Prym varieties: a survey

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Prym Varieties: A Survey

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Introduction. Prym varieties form a special class of principally polarized abelian varieties, more general than Jacobians, but still accessible geometrically. They were discovered by Wirtinger [**Wi**], then played a key role in the Schottky-Jung identities [**S-J**]. Interest in them was revived in the early 1970s by D. Mumford [**M**], and they have been studied actively since then. In these notes I'll try to give an overview of the current state of the theory.

1. Definition; the Schottky-Jung configuration. We start with two Riemann surfaces $C, \tilde{C}$ and an étale double covering $\pi: \tilde{C} \to C$. Let σ be the involution of $\tilde{C}$ that exchanges the two sheets of the covering π. Then σ acts on the Jacobian $J\tilde{C}$ of $\tilde{C}$, and $J\tilde{C}$ splits under this action into a "+" part (which is nothing but $\pi^* JC$) and a "−" part P, which is called the Prym variety of $(\tilde{C}, C)$. More precisely, one has

$$P = \operatorname{Im}(1 - \sigma^*) \subset J\tilde{C}.$$

We'll usually put $g(C) = g + 1$, so that $\tilde{C}$ has genus $2g + 1$ and P has dimension g.

The Prym variety P turns out to be a principally polarized abelian variety (I'll say for short p.p.a.v.); in fact, *the principal polarization of* $J\tilde{C}$ *induces twice a principal polarization on* P. This follows from a general (and simple) lemma about p.p.a.v.'s:

LEMMA. *Let u be an endomorphism of a p.p.a.v. A, and p a positive integer. Assume*

(i) *u is symmetric (i.e., $u = \hat{u}$, the dual variety $\hat{A}$ of A being identified with A via the principal polarization),*

(ii) *the kernel of u is connected,*

(iii) $u^2 = pu$.

Then the principal polarization of A induces p times a principal polarization on the image of u. □

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The lemma applies to $u = 1 - \sigma$, giving our assertion. $\square$

Actually much more is true. Using the special properties of the theta divisor of a Jacobian (essentially the Riemann singularity theorem), one can show that the theta divisor of $J\tilde{C}$ itself restricts to twice a theta divisor on P [M]. More precisely, let us fix a *theta-characteristic* on C, that is, a line bundle L on C such that $L^2 = \omega_C$. Such a theta-characteristic corresponds to a symmetric theta divisor Θ_L on JC, defined (set-theoretically) by

$$\Theta_L = \{\alpha \in JC | h^0(L \otimes \alpha) \geq 1\}.$$

The line bundle $\pi^* L$ is a theta-characteristic on $\tilde{C}$; we will assume that it is *even*, i.e., that $h^0(\pi^* L)$ is even. We'll denote by $\tilde{\Theta}_{\pi^* L}$ the corresponding theta divisor on $J\tilde{C}$. Then

THEOREM 1. *There exists a divisor Ξ on P, defining the principal polarization of P, such that*

$$\tilde{\Theta}_{\pi^* L}|_P = 2\Xi. \tag{1}$$

The relation of $\tilde{\Theta}_{\pi^* L}$ with JC is easier to establish. The covering π is defined by a "half-period" η on C (that is, a line bundle whose square is trivial). I claim

$$(\pi^*)^{-1}\tilde{\Theta}_{\pi^* L} = \Theta_L + \Theta_{L\otimes\eta}. \tag{2}$$

Let us check this set-theoretically: an element α of JC belongs to the left-hand side iff $H^0(\tilde{C}, \pi^*(L \otimes \alpha))$ is nonzero. But one has

$$H^0(\tilde{C}, \pi^*(L \otimes \alpha)) = H^0(C, L \otimes \alpha) \oplus H^0(C, L \otimes \eta \otimes \alpha),$$

which gives (2).

The relations (1) and (2) taken together deserve to be called the *Schottky-Jung configuration.* Let us see that they imply the classical Schottky-Jung identities. Let $\tilde{\theta}, \xi, \theta$ be nonzero sections of $H^0(J\tilde{C}, \mathscr{O}(\tilde{\Theta}_{\pi^* L})), H^0(P, \mathscr{O}(\Xi))$, and $H^0(JC, \mathscr{O}(\Theta_L))$. There exist constants a, b such that

$$\tilde{\theta}(x) = a\xi(x)^2 \quad \text{for all } x \text{ in } P; \tag{1'}$$

$$\tilde{\theta}(\pi^* y) = b\theta(y)\theta(y + \eta) \quad \text{for all } y \text{ in } JC. \tag{2'}$$

(These are equalities between elements of line bundles over P and JC.)

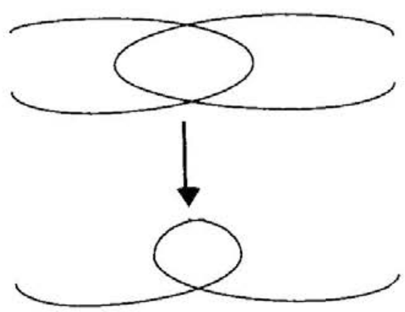
We want to apply both (1′) and (2′) to the same element of $J\tilde{C}$. This element must be of the form $\pi^* y$, with $y \in JC$ and $\pi^* y \in P$. Applying π_*, this implies $2y = 0$. Let us denote by JC_2 the group of points of order 2 in JC. One checks that for an element y of JC_2, the condition $\pi^* y \in P$ means that y is orthogonal to η for the natural pairing on JC_2. Therefore for $y \in \eta^\perp$ we get that

$$\xi(\pi^* y)^2 / \theta(y)\theta(y + \eta) \quad \textit{is independent of } y.$$

This is the classical form of the Schottky-Jung identities.

