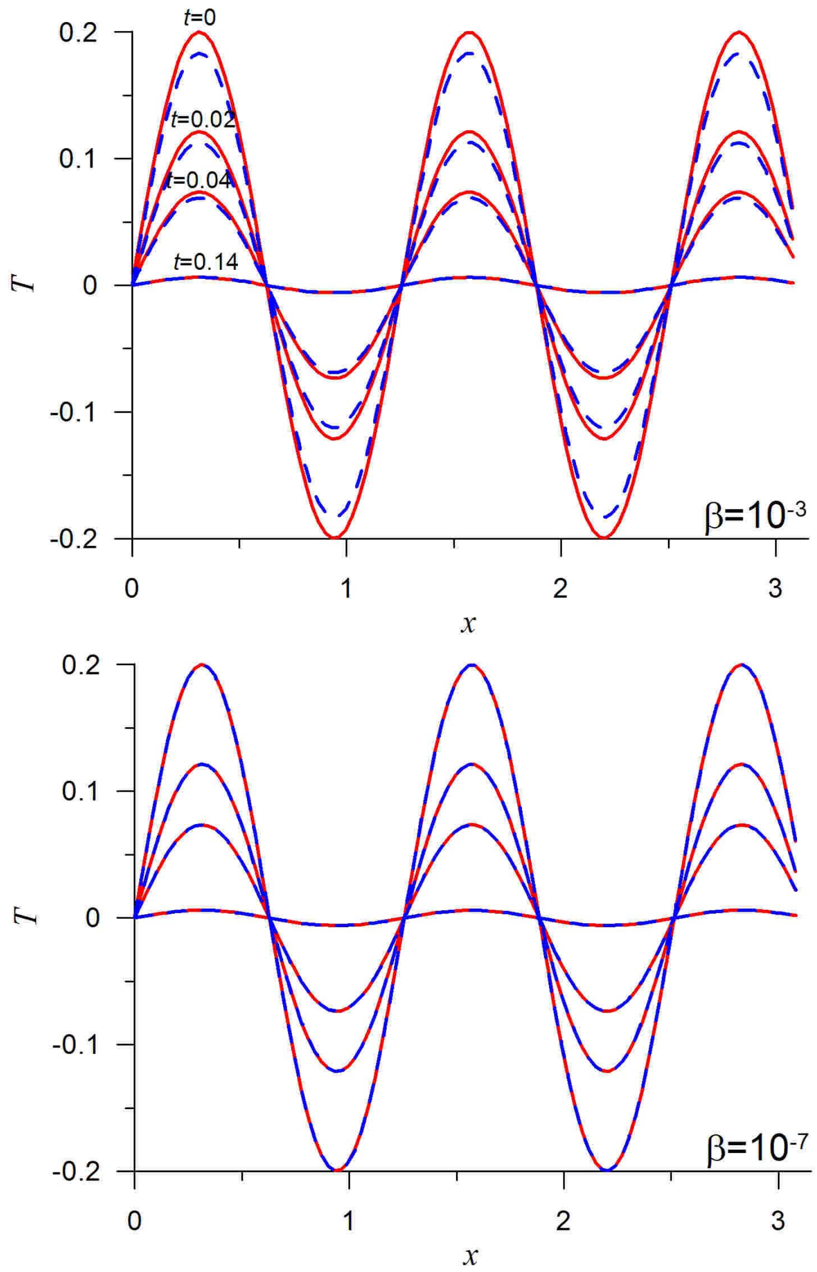
Solution

$$T_\beta(t,x) = A_n \exp\left(-(4n+1)^2 (1+\beta(4n+1)^4)^{-1} t\right) \sin((4n+1)x),$$

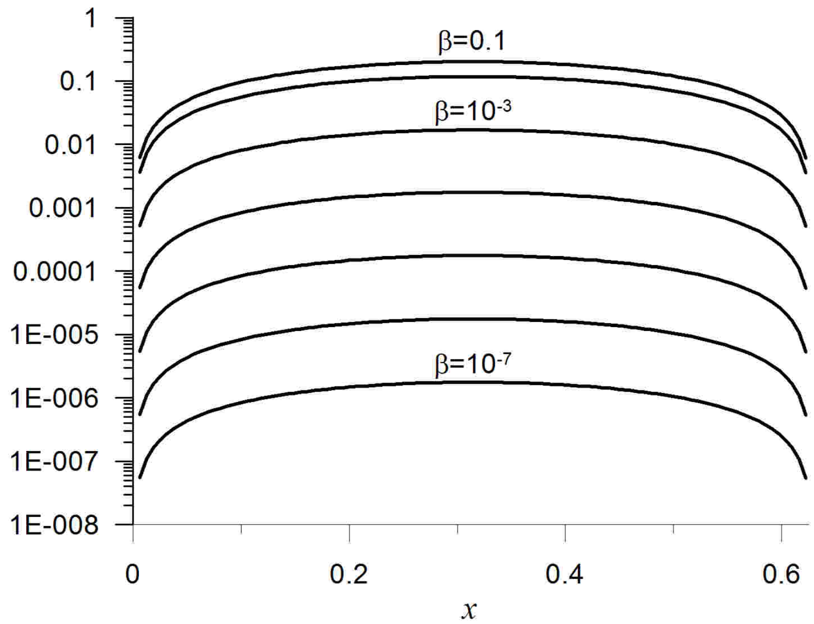
$$A_n = \frac{1}{4n+1} \exp(-(4n+1)^2 t^*) \exp^{-1}\left(-(4n+1)^2 (1+\beta(4n+1)^4)^{-1} t^*\right).$$

At $t = 0$ and $\beta \to 0$ the solution to the regularized problem approaches the initial condition for the direct problem.

Solution



Temperature residual



Ismail-Zadeh et al. 2016

MATHEMATICAL STATEMENT

Model domain $\Omega = (0, x_1 = l_1) \times (0, x_2 = l_2) \times (0, x_3 = h), \quad t \in (0, \vartheta)$

The boundary-value problem for flow velocity

$$-\nabla P + \nabla \cdot (\mu(T)[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]) + RaT\mathbf{e} = 0, \qquad \mathbf{x} \in \Omega$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{x} \in \Omega$$

$$\mathbf{u} = 0, \quad \mathbf{x} \in \Gamma_1, \quad \Gamma_1 \subset \partial\Omega \qquad \Gamma_1 \cap \Gamma_2 = \varnothing$$

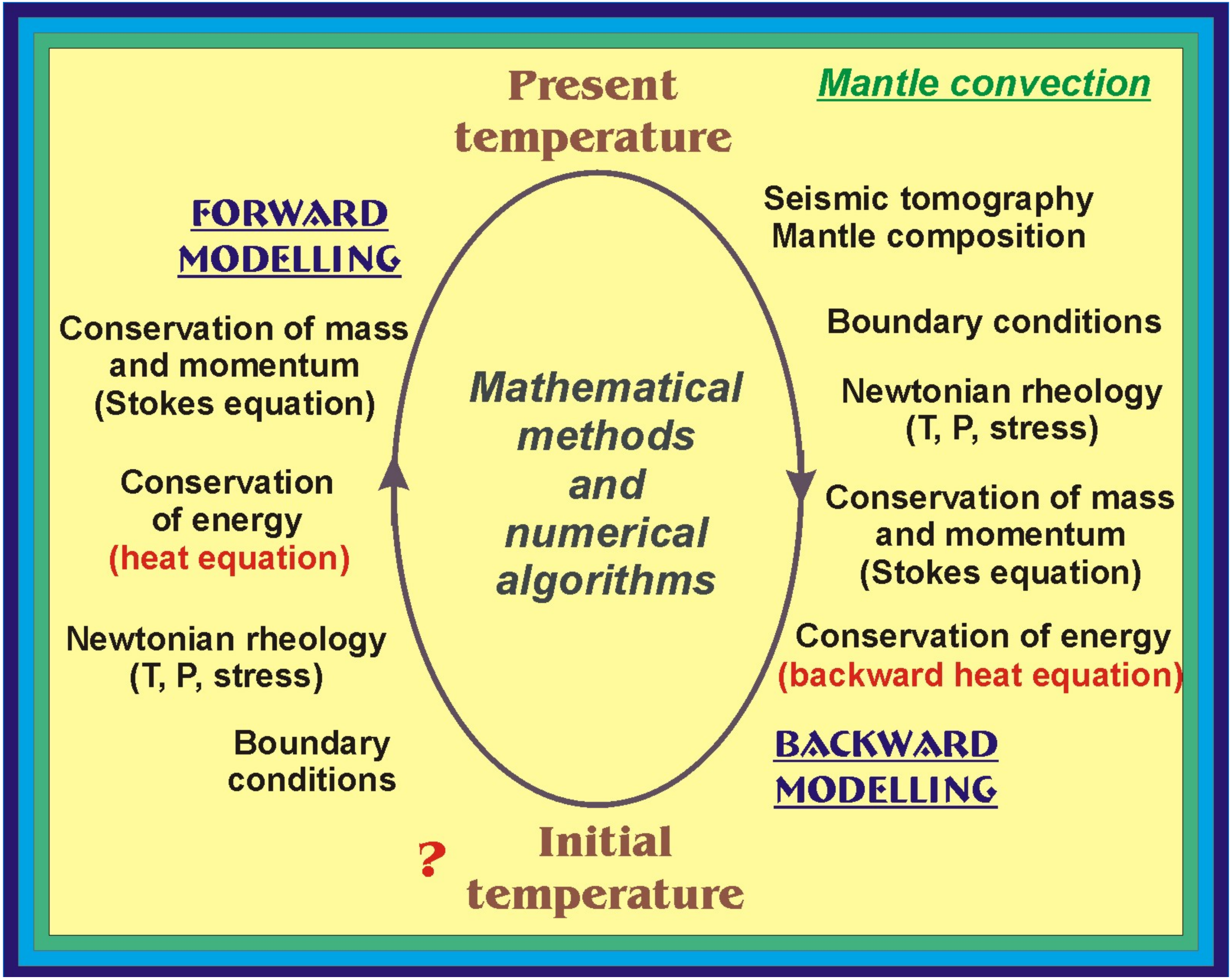
$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \partial \mathbf{u}_\tau / \partial \mathbf{n} = 0, \qquad \mathbf{x} \in \Gamma_2, \quad \Gamma_2 \subset \partial\Omega \qquad \Gamma_1 \cup \Gamma_2 = \partial\Omega$$

