

The Fibers of the Prym Map

Ron Donagi

Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104

Introduction

§1. Pryms

§1.1 Pryms and parity

§1.2 Bilinear and quadratic forms

§1.3 The Prym maps

§2. Polygonal constructions

§2.1 The n -gonal constructions

§2.2 Orientation

§2.3 The bigonal construction

§2.4 The trigonal construction

§2.5 The tetragonal construction

§3. Bielliptic Pryms

§4. Fibers of $\mathcal{P} : \mathcal{R}_6 \rightarrow \mathcal{A}_5$

§4.1 The structure

§4.2 Special fibers

(4.6) Cubic threefolds

(4.7) Jacobians

(4.8) Quartic double solids and the branch locus of \mathcal{P}

(4.9) Boundary behavior

§5. Fibers of $\mathcal{P} : \mathcal{R}_5 \rightarrow \mathcal{A}_4$

§5.1 The general fiber

Research partially supported by NSF grant No. DMS90-08113. This paper is in final form and no version of it will be submitted for publication elsewhere.

1991 *Mathematics Subject Classification*. Primary 14H42.

© 1992 American Mathematical Society
0271-4132/92 \$1.00 + \$.25 per page

§5.2 Isotropic subgroups

§5.3 Proofs

§5.4 Special fibers

(5.11) Nodal cubic threefolds

(5.12) Degenerations in \mathcal{RC}^+ (5.13) Degenerations in \mathcal{R}^2Q^+

(5.14) Jacobians

(5.15) The boundary

(5.16) Thetanulls

(5.17) Pentagons and wheels

§6. Other genera

Introduction

The Prym map

$$\mathcal{P} : \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}$$

sends a pair $(C, \tilde{C}) \in \mathcal{R}_g$, consisting of a curve $C \in \mathcal{M}_g$ and an unramified double cover \tilde{C} , to its Prym variety

$$P = \mathcal{P}(C, \tilde{C}) := \ker^0(\text{Nm} : J(\tilde{C}) \rightarrow J(C)).$$

Prym varieties and the Prym map are central to several approaches to the Schottky problem, e.g. [B1], [D3-D5], [Deb1], [vG], [vGvdG], [I], [M2], [W]. The purpose of this work is to describe the fibers of the Prym map. When $g = 5$ or 6 , these fibers turn out to have some beautiful, and perhaps unexpected, structure. We spend much of our effort in §§4, 5 on analyzing the picture in these two cases, both generically and over some of the natural special loci in \mathcal{A}_4 and \mathcal{A}_5 . In §6 we summarize some of what is known in other genera.

Here are some of the results. When $g = 6$, the map is generically finite of degree 27 [DS]. We show that its monodromy group equals the Weyl group WE_6 , and that the general fiber has on it a structure which is equivalent to the incidence correspondence on the 27 lines on a non-singular cubic surface (Theorem (4.2)). The map fails to be finite over some of the interesting loci in \mathcal{A}_5 , such as \mathcal{J}_5 (5-dimensional Jacobians) and \mathcal{C} (intermediate Jacobians of cubic threefolds). Finiteness is restored when \mathcal{P} is compactified (§1.3) and blown up (§4.2); the resulting finite fibers can be described very explicitly ((4.6), (4.7)). A similarly explicit description of the fibers is available over the locus of intermediate Jacobians of Clemens' quartic double solid ((4.8), following [C1], [DS]). The latter is the branch locus of $\mathcal{P} : \mathcal{R}_6 \rightarrow \mathcal{A}_5$ [D6].

When $g = 5$, we show (Theorems (5.1)-(5.3)) that the fiber $\mathcal{P}^{-1}(A)$, over generic $A \in \mathcal{A}_4$, is a double cover $F(\widetilde{X})$ of the Fano surface $F(X)$ of lines on a cubic threefold X . The correspondence between $A \in \mathcal{A}_4$ and the pair $(X, \delta) \in \mathcal{RC}^+$ consisting of the cubic threefold X and the non zero, “even”, point δ of order 2 in $J(X)$, is a birational equivalence of the moduli spaces. It also turns out that \mathcal{R}_5 has an involution λ which commutes with \mathcal{P} , inducing the sheet interchange on the double cover $F(\widetilde{X})$. This λ is quite exotic; for example, it interchanges double covers of trigonal curves with “Wirtinger” double covers of nodal curves (5.14). Again, we can describe the fiber in more detail over the three distinguished divisors in $\widetilde{\mathcal{A}}_4$: Jacobians, the boundary (= degenerate abelian varieties), and the locus θ_{null} of abelian varieties with a vanishing thetanull: in all three cases, the cubic threefold becomes nodal, and the covers (C, \widetilde{C}) in the fiber can be described. A particularly pretty picture arises for $\widetilde{\mathcal{P}}^{-1}(A)$, where $A \in \mathcal{A}_4$ is the unique 4-dimensional, non-hyperelliptic PPAV with 10 vanishing thetanulls. Varley [V] showed that all Humbert curves (with their natural double covers) are in this fiber. We observe that the corresponding cubic threefold is Segre’s 10-nodal cubic (4.8); this leads quickly to a complete description of the whole fiber, (5.17).