2. Pryms versus Jacobians. I said in the introduction that Pryms are more general than Jacobians. This requires some explanation. Let me denote by $\mathscr{A}_g$ the moduli space of p.p.a.v.'s of dimension g, by $\mathscr{J}_g$ the Jacobian locus in $\mathscr{A}_g$, by $\mathscr{P}_g$ the subset of $\mathscr{A}_g$ corresponding to Prym varieties, and by $\overline{\mathscr{P}}_g$ its closure. Then *$\overline{\mathscr{P}_g}$ is an irreducible subvariety of $\mathscr{A}_g$, of dimension $3g$ (for $g \geq 5$), containing $\mathscr{J}_g$; for $g \leq 5$ one has $\overline{\mathscr{P}}_g = \mathscr{A}_g$.* These facts are (essentially) due to Wirtinger **[Wi]**. Since curves of genus $g+1$ depend on $3g$ moduli, the assertion on $\dim(\mathscr{P}_g)$ means that the "Prym map" $(\tilde{C}, C) \to \operatorname{Prym}(\tilde{C}, C)$ is generically finite for $g \geq 5$, and dominant for $g \leq 5$; later I'll give much more precise statements (§4). Let me first prove the inclusion $\mathscr{J}_g \subset \overline{\mathscr{P}}_g$.

We start with a smooth curve X of genus g; we choose two distinct points on C—say, p and q—and denote by X' the curve obtained from X by identifying p and q. We construct an étale double cover $\tilde{X}'$ of X' by taking two copies of X and identifying point p of the first copy with point q of the second, and vice versa. The picture of this *Wirtinger cover* looks like this:



By deformation one can get a flat family $\mathscr{C} \to D$ of curves over the unit disk D, with $\mathscr{C}_t$ smooth for $t \neq 0$ and $\mathscr{C}_0 = X'$, and an étale double cover $\tilde{\mathscr{C}} \to \mathscr{C}$ such that $\tilde{\mathscr{C}}_0 = \tilde{X}'$. It is then quite easy to show that the Prym variety $\operatorname{Prym}(\tilde{\mathscr{C}}_t, \mathscr{C}_t)$ specializes to JX, which proves our assertion.

The Jacobian locus is not contained in $\mathscr{P}_g$, but some Jacobians are. Here are some examples.

(a) *Hyperelliptic curves.* Assume that the curve C is hyperelliptic, i.e., can be realized as a double cover of $\mathbf{P}^1$, with branch locus $B \subset \mathbf{P}^1$. It is well known that every étale double cover of C is obtained in the following way: one chooses a partition $B = B_1 \amalg B_2$ of B with $\#B_1$ and $\#B_2$ even. One denotes by C_i the double cover of $\mathbf{P}^1$ branched along B_i $(i = 1, 2)$, by σ_i the natural involution of C_i, and by $\tilde{C}$ the fibered product $C_1 \times_{\mathbf{P}^1} C_2$. Then $\sigma = (\sigma_1, \sigma_2)$ is a fixed-point free involution of $\tilde{C}$, and the quotient $\tilde{C}/\sigma$ is a 2-sheeted cover of $\mathbf{P}^1$ branched along B—hence isomorphic to C. Now in this situation it is easy to check that *the Prym variety of $(\tilde{C}, C)$ is isomorphic to $JC_1 \times JC_2$*, hence belongs to $\overline{\mathscr{J}_g}$—and to $\mathscr{J}_g$ iff $\#B_1$ or $\#B_2$ equals 2. Conversely, taking $C_1 = X$ and $C_2 = \mathbf{P}^1$ shows that every hyperelliptic Jacobian JX is a Prym.

(b) *Trigonal curves.* Assume now that C is trigonal, i.e., admits a base-point free linear system $|D|$ of degree 3 and (projective) dimension 1. Let $\mathscr{X}$

denote the variety of effective divisors E on C such that $\pi_*E \in |D|$. Then $\mathscr{X}$ has two components X and X', which are exchanged under the involution σ. The curve X is smooth, and the map $E \to \pi_*E$ from X to the projective line $|D|$ is 4-to-1. One can show **[R2]** that $\mathrm{Prym}(\tilde{C}, C)$ *is isomorphic to* JX, and conversely that the Jacobian of a tetragonal curve is the Prym of some trigonal curve.

(c) *Plane quintics.* Let us start here from a genus 5 curve X that is neither hyperelliptic nor trigonal. The canonical model of X is then defined by 3 quadratic equations $P = Q = R = 0$ in $\mathbf{P}^4$. Thus the linear system of quadrics in $\mathbf{P}^4$ containing X is a net (= projective plane) Π. The discriminant curve C (i.e., the subset of Π corresponding to singular quadrics) is defined by the equation $\det(xP + yQ + zR) = 0$, and is therefore of degree 5. Let us assume for simplicity that Π contains no rank 3 quadric. Then C is smooth, and the quadric corresponding to a point in C has two different rulings. These two rulings define an étale 2-sheeted covering $\pi: \tilde{C} \to C$. One can prove **[Ma]** that *the Prym variety of* $(\tilde{C}, C)$ *is isomorphic to* JX. The half-period η associated to π satisfies $h^0(\mathscr{O}_C(1) \otimes \eta) = 0$; conversely, any pair $(\tilde{C}, C)$, where C is a plane quintic and $\pi: \tilde{C} \to C$ a covering with $h^0(\mathscr{O}_C(1) \otimes \eta) = 0$, comes from a net of quadrics, so that $\mathrm{Prym}(\tilde{C}, C)$ is a Jacobian.