The initial-boundary-value problem for temperature

$$\partial T / \partial t + \mathbf{u} \cdot \nabla T = \nabla^2 T, \qquad t \in (0, \vartheta), \ \mathbf{x} \in \Omega$$

$$\sigma_1 T + \sigma_2 \partial T / \partial \mathbf{n} = T_*(t, \mathbf{x}), \qquad t \in (0, \vartheta), \ \mathbf{x} \in \partial\Omega$$

$$T(0, \mathbf{x}) = T_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega$$


Present temperature
Mantle convection
FORWARD MODELLING
Seismic tomography
Mantle composition
Conservation of mass and momentum (Stokes equation)
Boundary conditions
Mathematical methods and numerical algorithms
Newtonian rheology (T, P, stress)
Conservation of energy (heat equation)
Conservation of mass and momentum (Stokes equation)
Newtonian rheology (T, P, stress)
Conservation of energy (backward heat equation)
Boundary conditions
BACKWARD MODELLING
?
Initial temperature

QUASI-REVERSIBILITY METHOD

The final-boundary-value problem to define temperature in the past

$$\partial T_\beta / \partial t - \mathbf{u}_\beta \cdot \nabla T_\beta = \nabla^2 T_\beta - \beta \Lambda(\partial T_\beta / \partial t), \quad t \in [0, \vartheta], \; \mathbf{x} \in \Omega$$

$$\sigma_1 T_\beta + \sigma_2 \partial T_\beta / \partial \mathbf{n} = T_*, \qquad t \in (0, \vartheta), \; \mathbf{x} \in \partial\Omega$$

$$\sigma_1 \partial^2 T_\beta / \partial \mathbf{n}^2 + \sigma_2 \partial^3 T_\beta / \partial \mathbf{n}^3 = 0, \qquad t \in (0, \vartheta), \quad \mathbf{x} \in \partial\Omega$$

$$T_\beta(t = \vartheta, \mathbf{x}) = T_\vartheta(\mathbf{x}), \qquad \mathbf{x} \in \Omega$$

The regularization term: $\Lambda(\Upsilon) = \partial^4 \Upsilon / \partial x_1^4 + \partial^4 \Upsilon / \partial x_2^4 + \partial^4 \Upsilon / \partial x_3^4$

The regularization parameter is chosen to minimize the temperature misfit:

$$J_1 = \left\| T(t = \vartheta, \cdot; T_{\beta_k}(t = 0, \cdot)) - T_\vartheta(\cdot) \right\|_{L_2}$$

$$\beta_k = \beta_0 q^{k-1}, \quad \beta_0 = 10^{-2}, \quad q = 0.1, \quad k = 1, 2, \ldots$$

$$J_2 = \left\| T_{\beta_{k+1}}(0, \cdot; T_\vartheta) - T_{\beta_k}(0, \cdot; T_\vartheta) \right\|_{L_2}$$

Temperature misfits

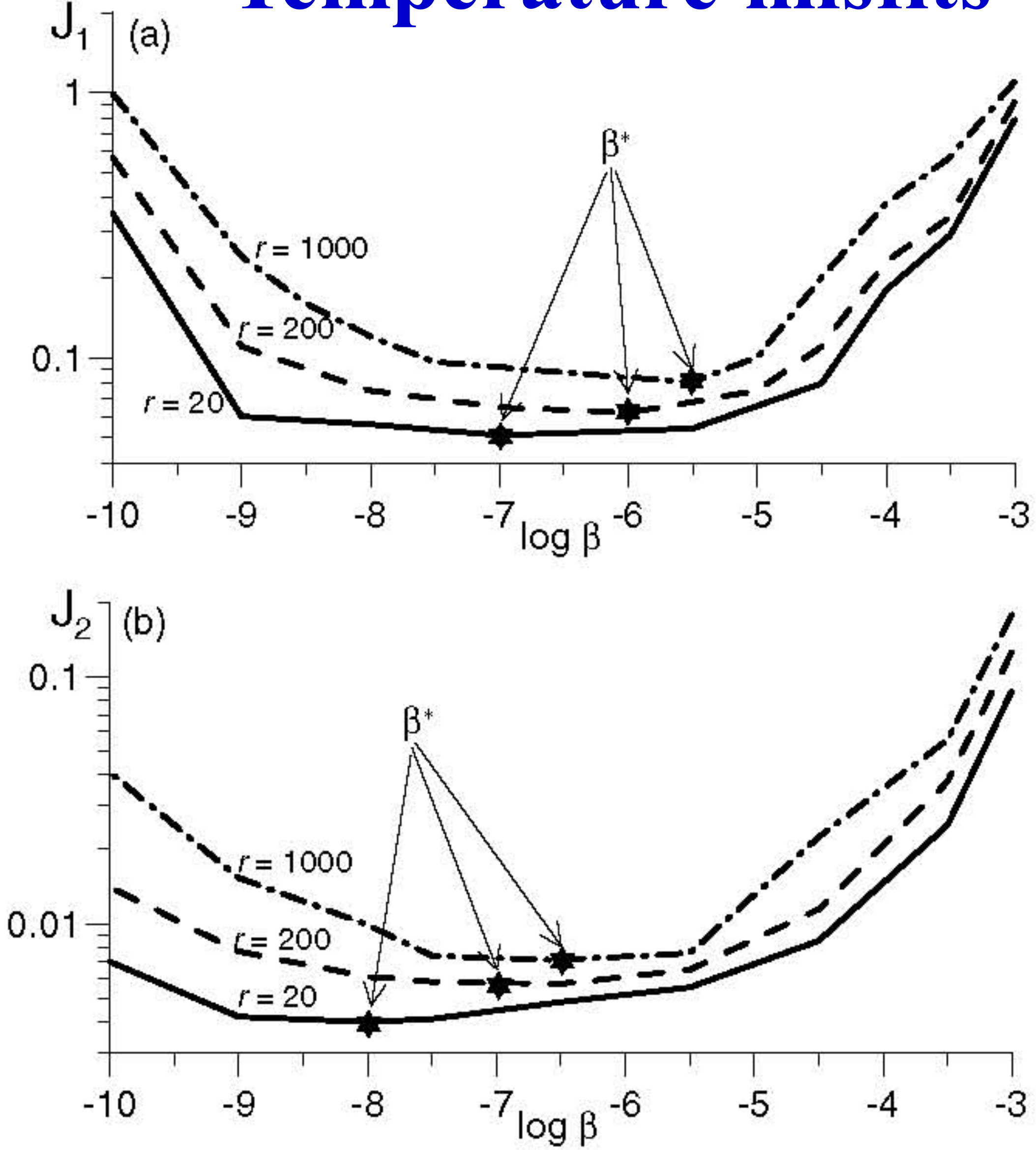


Figure 3. Temperature misfit (a) J_1 and (b) J_2 as functions of the regularization parameter β. The minimum of the temperature misfit is achieved at β^*, an optimal regularization parameter. Solid curves: $r = 20$; dashed curves: $r = 200$; and dash–dotted curves: $r = 1000$.

Ismail-Zadeh et al. 2007

NON-FOURIER LAWS OF HEAT CONDUCTION

C. Riemann modified the Fourier constitutive heat equation

$$\vec{Q} = -k\nabla T - \tau \; \partial\vec{Q} / \partial t$$

where Q **is the heat flux, and** k **is the coefficient of thermal conductivity**

$\tau = k / \left(\rho c_p v^2 \right)$ **is the thermal relaxation time,** ρ **is density,**

c_p **is the specific heat,** v **is the heat propagation velocity**

The modified heat conduction equation

$$\partial T / \partial t = \nabla^2 T + \tau \; \partial^2 T / \partial t^2$$

If the Fourier law is modified further by an addition of the second derivative of heat flux (Bubnov, 1976)

$$\vec{Q} = -k\nabla T + \beta \frac{\partial^2 \vec{Q}}{\partial t^2}$$

Time Interval for Data Assimilation

The time interval for the ***VAR*** data assimilation depends strongly on smoothness of the input data and the solution.

The time interval for the ***BAD*** data assimilation depends on the intensity of mantle convection: it is short for conduction-dominated heat transfer and becomes longer for advection-dominated heat flow. In the absence of thermal diffusion, the backwards advection of a low-density fluid in the gravity field will finally yield a uniformly stratified, inverted density structure, where the low-density fluid overlain by a dense fluid spreads across the lower boundary of the model domain to form a horizontal layer. Once the layer is formed, information about the evolution of the low-density fluid will be lost, and hence any forward modeling will be useless, because no information on initial conditions will be available.