For other values of g , the picture does not seem to be quite as rich. For $g \leq 4$, one can give a rather elementary description of the fibers using Masiewicki’s criterion [Ma] and Recillas’ trigonal construction [R]. When $g \geq 7$, the map is generically injective. ([FS], [K], [W]), but we show that it is never injective (§6).

The main tool used to analyze the Prym map is the tetragonal construction (§2.5), a triality on the locus of curves with a g_4^1 in \mathcal{R}_g , which commutes with \mathcal{P} . We exploit it consistently, together with standard facts [ACGH] on the existence of g_4^1 ’s on curves of low genus, to establish the various structures on the fibers of \mathcal{P} . In genus 5 this fits into a larger symmetry, indexed by the finite projective plane $\mathbf{P}^2(\mathbf{F}_2)$, which we describe in §5.2 and use to find the cubic threefold.

Almost all the results in this work were announced in [D1]. Since then, several preliminary manuscripts have circulated, but most of these results have not been published before. Several interesting recent developments concerning closely related questions, especially Clemens’ notes [C2] and Izadi’s thesis [I], convinced me that these ideas may still be useful, and should be published. The present work, then, provides the details for almost everything in [D1]. The main exception are the results on the Andreotti-Mayer locus, which have since appeared (in a corrected form) in [Deb1] and in [D5]. I include here only the underlying idea, which is the systematic application of the tetragonal construction to double covers of bielliptic curves (§3).

As mentioned above, several beautiful extensions of our results have recently been obtained by Clemens [C2] and Izadi [I]. Their basic idea is that the cubic threefold X associated to an abelian variety $A \in \mathcal{A}_4$ can be realized concretely inside the van Geemen-van der Geer linear system Γ_{00} on A , through use of Clemens' quartic double solids. The period map \mathcal{J} for these is analyzed in [D6], and the fiber $\mathcal{J}^{-1}(A)$ turns out to be a certain cover of the cubic threefold X . Clemens constructs a map

$$c : \mathcal{J}^{-1}(A) \rightarrow \Gamma_{00}$$

whose image is X . He conjectures, and Izadi proves, that the projective dual X^* of X can be recovered as an irreducible component of the branch locus of the rational map from A to Γ_{00}^* determined by the linear system Γ_{00} . This concrete model of X leads to several interesting applications:

- Over $A \in \partial\mathcal{A}_5$, which is a \mathbb{C}^* -extension of $A_0 \in \mathcal{A}_4$, Izadi obtains the cubic surface (of Theorem (4.2)) as hyperplane section of the cubic threefold X of A_0 . (cf. (4.9) for some more details.)
- The Abel-Prym models of the six genus-5 curves making up a $\mathbb{P}^2(\mathbb{F}_2)$ -diagram (§5.2) can be realized as the intersection $\Theta_a \cap \Theta_{-a} \cap H$ of two theta-translates with a divisor in Γ_{00}
- Izadi is able to describe precisely where our birational map $\mathcal{A}_4 \sim \mathcal{RC}^+$ fails to be an isomorphism.
- She is also able to verify some of the [vGvdG] conjectures in genus 4.

A second area of current activity is conjecture (6.5.1), which says that all non-injectivity of the Prym maps is due to the tetragonal construction. For non-hyperelliptic, non-trigonal and non-bielliptic curves of genus ≥ 13 , this was proved in [Deb2]. The generic bielliptic case, $g \geq 10$, is in [N]. Radionov [Ra] has recently proved that for $g \geq 7$ the graph of the tetragonal construction provides at least an irreducible component of the non-injectivity locus of \mathcal{P} .

Some of the results of the present work were used in [D3] and [D4] to study the Schottky-Jung loci. This leads to a proof, which I hope to publish in the near future, of the Schottky-Jung conjecture in genus 5, i.e. that the Schottky-Jung equations in genus 5 characterize Jacobians. An exciting new idea in [vGP] is the interpretation of Schottky-Jung and tetragonal-type identities via rank-2 vector bundles; we wonder whether the results on the geometry of the Prym map will also admit interpretations in terms of the geometry of the moduli space of vector bundles.

It is a pleasure to acknowledge many beneficial conversations on the

subject of Prym varieties which I have had over the years with Arnaud Beauville, Roy Smith, Robert Varley, and especially Herb Clemens, who introduced me to Prym geometry and to his double solids, and who has had a profound motivating effect on my thinking.