Using Mumford's work, Shokurov **[Sh]** proved that these examples are essentially the only ones.

THEOREM 2. *Assume* $g \neq 4$. *If* $\mathrm{Prym}(\tilde{C}, C)$ *is a Jacobian, then* C *is hyperelliptic, or trigonal, or* C *is a plane quintic and* $h^0(\mathscr{O}_C(1) \otimes \eta) = 0$.

The proof is based on Mumford's analysis of the singularities of the Θ divisor **[M]**. A Jacobian has a highly singular Θ divisor: the dimension of the singular locus $\mathrm{Sing}(\Theta)$ is at least $g - 4$. This is a consequence of the Riemann singularity theorem, which gives a parametrization of $\mathrm{Sing}(\Theta)$ in terms of special divisors on the curve. For Prym varieties, one deduces from Theorem 1 a geometric description of $\mathrm{Sing}(\Theta)$. Using that description Mumford was able to prove the following result **[M]**: *If* $\mathrm{Prym}(\tilde{C}, C)$ *satisfies* $\dim \mathrm{Sing}(\Theta) \geq g - 4$, *one of the following possibilities occurs*:

(i) *C is hyperelliptic.*

(ii) *C is trigonal.*

(iii) *C is a plane quintic, and* $h^0(\mathscr{O}_C(1) \otimes \eta) = 0$.

(iv) *C is bielliptic (i.e., admits a 2-to-1 map onto an elliptic curve).*

(v) *C is a genus 5 curve with a line bundle L such that* $L^{\otimes 2} \cong \omega_C$, $h^0(L) = 2$, *and* $h^0(L \otimes \eta) = 0$.

Assume now that $\mathrm{Prym}(\tilde{C}, C)$ is a Jacobian, of dimension $g \neq 4$. Then we are in one of the situations (i)–(iv), and we want to rule out case (iv). Shokurov does it by considering the points of multiplicity ≥ 3 of Θ; this approach works for $g \geq 7$. Alternatively one can use the detailed analysis of $\mathrm{Sing}(\Theta)$ in this case given in **[D1]** to get the result for all $g \neq 4$. $\square$

REMARKS. (1) In case (v) ($g = 4$), the Prym might be the Jacobian of a genus 4 curve with one vanishing thetanull (or in other words, one g_3^1 such that $2g_3^1 \equiv K$). I do not know how to rule out this possibility. The only way I can imagine would be by proving that the tangent cone at the double point is a rank 4 quadric. This brings up the difficult (but interesting) problem of describing the tangent cone to Θ at an exceptional singularity (see next remark).

(2) We now know a lot more about $\mathrm{Sing}(\Theta)$. If C is generic, Welters **[W1]** and Debarre **[D2]** have shown that $\mathrm{Sing}(\Theta)$ is irreducible of dimension $g-6$ for $g \geq 7$, finite for $g = 6$, and empty for $g \leq 5$. On an arbitrary Prym, two types of singularities may appear on the theta divisor **[M]**. The *stable singularities*, which exist provided $g \geq 6$, are a specialization of the generic case; each component of their locus $\mathrm{Sing}_{\mathrm{st}}(\Theta)$ has dimension $\geq g - 6$. The tangent cone to Θ at such a singular point has a nice geometric description. On the other hand, the presence of special divisors on some particular curves C creates *exceptional singularities* of Θ, which are harder to control: the dimension of their locus $\mathrm{Sing}_{\mathrm{ex}}(\Theta)$ may be arbitrary, and I don't even know in general how to compute their multiplicity.

3. Prym varieties as intermediate Jacobians. Let me first recall one possible definition of the Jacobian of an algebraic curve C. We start from the Hodge decomposition

$$H^1(C, \mathbf{C}) = H^{1,0} \oplus H^{0,1};$$

the subgroup $H^1(C, \mathbf{Z})$ of $H^1(C, \mathbf{C})$ projects as a lattice in each direct factor. We put $JC = H^{0,1}/\mathrm{Im}\, H^1(C, \mathbf{Z})$; the cup-product on $H^1(C, \mathbf{Z})$ defines the principal polarization of JC.

Now let X be a smooth threefold; assume for simplicity that X has no nonzero holomorphic 3-form. Then the Hodge decomposition of $H^3(X, \mathbf{C})$ reduces to

$$H^3(X, \mathbf{C}) = H^{2,1} \oplus H^{1,2}.$$

Again the image of $H^3(X, \mathbf{Z})$ in $H^{1,2}$ is a lattice; we define the intermediate Jacobian JX of X as the complex torus $H^{1,2}/\mathrm{Im}\, H^3(X, \mathbf{Z})$, with the principal polarization defined by the cup-product on $H^3(X, \mathbf{Z})$.

The role of intermediate Jacobians is quite analogous to the role played by Jacobians in the theory of curves. Their importance stems in particular from three different aspects.

(a) *Curves on threefolds.* As in the case of curves, there is a homomorphism φ from the group of 1-dimensional cycles on X with zero homology class into JX. This homomorphism is "algebraic" in the following sense. Let $(C_t)_{t \in T}$ be an algebraic family of curves on X, parametrized by a connected variety T. Choose a base-point $t_0 \in T$, and define the *Abel-Jacobi map* $\alpha \colon T \to JX$ by $\alpha(t) = \varphi(C_t - C_{t_0})$. Then α is an algebraic map (which depends on the choice of t_0 only up to translation).