Time Interval for Data Assimilation

The ***QRV*** method can provide stable results within the characteristic thermal diffusion time interval. However, the length of the time interval for QRV data assimilation depends on several factors.

$$\left\| T(t,x) - T_\beta(t,x) \right\| \le \tilde{C}\delta \exp[\beta^{-1/2}(t^* - t)] + \beta L_d t, \quad 0 \le t \le t^*$$

where constant $\tilde{C}$ is determined from the a priori known parameters of the backward heat conduction problem. For the given regularization parameter, errors in the input data δ, and smoothness parameter L_d, it is possible to evaluate the time interval $0 \le t \le t^*$ of data assimilation for which the temperature misfit would not exceed a prescribed value.

Errors in Data Assimilation

Apart from the errors associated with the numerical modeling (model, discretization, and iteration errors),

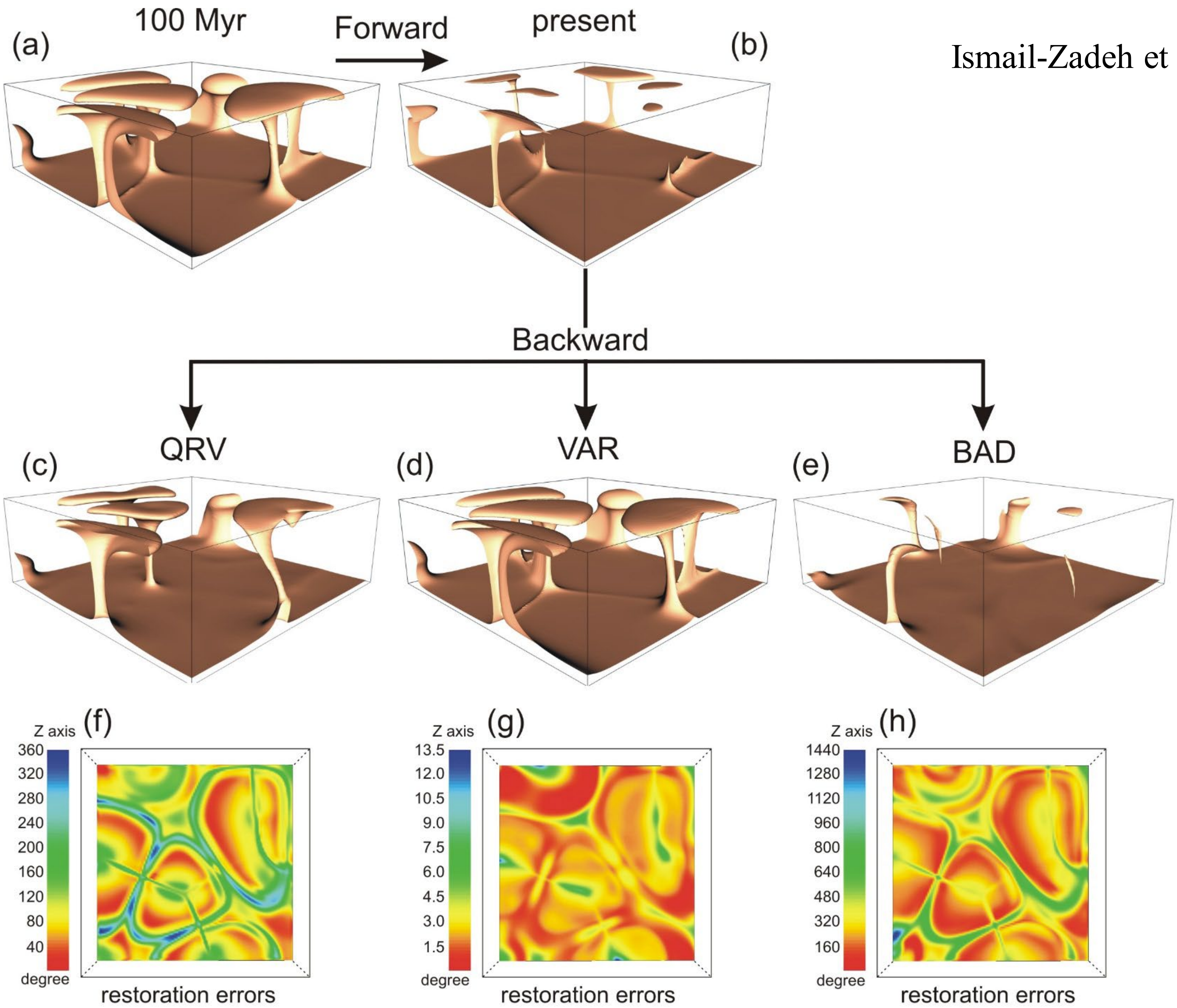
there are at least two sources of errors in data assimilation:

(i) *data misfit associated with the uncertainties in the present temperature distribution and/or in the surface movements* and

(ii) *errors associated with the uncertainties in initial and boundary conditions*.

Comparison GRV, VAR, and BAD methods

Ismail-Zadeh et al. 2007



Data Assimilation Methods

Table 1. Comparison of methods for data assimilation in models of mantle dynamics.

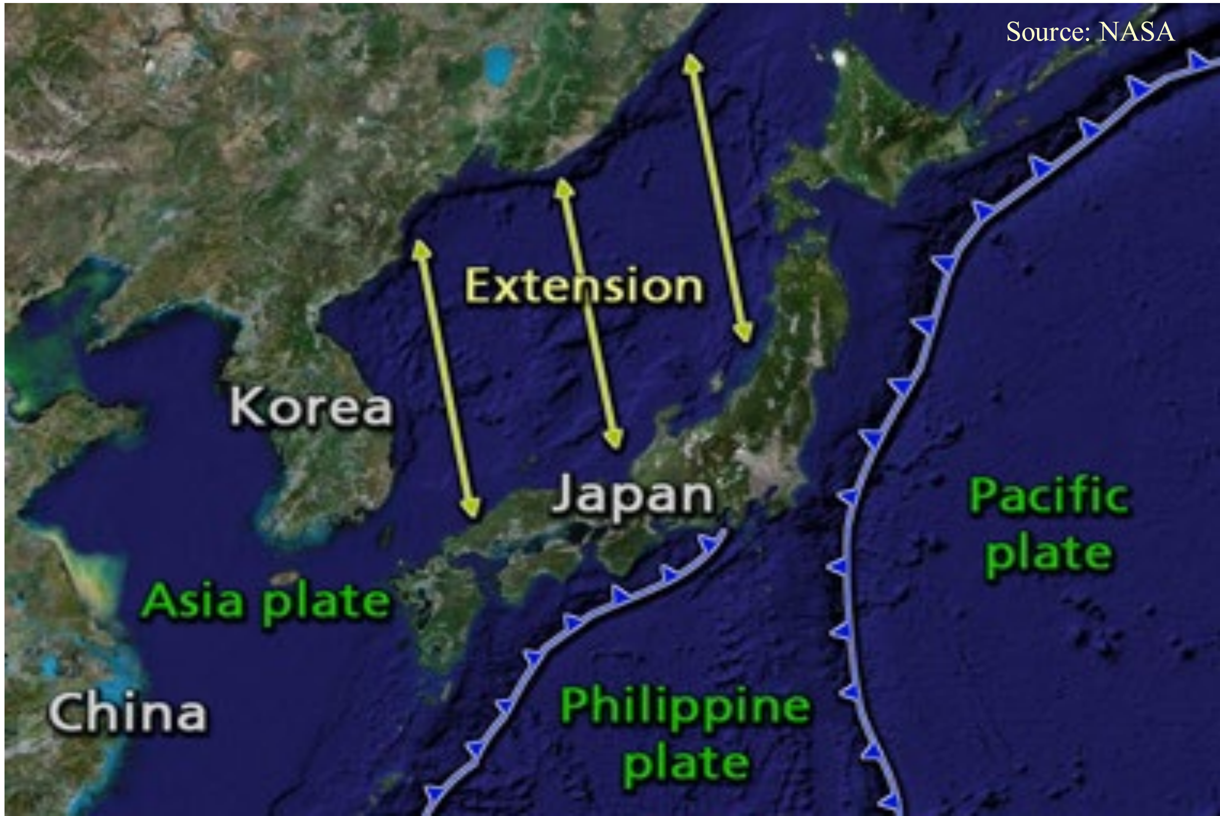
	QRV method	VAR method	BAD method
Method	Solving the regularized backward heat problem with respect to parameter β	Iterative sequential solving of the direct and adjoint heat problems	Solving of heat advection equation backward in time
Solution's stability	Stable for parameter β to numerical errors (see text; also in[a]) and conditionally stable for parameter β to arbitrarily assigned initial conditions (numerically[b])	Conditionally stable to numerical errors depending on the number of iterations (theoretically[c]) and unstable to arbitrarily assigned initial conditions (numerically[d])	Stable theoretically and numerically
Solution's convergence	Numerical solution to the regularized backward heat problem converges to the solution of the backward heat problem in the special class of admissible solutions[e]	Numerical solution converges to the exact solution in the Hilbert space[f]	Not applied
Solution's accuracy[g]	Acceptable accuracy for both synthetic and geophysical data	High accuracy for synthetic data.	Low accuracy for both synthetic and geophysical data in conduction-dominated mantle flow
Time interval for data assimilation[h]	Limited by the characteristic thermal diffusion time	Limited by the characteristic thermal diffusion time and the accuracy of the numerical solution	No specific time limitation; depends on mantle flow intensity
Analytical work	Choice of the regularizing operator	Derivation of the adjoint problem	No additional analytical work
Algorithmic work	New solver for the regularized equation should be developed	No new solver should be developed	No new solver should be developed

[a]Lattes & Lions 1969; [b]See Fig. 3 and relevant text in the paper; [c]Ismail-Zadeh *et al.* 2004a; [d]Ismail-Zadeh *et al.* 2006; [e]Tikhonov & Arsenin, 1977; [f]Tikhonov & Samarskii 1990; [g]See Table 2; [h]See text for details.

Ismail-Zadeh et al. 2016

Restoring Mantle Evolution Beneath the Japanese Islands

Mechanisms of opening of the Japan Sea

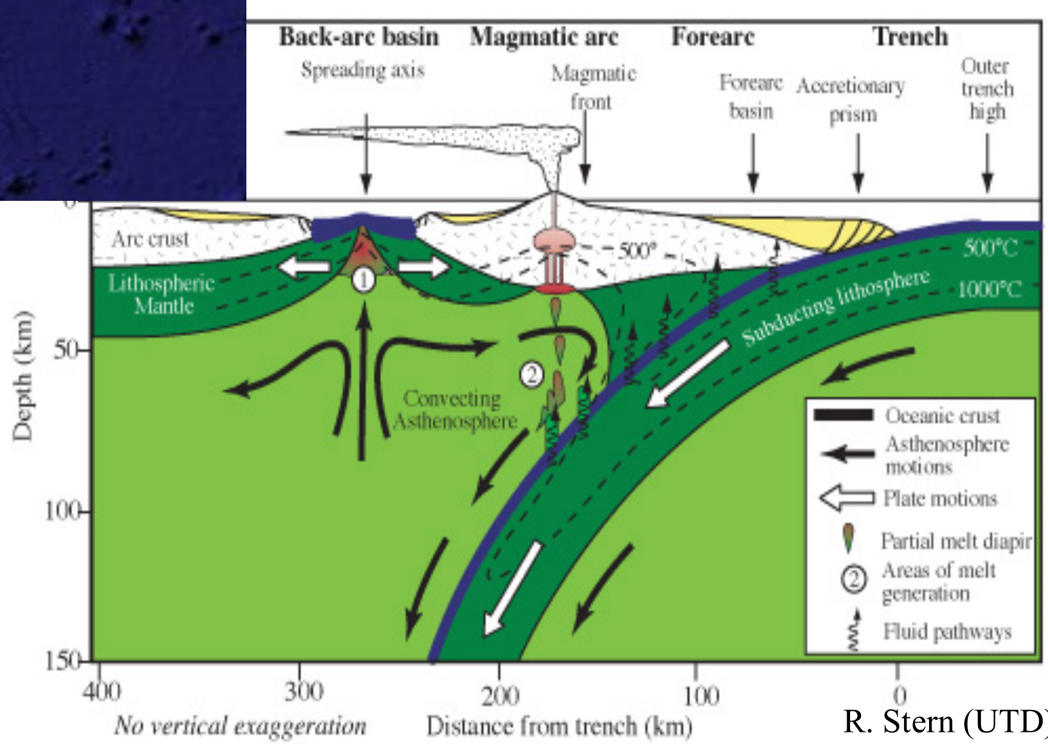


(1) Mantle diapirism caused by melting of the subducted slab (e.g. Karig, 1971);

(2) Induced convection (e.g., McKenzie, 1969; Toksöz and Bird, 1977)

(3) Rollback of the subducted lithopshere (e.g., Uyeda and Kanamori, 1979; Dewey, 1980)

(4) Injection of asthenosphere unrelated to the subduction process (e.g., Miyashiro, 1986; Tatsumi, Maruyama and Nohda, 1990).



R. Stern (UTD)

Mathematical Statement of the Problem

Model domain $\Omega = (0, x_1 = l_1) \times (0, x_2 = l_2) \times (0, x_3 = h), \quad t \in (0, \vartheta)$

The boundary-value problem for flow velocity

$$-\nabla P + \nabla \cdot (\eta(T)[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]) + (RaT - La\Phi(T))\mathbf{e} = 0, \quad \mathbf{x} \in \Omega$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{x} \in \Omega$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \partial \mathbf{u}_\tau / \partial \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega \qquad Ra = \frac{\alpha \rho_* g \Delta T h^3}{\eta_* \kappa}, \quad La = \frac{Ra}{\alpha \Delta T}$$

Phase changes at 410km and 660 km boundaries Christensen & Yuen (1989)

$$\rho(\mathbf{x}, T) = \rho_*(1 - \alpha T + \Phi(T)), \quad \Phi(T) = \sum_{i=1}^{2} a_i \Phi_i(T)$$

$$\Phi_i(\pi) = \frac{1}{2}\left[1 + \tanh \frac{\pi_i}{w_i}\right], \quad \pi_i = z_i - x_3 - \gamma_i (T - T_i), \quad i = 1, 2$$

where ρ_* is the reference density; α is the coefficient of thermal expansion;
$a_1 = 0.05; a_2 = 0.09;$ $d = 800$ km; $z_1 = 410$ km; $z_2 = 660$ km; $w_1 = w_2 = 10$ km;
$\gamma_1 = 4$ MPa K^{-1} and $\gamma_2 = -2$ MPa K^{-1}; $T_1 = 1790$ K; $T_2 = 1858$ K;
g is the acceleration due to gravity.

Mathematical Statement of the Problem

Model domain $\Omega = (0, x_1 = l_1) \times (0, x_2 = l_2) \times (0, x_3 = h), \quad t \in (0, \vartheta)$

The initial-boundary-value problem for temperature

$$\frac{DT}{Dt} + \mathbf{A}^{-1}\mathbf{B} Di^* \; Ra \; u_3 \; (T + T_{as}) = \nabla^2 T + \mathbf{A}^{-1} Di^* \; \eta \sum_{i,j=1}^{3} (e_{ij})^2 \quad t \in (0, \vartheta), \; \mathbf{x} \in \Omega$$

$$\mathbf{A} = \left[1 + \left(a_1 \frac{d\Phi_1}{d\pi_1} \bar{\gamma}_1^2 + a_2 \frac{d\Phi_2}{d\pi_2} \bar{\gamma}_2^2 \right) Di^* \; La \; (T + T_{as}) \right]$$

$$Di^* = \eta\kappa / (\rho c d^2 T_{ref})$$

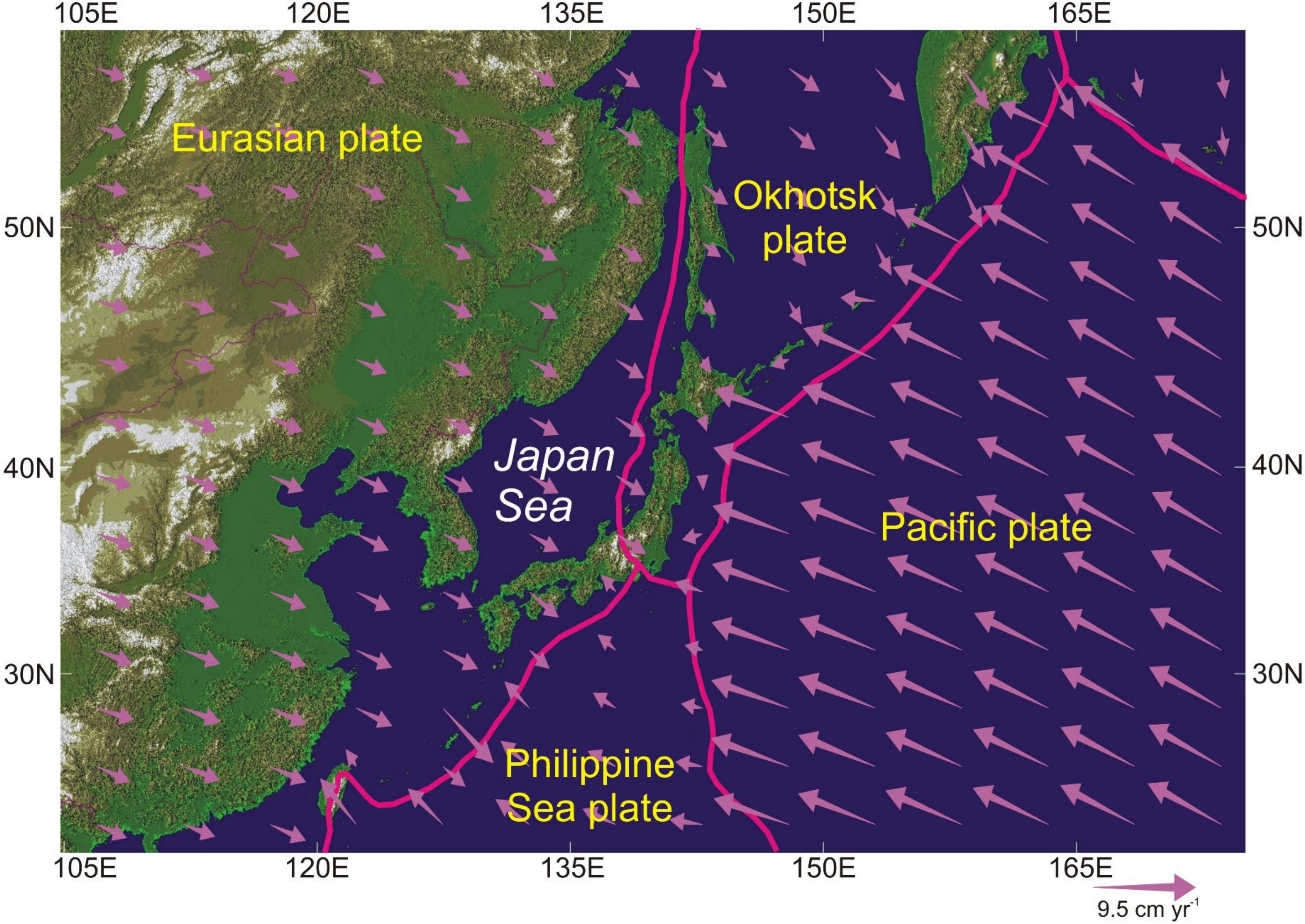
$$\mathbf{B} = \left[1 + \frac{La}{Ra} \left(a_1 \frac{d\Phi_1}{d\pi_1} \bar{\gamma}_1 + a_2 \frac{d\Phi_2}{d\pi_2} \bar{\gamma}_2 \right) \right]$$