(b) *The Torelli problem.* This is the question of whether the Hodge structure on $H^3(X, \mathbf{Z})$ determines the threefold X. The data of this Hodge structure amounts to the data of the p.p.a.v. JX; therefore the problem is to recover X from its intermediate Jacobian.

(c) *Rationality questions.* The intermediate Jacobian of a rational threefold is a Jacobian or a product of Jacobians **[C-G]**. Therefore if $JX \notin \overline{\mathscr{J}_g}$, the threefold X is not rational: this is the *Clemens-Griffiths criterion*, one of the few known ways to prove the irrationality of a given threefold (see **[B5]**).

Prym varieties appear as intermediate Jacobians of a particular class of threefolds, the *conic bundles*. We'll say that the threefold X is a conic bundle if it admits a morphism f onto a rational surface S such that each smooth fibre of f is a rational curve, and each singular fibre is a union of two rational curves meeting transversally. It is then easy to show that the set of points s in S such that $f^{-1}(s)$ is singular is a smooth curve $C \subset S$. The pair of rational curves above each point of C define an étale 2-sheeted covering $\pi \colon \tilde{C} \to C$. Then

THEOREM 3 (MUMFORD). *The intermediate Jacobian of X is isomorphic to the Prym variety $P = \operatorname{Prym}(\tilde{C}, C)$.*

The isomorphism can be defined as follows. For $\tilde{s} \in \tilde{C}$, let us denote by $l_{\tilde{s}}$ the corresponding component of $f^{-1}(\pi\tilde{s})$. Then $(l_{\tilde{s}})_{\tilde{s}\in\tilde{C}}$ is a family of curves on X, and therefore gives rise to an Abel-Jacobi map $\alpha \colon \tilde{C} \to JX$ (defined up to translation), which extends to a homomorphism $\beta \colon J\tilde{C} \to JX$. Let s, t be two points of C, with $\pi^{-1}(s) = \{\tilde{s}, \tilde{s}'\}$ and $\pi^{-1}(t) = \{\tilde{t}, \tilde{t}'\}$. Then one has

$$\beta(\pi^*[s] - \pi^*[t]) = l_{\tilde{s}} + l_{\tilde{s}'} - l_{\tilde{t}} - l_{\tilde{t}'} = f^*([s] - [t]);$$

but the 0-cycle $[s] - [t]$ on S is linearly equivalent to 0, hence its image under β is 0. By linearity we conclude that β vanishes on $\pi^* JC$, hence factors through the quotient $J\tilde{C}/\pi^* JC$, which is isomorphic to P. I have to refer to **[B2]** or **[T2]** for the proof that β induces in fact an isomorphism from P onto JX. $\square$

Now let me give a few examples.

(a) *The cubic threefold.* Let X be a smooth cubic hypersurface in $\mathbf{P}^4$. We choose a line l contained in X, and denote by X_l the blow-up of X along l. The projection from l onto a generic $\mathbf{P}^2 \subset \mathbf{P}^4$ defines a morphism $f \colon X_l \to \mathbf{P}^2$. For $t \in \mathbf{P}^2$, the plane spanned by l and t intersects X along a cubic, which is the union of l and a conic; the fibre $f^{-1}(t)$ is nothing but that conic. One checks easily that the discriminant curve C is a quintic in $\mathbf{P}^2$; it is smooth if l has been chosen general enough, and the covering π satisfies $h^0(\mathscr{O}_C(1) \otimes \eta) = 1$. One deduces from Theorem 2 that JX $(= JX_l)$ is not a Jacobian, and then from the Clemens-Griffiths criterion that X *is not rational.* This was one of the first examples known of a unirational but nonrational threefold **[C-G]**.

A detailed analysis of the Prym variety JX shows in fact that the Θ divisor of JX has only one singular point, which has multiplicity 3, and that the projective tangent cone to Θ at this point is isomorphic to X **[B4]**. This gives the Torelli theorem for the cubic (first proved in **[T1]** and **[C-G]**). The Fano surface F_X, which parametrizes lines on X, has a simple interpretation in terms of the covering $\tilde{C} \to C$; this may be used for instance to prove that the Abel-Jacobi map $\alpha: F_X \to JX$ is an embedding **[B3]**.

(b) *The intersection of* 3 *quadrics in* $\mathbf{P}^6$. Now let X be a smooth complete intersection of 3 quadrics in $\mathbf{P}^6$. As in §2, example (c), we denote by Π the net of quadrics in $\mathbf{P}^6$ containing X and by C the discriminant curve, which is now of degree 7. We'll again assume for simplicity that all points of C correspond to rank 6 quadrics, so that we have an étale 2-sheeted covering $\pi: \tilde{C} \to C$ defined by the two rulings of these quadrics.

Let us choose again a line l contained in X. For a general point x of X, there is exactly one quadric $f(x)$ of Π containing the plane spanned by l and x. One thus defines a rational map $X \to \mathbf{P}^2$, which extends to a morphism $f: X' \to \mathbf{P}^2$, where X' is obtained from X by some innocuous blowing up and down (here "innocuous" means $JX = JX'$). Let q be a quadric of Π. The fibre $f^{-1}(q)$ is the set of planes in $\mathbf{P}^6$ contained in q and containing l; this is a rational curve if q is smooth, and the union of two rational curves (one for each ruling of q) if q is singular. We therefore conclude from Theorem 3 that JX *is isomorphic to* $\operatorname{Prym}(\tilde{C}, C)$.