$$Di = Di^* Ra$$

$$\sigma_1 T + \sigma_2 \partial T / \partial \mathbf{n} = T_*(t, \mathbf{x}) \qquad T(0, \mathbf{x}) = T_0(\mathbf{x})$$

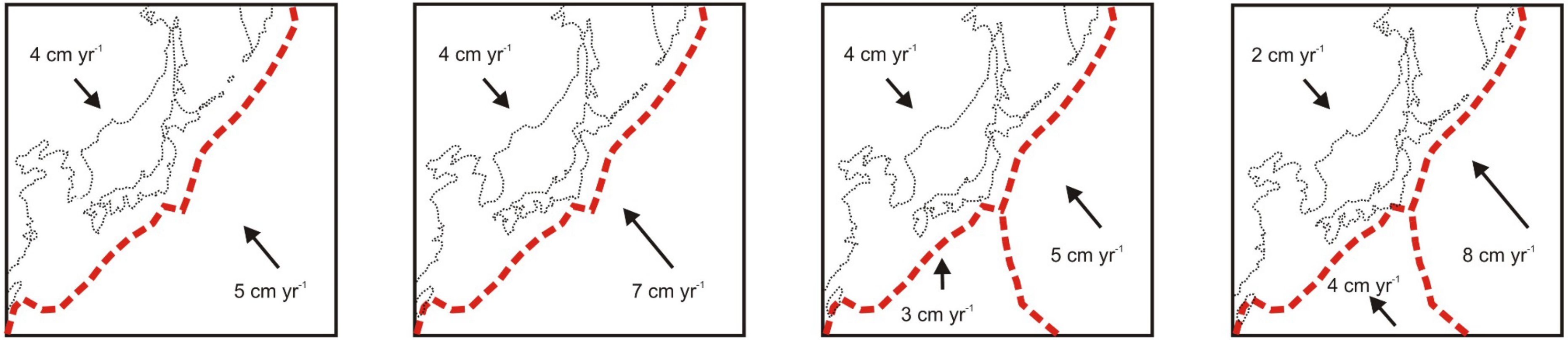
Viscosity law $\eta(T, z) = \exp\left(\frac{E_a + V_a \rho g z}{RT} \right)$

Present Plate Motion



The data are taken from the Actual Plate Kinematic and Deformation Model (APKIM2005) derived from various geodetic data. As the Philippine Sea plate is not covered by sufficient geodetic stations, the data are taken from PB2002 (Peter Bird's model, 2002).

Present and Past Plate Motion in the Model



38.9 Ma | 25.7 Ma | 12.6 Ma | Present

Boundary Conditions

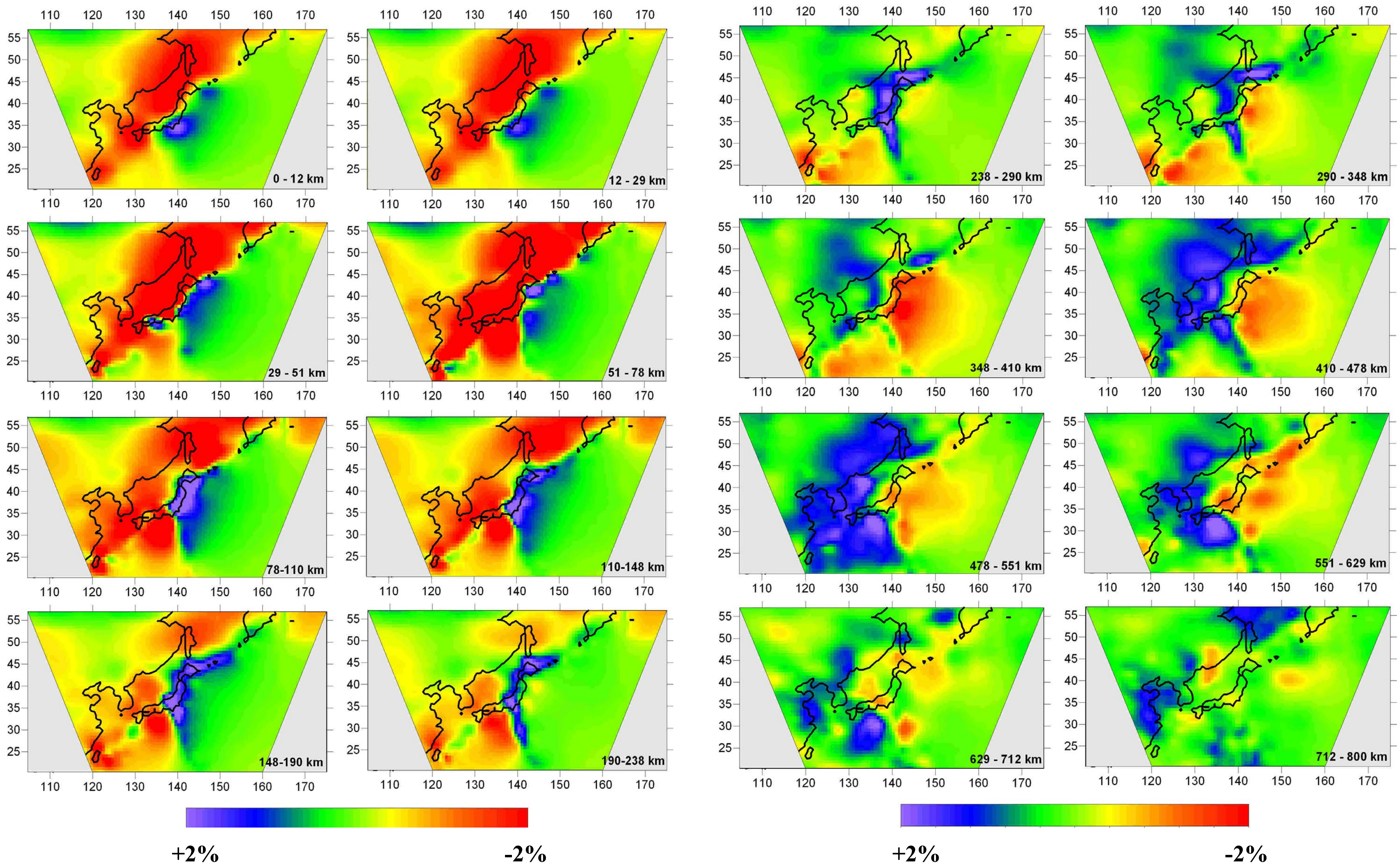
Conditions at the ***top surface*** of the model boundary are prescribed velocity and fixed temperature

Conditions at the ***lower surface*** of the model boundary are no-slip and fixed temperature

Conditions at all ***side boundaries***: $\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0, \quad \frac{\partial T}{\partial \mathbf{n}} = 0$

$\frac{\partial P}{\partial \mathbf{n}} = 0$ at all boundary faces.