A few consequences: first of all, X is not rational (though it is unirational). Second, the Torelli theorem holds for these varieties: this is because one can recover the pair $(\tilde{C}, C)$ from the associated Prym (**[D4]**, see §4), then reconstruct X from $(\tilde{C}, C)$ (**[B2]** or **[T3]**). Finally the surface of conics lying in X can be described nicely from the point of view of Pryms **[B3]**.

All this set-up generalizes in a straightforward way to the intersection of 3 quadrics in $\mathbf{P}^{2n}$. The intermediate Jacobian is isomorphic to $\operatorname{Prym}(\tilde{C}, C)$, where C is a plane curve of degree $2n+1$ (for $n = 2$ this is example (c) of §2). The Torelli theorem still holds.

(c) *Other examples.* Most of the classical examples of unirational threefolds are not conic bundles, but can be specialized to conic bundles by acquiring a certain number of double points. This allows us to prove *generic irrationality* results, i.e., to show that a member of these families which is general enough is not rational **[B2]**. I'd like also to mention the paper **[B-CT-S-SD]**, where we give an example of a conic bundle X that is not rational, but such that $X \times \mathbf{P}^3$ is rational.

4. The Prym map.

(a) *Generic injectivity.* We denote by $\mathscr{R}_p$ the moduli space of étale 2-sheeted coverings $\pi: \tilde{C} \to C$, with $g(\tilde{C}) = p$; equivalently this is the moduli space of curves $C \in \mathscr{M}_p$ with a (nonzero) half-period $\eta \in H^1(C, \mathbf{Z}/2)$. It is a finite cover of $\mathscr{M}_p$, of degree $2^{2p} - 1$.

By associating to a pair $(\tilde{C}, C)$ its Prym variety, we define a morphism

$$\mathfrak{p}_g : \mathscr{R}_{g+1} \to \mathscr{A}_g,$$

called the *Prym map*. It should be thought of as an analogue of the period map (or Torelli map) $J: \mathcal{M}_g \to \mathcal{A}_g$, which associates to a curve its Jacobi variety. The Torelli theorem asserts that J is injective—it is even an embedding **[O-S]**. For the Prym map only a weaker statement is true:

THEOREM 4. *For $g \geq 6$, the Prym map $\mathfrak{p}_g: \mathcal{R}_{g+1} \to \mathcal{A}_g$ is generically injective.*

This result was first proved by Friedman-Smith **[F-S]** and, for $g \geq 8$, by Kanev **[K]**, using degeneration arguments. A more geometric proof was obtained later by Welters **[W3]**, and recently by Debarre **[D2]**. Both proofs mimic known proofs of the Torelli theorem. For any p.p.a.v. (A, Θ), define a subvariety Σ_A of A by

$$\Sigma_A = \{a \in A | a + \operatorname{Sing}(\Theta) \subset \Theta\}.$$

Welters shows that when (A, Θ) is the Prym of a pair $(\tilde{C}, C)$, with C general enough and $g(C) \geq 17$, Σ_A is the union of the surface

$$\tilde{C} - \tilde{C} := \{[x] + [y] - [\sigma x] - [\sigma y] | x, y \in \tilde{C}\}$$

and possibly of some components of dimension ≤ 1. It is then easy to recover $\tilde{C}$ and C from the surface $\tilde{C} - \tilde{C}$. Observe that for a Jacobian JC, Σ_{JC} is the surface $C - C$ by another result of Welters **[W2]**.

Debarre considers the tangent cones to the singular points of Θ. The tangent space to $\operatorname{Prym}(\tilde{C}, C)$ at any point can be canonically identified to the dual of $H^0(C, \omega_C \otimes \eta)$. Let us denote by $\mathbf{P}$ the projective space $\mathbf{P}(H^0(C, \omega_C \otimes \eta)^*)$. The linear system $|\omega_C \otimes \eta|$ maps C into $\mathbf{P}$, and this map is an embedding if C has no g_4^1; the image is the *half-canonical model* of C. For each singular point a of Θ, the projectivization of the tangent cone to Θ at a is a hypersurface in $\mathbf{P}$. For $g \geq 7$, Debarre shows that *the intersection of these projective tangent cones is the half-canonical curve* (when C is generic). This should be compared to Green's theorem, which says that for all Jacobians except the well-known exceptions (hyperelliptic, trigonal, plane quintics), the intersection of the projectivized tangent cones to the theta divisor is the canonical curve.

Contrary to the case of Jacobians, both results do not hold for every Prym. If C is a plane curve and $h^0(\mathcal{O}_C(2) \otimes \eta) = 0$, the Prym variety P satisfies $\Sigma_P = \{0\}$. If C is trigonal, say of genus ≥ 8, the intersection of the tangent cones is the tetragonal curve X such that $\operatorname{Prym}(\tilde{C}, C) = JX$ (§2, example (b)); therefore there exist singular points of Θ such that the corresponding projective tangent cones do not contain the half-canonical model of C. In both cases the trouble comes from the exceptional singularities of the theta divisor (§2, Remark 2).

(b) *Non-injectivity*. The Prym map is *never* injective. This is because of Donagi's *tetragonal construction*, which I am now going to explain. We start with a pair $(\tilde{C}, C)$, where the curve C is tetragonal, i.e., admits a linear system $|D|$ of degree 4 and dimension 1. For simplicity I will assume that every member of $|D|$ has at most one double point. As in §2, example (b), we consider the variety $\mathscr{X}$ of effective divisors on C such that $\pi_*E \in |D|$. It is again the disjoint union of two smooth curves, say $\tilde{C}_1$ and $\tilde{C}_2$, but contrary to the trigonal case the natural involution of $\mathscr{X}$ preserves each $\tilde{C}_i$ and induces on $\tilde{C}_i$ a fixed-point-free involution σ_i. We put $C_i = \tilde{C}_i/\sigma_i$. The map $E \to \pi_*E$ from $\tilde{C}_i$ onto $|D| = \mathbf{P}^1$ defines a 4-to-1 map $C_i \to \mathbf{P}^1$. So we have associated to $(\tilde{C}, C)$ two other pairs $(\tilde{C}_1, C_1)$ and $(\tilde{C}_2, C_2)$, with C_1 and C_2 again tetragonal. These three pairs are not isomorphic in general. However, Donagi proves that *the Prym varieties* $\mathrm{Prym}(\tilde{C}, C)$, $\mathrm{Prym}(\tilde{C}_1, C_1)$, *and* $\mathrm{Prym}(\tilde{C}_2, C_2)$ *are isomorphic* **[Do1]**. Note that the construction is symmetric: if we perform it on $(\tilde{C}_1, C_1)$, we'll get back the two other pairs $(\tilde{C}, C)$ and $(\tilde{C}_2, C_2)$.

In **[Do1]** Donagi conjectures that the tetragonal construction is the only obstruction to the injectivity of the Prym map. At that time this conjecture seemed to me overoptimistic. It looks now more reasonable in view of the recent results of Debarre **[D3, D4]**:

THEOREM 5. *Let* $(\tilde{C}, C)$ *and* $(\tilde{C}', C')$ *be two pairs whose Prym varieties are isomorphic.*

(a) *If* C *is tetragonal of genus* ≥ 13 *(but not bielliptic), then either* $(\tilde{C}', C')$ *is isomorphic to* $(\tilde{C}, C)$, *or it is obtained from* $(\tilde{C}, C)$ *by the tetragonal construction.*

(b) *If* C *is a plane curve of degree* ≥ 7, *then* $(\tilde{C}', C')$ *is isomorphic to* $(\tilde{C}, C)$.

Let me also mention that the differential of the Prym map behaves nicely at nontetragonal curves. At a pair $(\tilde{C}, C)$ with Prym variety P, the transpose of the tangent map

$$T_{(\tilde{C},C)}\mathfrak{p}_g : T_{(\tilde{C},C)}(\mathscr{R}_{g+1}) \to T_P(\mathscr{A}_g)$$

can be identified with the natural homomorphism from $S^2H^0(C, \omega_C \otimes \eta)$ into $H^0(C, \omega_C^{\otimes 2})$. It follows from **[G-L]** that this map is surjective (i.e., that $\mathfrak{p}_g$ *is an immersion at* $(\tilde{C}, C)$) if C has no g^1_4 and $g(C) \geq 10$.

(c) *The Prym map from* $\mathscr{R}_6$ *to* $\mathscr{A}_5$. We now consider the low genus case $g \leq 5$. The Prym map is no longer injective, it is actually *generically surjective*—this can be deduced from the expression for its codifferential given above **[B2]**. However the story does not stop here: the description of the fibres of $\mathfrak{p}_g$ involves some beautiful geometry.

For $g = 5$ one has $\dim \mathscr{R}_6 = \dim \mathscr{A}_5 = 15$, so the Prym map is generically finite. Its degree has been computed in **[D-S]**: it is 27. Let me quote here Donagi-Smith: "Wake an algebraic geometer in the dead of night, whispering: '27'. Chances are, he will respond: lines on a cubic surface...." Indeed Donagi proved that *the fibre of* $\mathfrak{p}_5$ *has the structure of the* 27 *lines on a cubic surface.* To explain what this means, let us say that two pairs $(\tilde{C}, C)$ and $(\tilde{C}', C')$ in $\mathscr{R}_6$ are *incident* if one is deduced from the other by a tetragonal construction. Then for (A, Θ) generic in $\mathscr{A}_5$, the fibre $\mathfrak{p}_5^{-1}(A, \Theta)$ endowed with this incidence relation is isomorphic to the set of lines on a (smooth) cubic surface with the usual incidence relation. For instance, a general curve of genus 6 has 5 g^1_4, giving rise to ten elements $(\tilde{C}_i, C_i)_{1\leq i\leq 10}$, with $(\tilde{C}_i, C_i)$ incident to $(\tilde{C}_{i+5}, C_{i+5})$; this corresponds to the fact that a line l on a cubic surface S has 10 incident lines $l_1, \dots, l_{10}$, distributed into 5 planes $\Pi_1, \dots, \Pi_5$ such that $\Pi_i \cap S = l \cup l_i \cup l_{i+5}$.

A consequence is that the Galois group of $\mathscr{R}_6$ over $\mathscr{A}_5$ is the Galois group of the 27 lines (the Weyl group of the root system E_6). Another consequence of this result is that it severely restricts the possible degenerations of the fibre, allowing us, for instance, to study the branch locus of $\mathfrak{p}_5$. I refer to Donagi's forthcoming book **[Do2]**, and in the meantime to the announcement **[Do1]**, for more details on the fascinating geometry of the situation.