P- wave seismic velocity anomalies



Obayashi et al., EPSL, 2006

Inferring mantle temperatures from body wave seismic tomography

• Calculation of ***anharmonic velocities*** (based on laboratory measurements of density and elastic parameters of the main rock-forming minerals at various *P* and *T* conditions)

Mantle composition

depth < 100 km:

75% Ol + 7% CPx + 17% OPx + 1% Gt (Ritter, 2005)

100 km < *depth* < 300 km

58% Ol + 16% CPx + 14% OPx + 12% Gt (Green & Falloon, 1998)

depth > 300 km

60% Ol + 25% CPx + 15% Gt (Agee, 1993)

Lithospheric slab composition

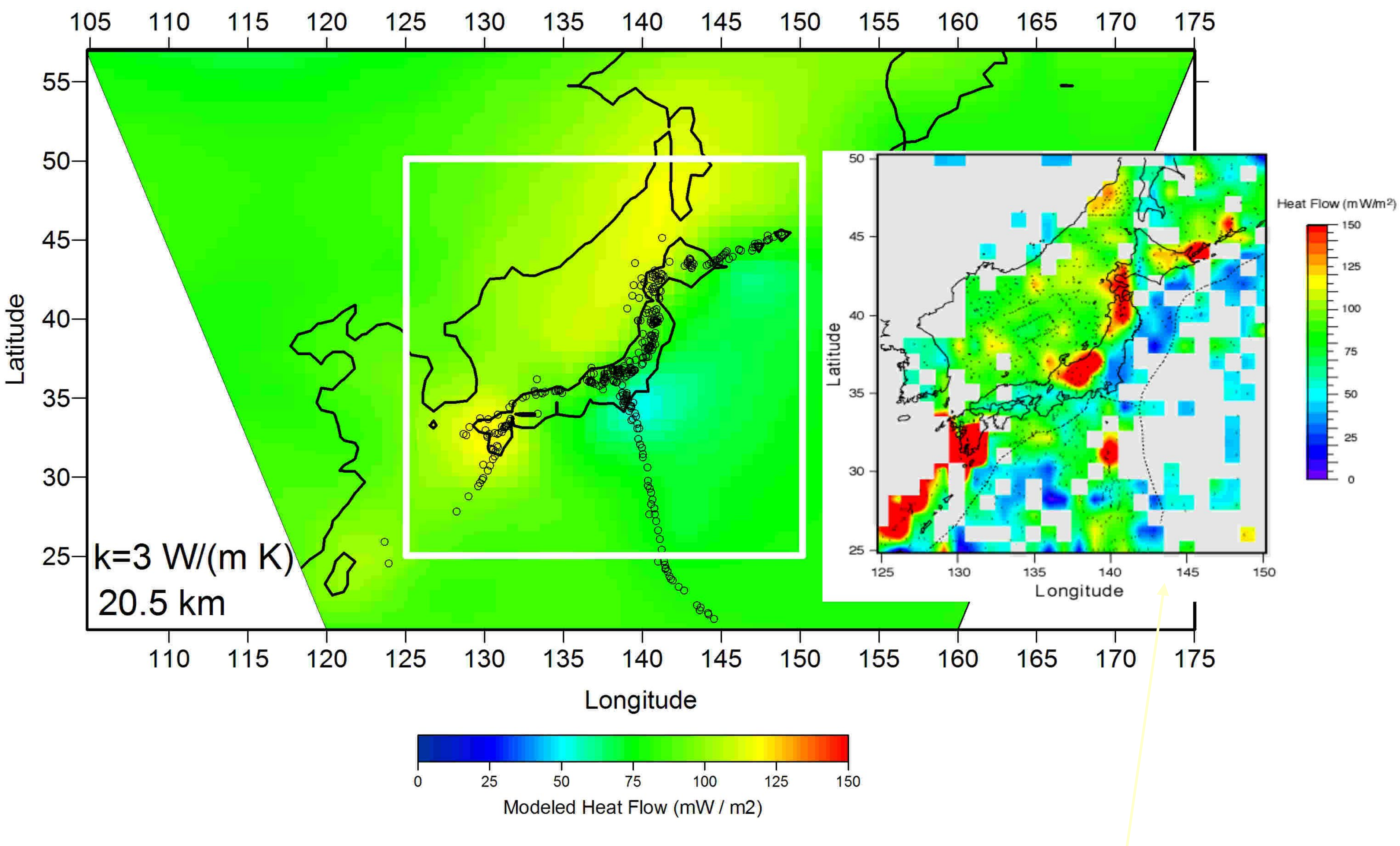
66% Ol + 18% CPx + 14% OPx +2% Gt (Ringwood & Irifune, 1988)

• Calculation of ***anelasticity*** effects (due to velocity dispersion and seismic wave attenuation)

• Calculation of the effect of the presence of ***melt*** on seismic velocity

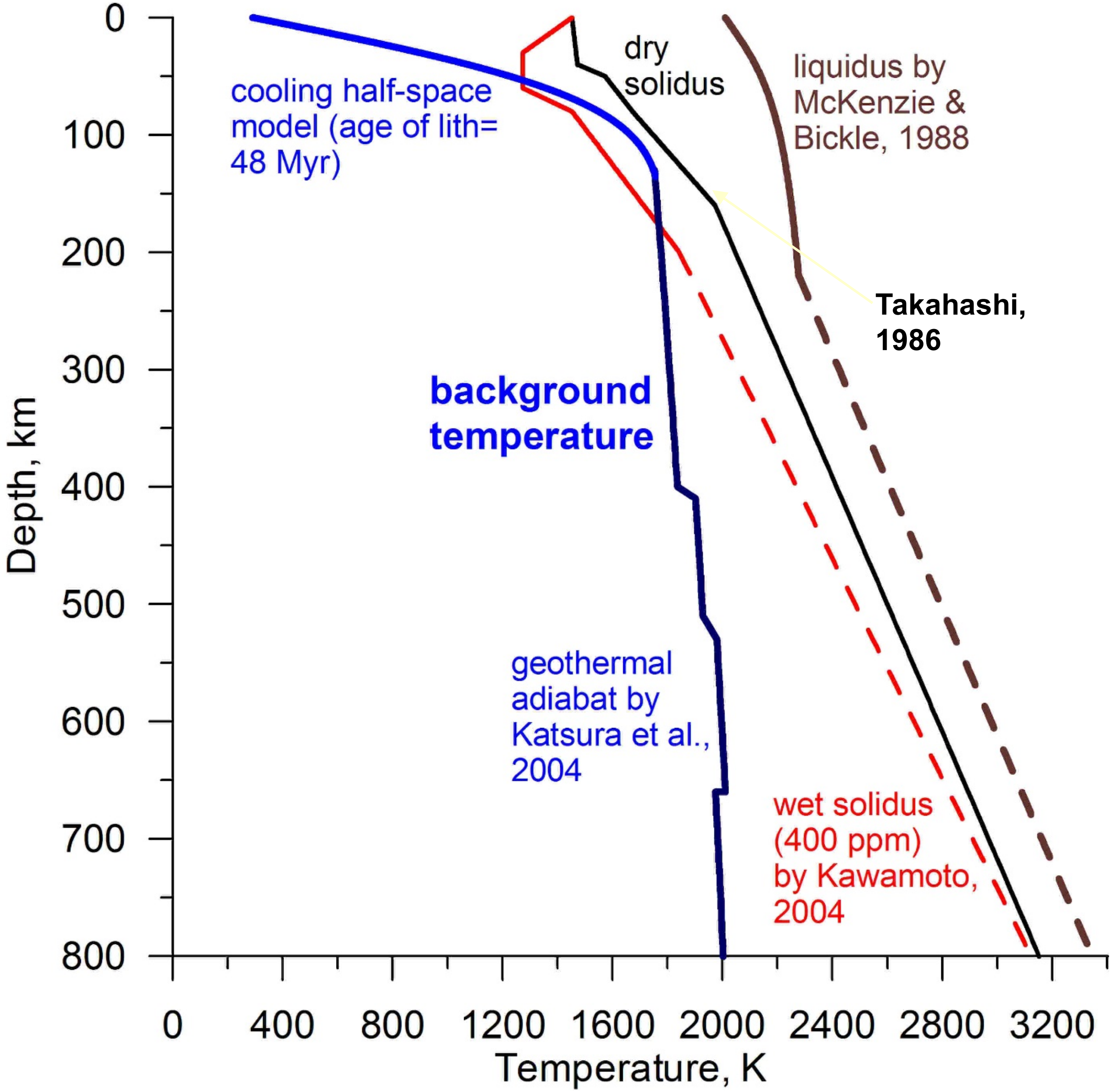
Method is described by Goes et al. (2000), Ismail-Zadeh et al. (PEPI, 2005)

Predicted vs. measured heat flow



Heat flow measurement by Y. Furukawa (Volcanological Laboratory, Kyoto University) http://kagi.coe21.kyoto-u.ac.jp/en/tidbit/tidbit28.html

Background temperature

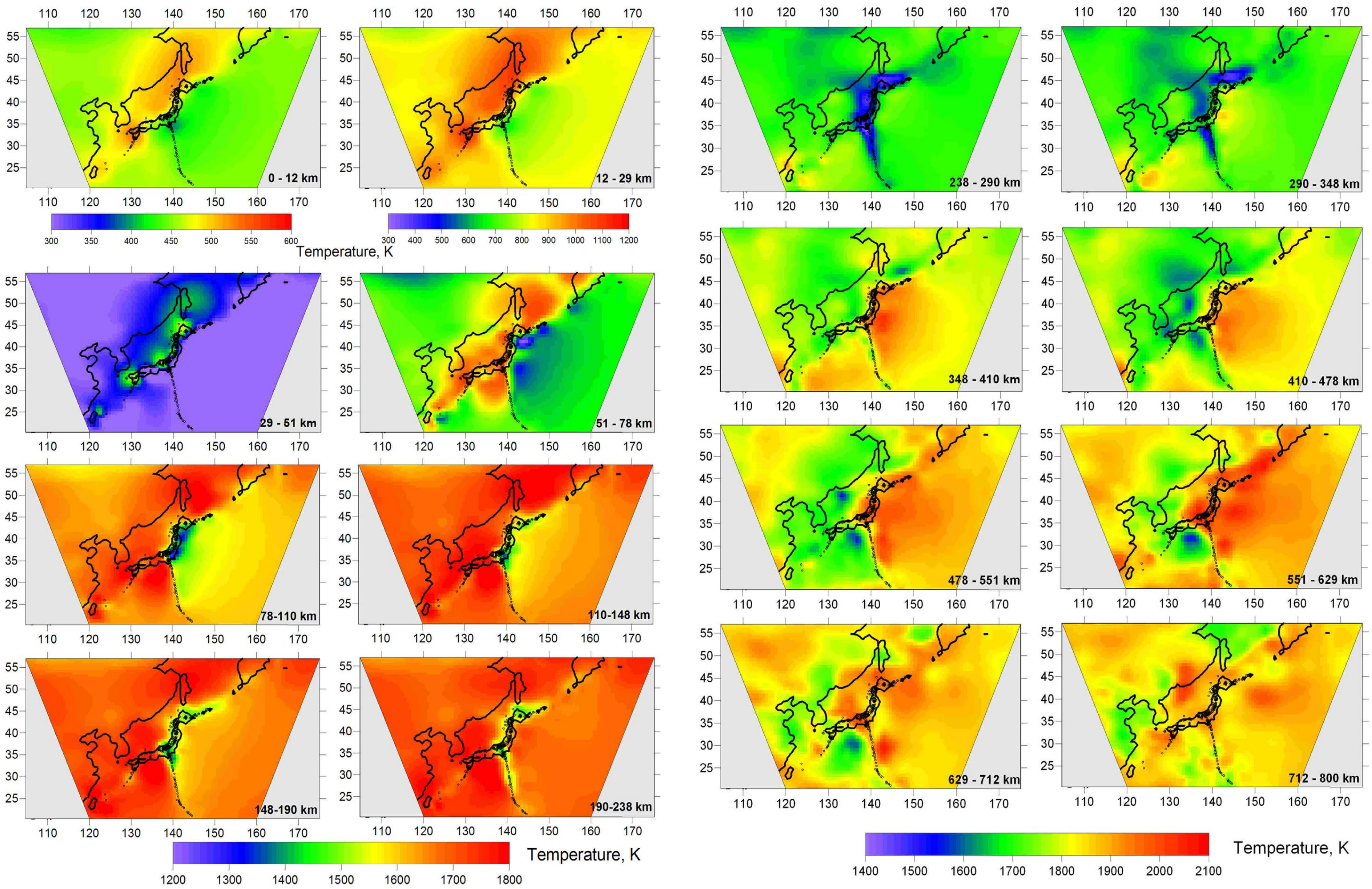


Age of the lithosphere

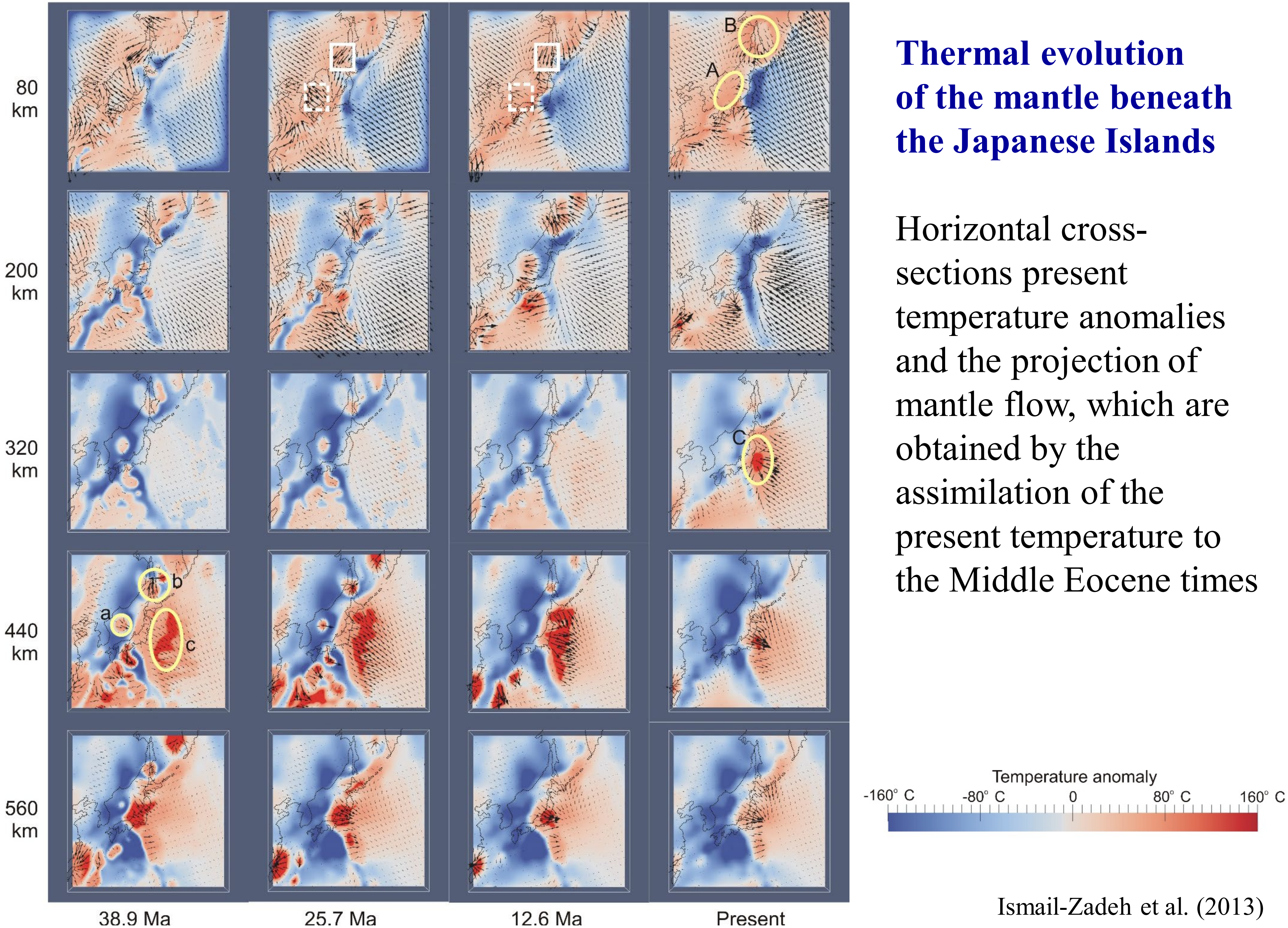
$$t = \frac{k^2 T_m^2}{\pi \kappa Q_s}$$

Considering k = 3 W m^{-1} K^{-1}, T_m =1330 C, κ = 10^{-6} m^2 s^{-1}, and Q_s = 0.07 W m^{-2}, we obtain the age of the lithosphere to be 48 Myr.

Temperature in the upper mantle (the initial condition)



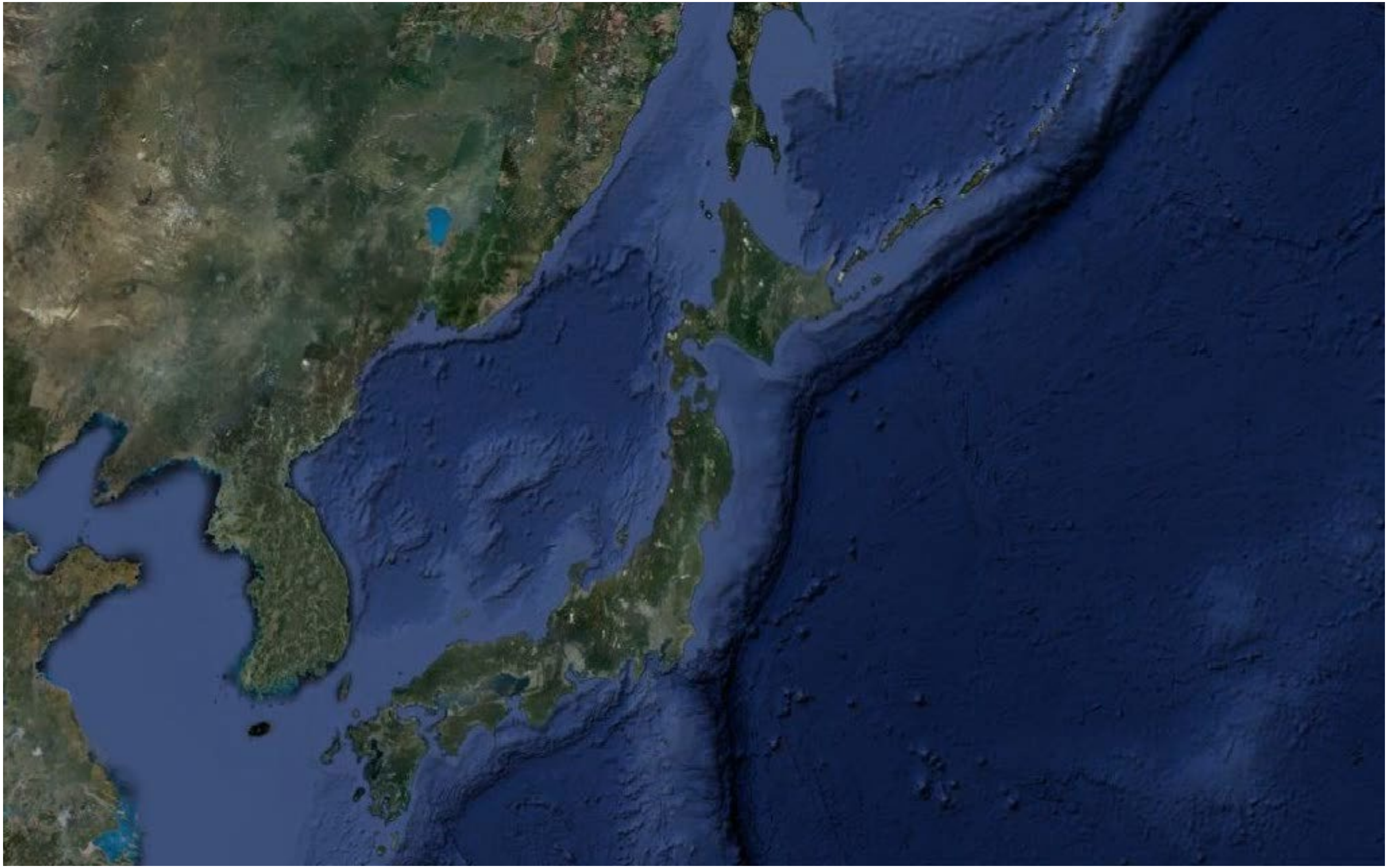
Thermal evolution of the mantle beneath the Japanese Islands