(d) *The Prym map from $\mathscr{R}_5$ to $\mathscr{A}_4$.* The structure of $\mathfrak{p}_4$ has been worked out again by Donagi **[Do1, Do2]**. To explain his result, let us first observe that the intermediate Jacobian of a cubic threefold X has a canonical symmetric Θ divisor, defined by the condition that 0 is its unique singular point. We then define the *parity* of a half-period (i.e., a point of order 2) on JX as the parity of the multiplicity of Θ at this point. We denote by $\mathscr{C}$ the moduli space of (smooth) cubic threefolds with an even half-period. Let (X, ε) be a point of $\mathscr{C}$; the half-period ε defines a degree 2 isogeny $\varphi\colon J^\varepsilon \to JX$. The Fano surface F_X of lines in X is embedded into JX through the Abel-Jacobi mapping (§3, example (b)); the surface $F_{X,\varepsilon} := \varphi^{-1}(F_X)$ is smooth and irreducible, and φ induces an étale 2-sheeted covering $F_{X,\varepsilon} \to F_X$.

Donagi constructs a *birational map* $\kappa\colon \mathscr{A}_4 \to \mathscr{C}$, then proves that *the fibre of* $\mathfrak{p}_4$ *at a general* $(A, \Theta) \in \mathscr{A}_4$ *is isomorphic to the surface* $F_{\kappa(A)}$.

(e) $g \leq 3$. The lower genus cases are easier—because we are now dealing with Jacobians—but still interesting. Let (A, Θ) be a p.p.a.v. We'll denote by ι the involution $a \to -a$ on A. If (A, Θ) is the Prym of a pair $(\tilde{C}, C)$, the curve $\tilde{C}$ can be embedded in A in such a way that ι induces on $\tilde{C}$ the involution σ (choose an element δ of JC_2 such that $(\delta.\eta) = 1$ and consider the map $x \to x - \sigma x + \pi^*\delta$).

If $\dim(A) = 3$, there are exactly two translates of the theta divisor, say Θ_a and Θ_{-a}, which contain $\tilde{C}$, and one has $\tilde{C} = \Theta_a \cap \Theta_{-a}$. Conversely, for any a in A, one deduces easily from §2, example (b), that (A, Θ) is isomorphic to $\mathrm{Prym}(\tilde{C}, \tilde{C}/\iota)$, where $\tilde{C}$ is the curve $\Theta_a \cap \Theta_{-a}$. Up to translation, this curve depends only on the element $\pm 2a$ in the Kummer variety $\mathrm{Km}(A) := A/\{\pm 1\}$. It follows that *the fibre* $\mathfrak{p}_3^{-1}(A, \Theta)$ *is isomorphic to* $\mathrm{Km}(A)$ **[R1]**.

If $\dim(A) = 2$, the curve $\tilde{C}$ belongs to the linear system $|2\Theta|$. Conversely, any curve $\tilde{C}$ in $|2\Theta|$ is symmetric, and (A, Θ) is isomorphic to $\mathrm{Prym}(\tilde{C}, \tilde{C}/\iota)$ (at least if $\tilde{C}$ is smooth). Taking into account the action of the group JC_2 on $|2\Theta|$ by translation, one sees that *the fibre* $\mathfrak{p}_2^{-1}(A, \Theta)$ *is birationally isomorphic to the quotient* $|2\Theta|/JC_2$. A biregular description of this fibre, based on a detailed analysis of the above birational map, can be found in **[V]**.

Finally the case $g = 1$ is an easy consequence of the explicit description for Pryms of the hyperelliptic curve given in §2, example (a); I leave it as an exercise for the reader.

(f) *Prym is proper.* The above statements must be taken with a grain of salt: if one sticks to Prym of smooth curves, the fibre of $\mathfrak{p}_g$ for $g \leq 4$ will be only an open set in the compact variety I have described. However, one of the key ingredients in the study of the Prym map is its compactification, often referred to by the motto "Prym is proper." This means that by accepting coverings with mild singularities, we'll get a moduli space $\overline{\mathscr{R}}_{g+1}$ (containing $\mathscr{R}_{g+1}$ as a dense open subset) and an extended Prym map $\overline{\mathfrak{p}}_g : \overline{\mathscr{R}}_{g+1} \to \mathscr{A}_g$, which is *proper.*

The points we have to add in $\overline{\mathscr{R}}_{g+1}$ are pairs $(\tilde{C}, C)$ of stable curves (this essentially means that their only singularities are ordinary double points), such that C is the quotient of $\tilde{C}$ by an involution σ. Two cases occur: if σ exchanges some components of $\tilde{C}$ one gets a Wirtinger-type covering (see §2), whose Prym is a Jacobian or a product. If σ preserves all components of $\tilde{C}$ one gets an *admissible cover*: σ fixes all the nodes (but no smooth point), and preserves the two branches at each node. For such a covering one can define a Prym variety exactly as in the nonsingular case, and all the properties we have previously found extend in a (more or less) straightforward way **[B1]**. These generalized Pryms appear naturally in the constructions we have already seen: e.g., as Jacobians of tetragonal curves when the g^1_4 has some higher-order ramification, as intermediate Jacobians of the (smooth) intersection of a net of quadrics in $\mathbf{P}^{2n}$ when at least one quadric of the net has a singular line, etc.

Let us now assume for simplicity that $\tilde{C}$ and C are irreducible, each of them with ν nodes. Let N and $\tilde{N}$ be the normalizations of C and $\tilde{C}$. The map $\pi : \tilde{C} \to C$ induces a covering $\pi' : \tilde{N} \to N$, ramified at the 2ν points of $\tilde{N}$ which dominate a singular point of $\tilde{C}$. Let σ' be the corresponding involution of $\tilde{N}$. In **[M]**, Mumford defines an abelian variety $R = \mathrm{Prym}(\tilde{N}, N)$ as the image of the endomorphism $1 - \sigma'^*$ of $J\tilde{N}$. Let $P = \mathrm{Prym}(\tilde{C}, C)$; there exists an isogeny $P \to R$ with kernel $(\mathbf{Z}/2)^{\nu-1}$ **[B1]**. Therefore if $\nu = 1$, we find $P = R$; this explains why the Prym of a covering $\pi' : \tilde{N} \to N$ with two ramification points has a principal polarization, while in the general case one has to choose a pairing between the ramification points in order to get a principal polarization on a suitable cover of R.