Horizontal cross-sections present temperature anomalies and the projection of mantle flow, which are obtained by the assimilation of the present temperature to the Middle Eocene times

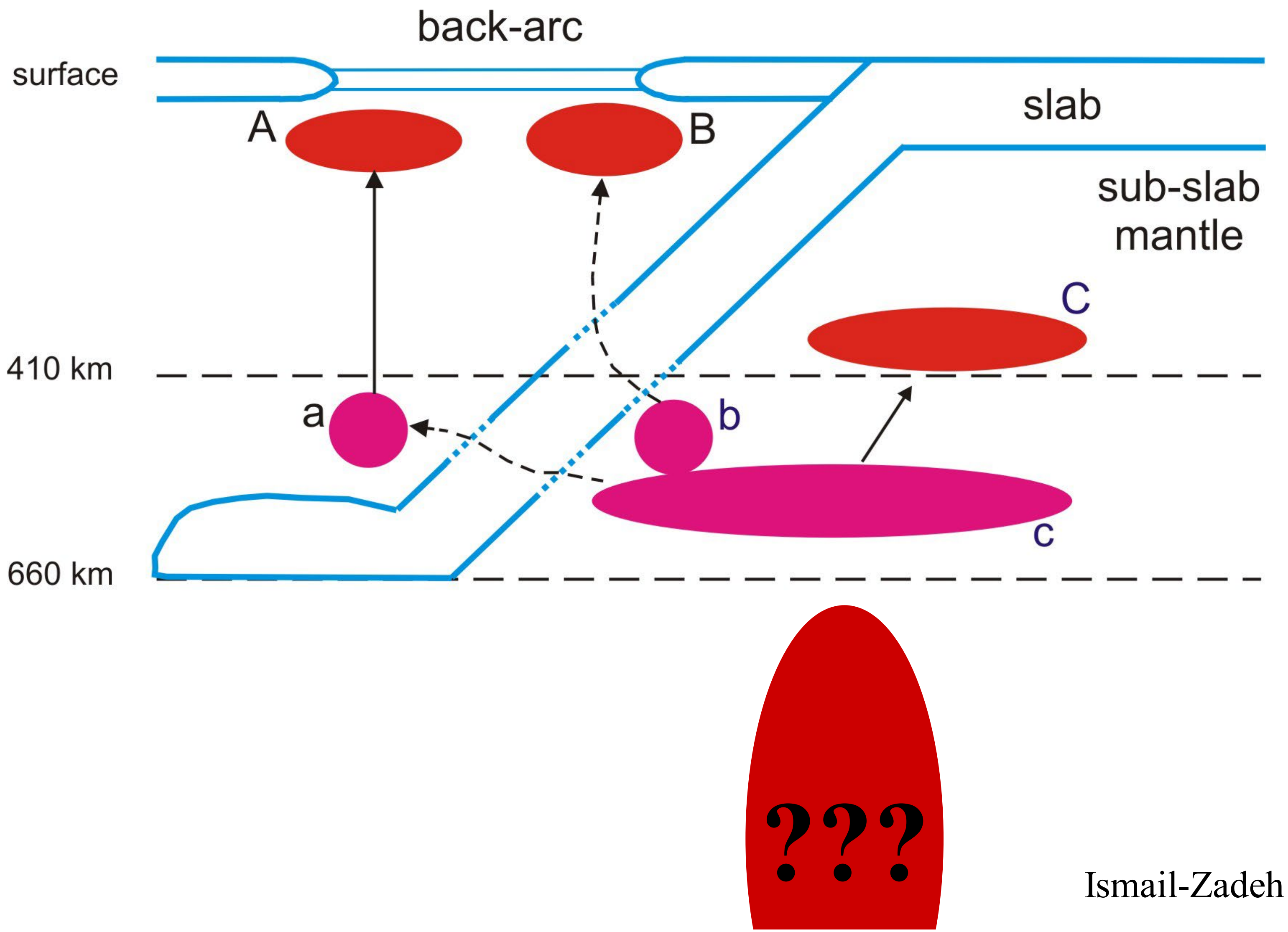
Ismail-Zadeh et al. (2013)

Evolution of the uppermost mantle



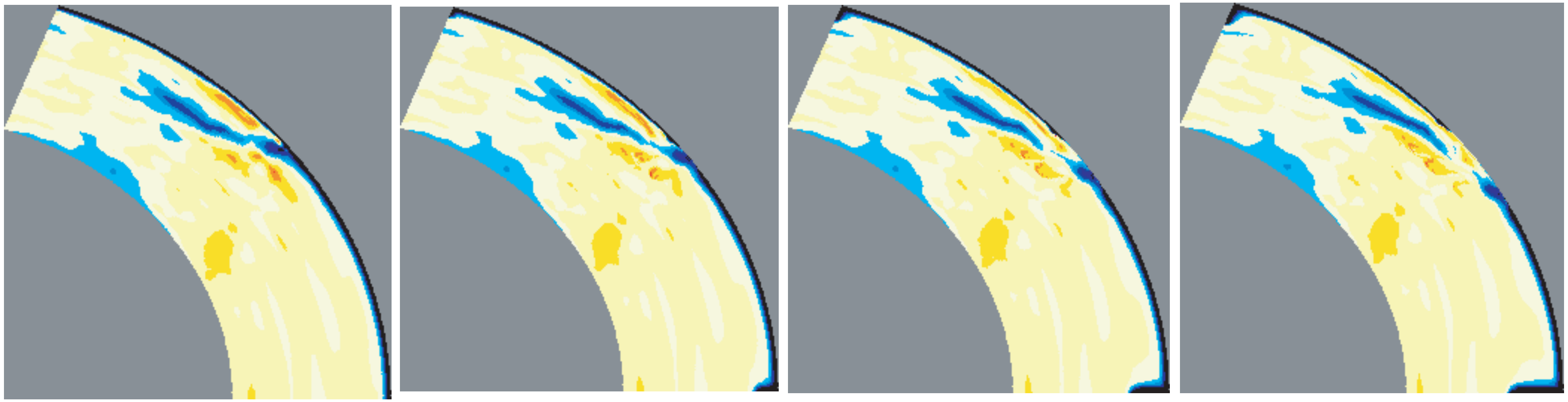
Visualization of Lithosphere Subduction – iBook can be downloaded from the Apple Store

Schematic Representation of the Thermal Evolution beneath the Japanese Islands and their Surroundings

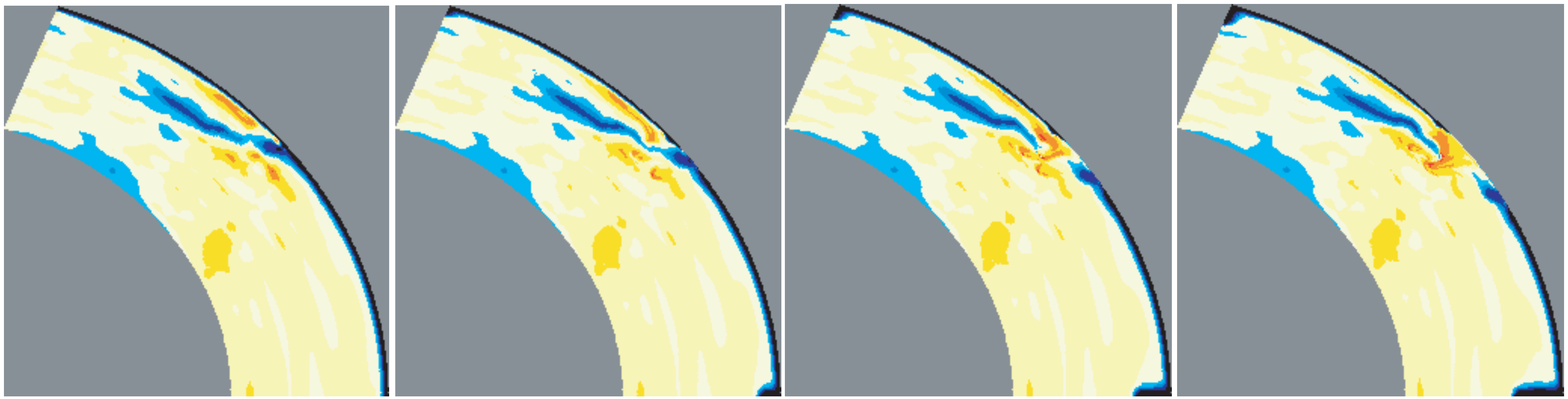


Ismail-Zadeh et al. (2013)

Newtonian



Non-Newtonian



Calculations by S. Honda