5. The Schottky problem for Pryms. The usual Schottky problem is the problem of characterizing Jacobians among all p.p.a.v.'s. I have to refer to other talks in this volume, or to the report **[B6]**, for a survey of the recent progress on this subject. Let me just mention three approaches which have been successful:

(a) *The Schottky-Jung approach* gives explicit equations for the subvariety $\mathcal{J}_g$ of $\mathcal{A}_g$, in terms of the *theta-constants* $\theta[{}^p_q](\tau)$ associated to a p.p.a.v. (those are essentially the values of the theta function at the points of order two).

(b) *The Andreotti-Mayer theorem* says that $\mathcal{J}_g$ is a component of the subvariety $\mathcal{N}_g^{g-4}$ of $\mathcal{A}_g$ consisting of p.p.a.v.'s (A, Θ) such that $\dim \operatorname{Sing}(\Theta) \geq g-4$ (cf. §2).

(c) *The trisecant approach* is based on the reducibility properties of the intersections $\Theta \cap \Theta_a$, which can be interpreted in terms of trisecants to the Kummer variety. For an indecomposable p.p.a.v. (A, Θ), the global sections of the line bundle $\mathcal{O}_A(2\Theta)$ (or equivalently the second-order theta functions) define a morphism $\psi \colon A \to \mathbf{P}^N$ $(N = 2^g - 1)$; the image of ψ is the Kummer variety $K(A, \Theta)$, isomorphic to $A/\{\pm 1\}$. The two following conditions turn out to be equivalent:

(i) There exist nonzero distinct elements a, x, y of A such that

$$\Theta \cap \Theta_a \subset \Theta_x \cup \Theta_y.$$

(ii) The Kummer variety $K(A, \Theta)$ admits a trisecant.

They are satisfied by Jacobians, and one hopes that they characterize Jacobians (*trisecant conjecture*). I have to refer to **[B6]** for a discussion of this approach. Let me just mention that by (cleverly) specializing the points a, x, y to 0, condition (i) becomes the celebrated KP equation, which indeed characterizes Jacobians **[S, A-D]**.

I would like now to consider the analogous problem for Pryms, i.e., to try to characterize Prym varieties among all p.p.a.v.'s. It would be very nice to carry out the first approach for Pryms, that is, to find explicit polynomials in the theta-constants vanishing on $\mathcal{P}_g$: in fact, through the Schottky-Jung identities (§1) this would give new equations for $\mathcal{J}_g$. Unfortunately nothing seems to be known in this direction.

On the other hand, the second and third approaches work nicely for Pryms, in striking analogy with the case of Jacobians. We already saw (§2) that $\mathcal{P}_g$ is contained in the locus $\mathcal{N}_g^{g-6}$ of p.p.a.v.'s with $\dim \operatorname{Sing}(\Theta) \geq g-6$; Debarre has proved that *$\mathcal{P}_g$ is a component of $\mathcal{N}_g^{g-6}$* **[D5]**. On a Prym (P, Θ), some intersections $\Theta \cap \Theta_a \cap \Theta_b$ are reducible, and in fact contained in some union $\Theta_x \cup \Theta_y$, with a, b, x, y distinct and $\neq 0$; this implies as above that *the Kummer variety $K(P, \Theta)$ admits quadrisecant planes.* It is proved in **[B-D]** that *$\mathcal{P}_g$ is a component of the locus of p.p.a.v.'s (A, Θ) such that $K(A, \Theta)$ has a quadrisecant plane* (or such that some intersection $\Theta \cap \Theta_a \cap \Theta_b$ is reducible). Here however the existence of a quadrisecant plane does not characterize Pryms: if a p.p.a.v. (A, Θ) contains an elliptic curve E such that $(\Theta . E) = 2$, there is a plane intersecting $K(A, \Theta)$ along a conic (namely, the image of E in $K(A, \Theta)$).

Contrary to the case of Jacobians, one cannot in general specialize the 4 points a, b, x, y simultaneously to 0. This becomes possible if (and only if) the curves $\tilde{C}, C$ defining the Prym are singular. One then obtains a differential equation analogous to the KP equation, called the BKP equation. One might hope that this equation characterizes Pryms of singular curves; however, because of the example quoted above (and others of the same kind), one has to impose some nondegeneracy conditions on the equation. A result of this type has been announced by Shiota.

Let me conclude with the following diagram.

	dim Sing(Θ)	reducibility of	Kummer has
Jacobians	$g-4$	$\Theta \cap \Theta_a$	trisecants
Pryms	$g-6$	$\Theta \cap \Theta_a \cap \Theta_b$	quadrisecant planes

This picture irresistibly suggests a question: what's next? Can we find a stratification $\mathcal{J}_g \subset \mathcal{P}_g \subset ?_g \subset \cdots \subset \mathcal{A}_g$, made of geometrically defined subvarieties of $\mathcal{A}_g$, fitting nicely into the above picture? At the moment this is totally unknown. One may hope that the results of Kanev on Prym-Tjurin varieties will eventually shed some light on this problem.

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