Notation

Moduli Spaces:

\mathcal{M}_g :	curves of genus g .
$\bar{\mathcal{M}}_g$:	the Deligne-Mumford compactification.
\mathcal{A}_g :	g -dimensional principally polarized abelian varieties (PPAV).
\mathcal{J}_g :	the closure in \mathcal{A}_g of the locus of Jacobians.
$\mathcal{Q} \subset \mathcal{M}_6$:	plane quintic curves.
$\mathcal{C} \subset \mathcal{A}_5$:	(Intermediate Jacobians of) cubic threefolds.
\mathcal{RA}_g :	pairs (A, μ) , $A \in \mathcal{A}_g$, $\mu \in A_2$ a non-zero point of order 2.
$\mathcal{R}_g, \mathcal{RQ}, \mathcal{RC}$:	the pullback of the cover $\mathcal{RA}_g \rightarrow \mathcal{A}_g$ to $\mathcal{M}_g, \mathcal{Q}, \mathcal{C}$ respectively.
$\bar{\mathcal{R}}_g$:	the Deligne-Mumford compactification of \mathcal{R}_g .
$\bar{\mathcal{R}}_g$:	the open subset of $\bar{\mathcal{R}}_g$ of Beauville-allowable double covers (§1.3).

Maps

$\mathcal{P} : \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}$:	the Prym map.
$\bar{\mathcal{P}} : \bar{\mathcal{R}}_g \rightarrow \mathcal{A}_{g-1}$:	Beauville's proper version of \mathcal{P} .
$\bar{\bar{\mathcal{P}}} : \bar{\bar{\mathcal{R}}}_g \rightarrow \bar{\mathcal{A}}_{g-1}$:	a compactification of \mathcal{P} , where $\bar{\mathcal{A}}_{g-1}$ denotes (Satake's compactification, or) an appropriate toroidal compactification.
$\Phi, \Psi, \varphi, \psi$:	canonical, Prym canonical, Abel-Jacobi and Abel-Prym maps of a curve.

We work throughout over the complex number field \mathbb{C} .

§1 Pryms.

§1.1 Pryms and parity.

Let

$$\pi : \tilde{C} \rightarrow C$$

be an unramified, irreducible double cover of a curve $C \in \mathcal{M}_g$. The genus of \tilde{C} is then $2g - 1$, and we have the Jacobians

$$J := J(C), \quad \tilde{J} := J(\tilde{C})$$

of dimensions $g, 2g - 1$ respectively, and the norm homomorphism

$$\text{Nm} : \tilde{J} \longrightarrow J.$$

Mumford shows [M2] that

$$\text{Ker}(\text{Nm}) = P \cup P^-$$

where $P = \mathcal{P}(C, \tilde{C})$ is an abelian subvariety of \tilde{J} , called the Prym variety, and P^- is its translate by a point of order 2 in \tilde{J} . The principal polarization on \tilde{J} induces twice a principal polarization on the Prym. This appears most naturally when we consider instead the norm map on line bundles of degree $2g - 2$,

$$\text{Nm} : \text{Pic}^{2g-2}(\tilde{C}) \rightarrow \text{Pic}^{2g-2}(C).$$

Let $\omega_C \in \text{Pic}^{2g-2}(C)$ be the canonical bundle of C .

Theorem 1.1 (Mumford [M1], [M2])

- (1) The two components P_0, P_1 of $\text{Nm}^{-1}(\omega_C)$ can be distinguished by their parity:

$$P_i = \{L \in \text{Nm}^{-1}(\omega_C) \mid h^0(L) \equiv i \pmod{2}\}, \quad i = 0, 1.$$

- (2) Riemann's theta divisor $\tilde{\Theta}' \subset \text{Pic}^{2g-2}(\tilde{C})$ satisfies

$$\tilde{\Theta}' \supset P_1$$

and

$$\tilde{\Theta}' \cap P_0 = 2\Xi'$$

where $\Xi' \subset P_0$ is a divisor in the principal polarization on P_0 .

§1.2 Bilinear and quadratic forms.

Let $X \in \mathcal{A}_g$ be a PPAV, and let Y be a torser (=principal homogeneous space) over X . By theta divisor in Y we mean an effective divisor whose translates in X are in the principal polarization. X acts by translation on the variety Y' of theta divisors in Y , making Y' also into an X -torser. In X' there is a distinguished divisor

$$\Theta' := \{\Theta \subset X \mid \Theta \ni 0\} \subset X'$$

which turns out to be a theta divisor, $\Theta' \in X''$. In particular, we have a natural identification $X'' \approx X$ sending Θ' to 0. Let X_2 be the subgroup of points of order 2 in X . Inversion on X induces an involution on X' ; the invariant subset X'_2 , consisting of symmetric theta divisors in X , is an X_2 -torser. Let \langle, \rangle denote the natural \mathbf{F}_2 -valued (Weil) pairing on X_2 . On X'_2 we have an \mathbf{F}_2 -valued function

sending $\Theta \in X'$ to its multiplicity at $0 \in X$, taken mod. 2.

Theorem 1.2 [M1] The function q_X is quadratic. Its associated bilinear form, on X_2 , is \langle , \rangle . When (X, Θ) vary in a family, $q_X(\Theta)$ is locally constant.

When X is a Jacobian $J = J(C)$, these objects have the following interpretations:

$$\begin{aligned} J' &\approx \text{Pic}^{g-1}(C) && \text{(use Riemann's theta divisor)} \\ J_2 &\approx \{L \in \text{Pic}^0(C) = J \mid L^2 \approx \mathcal{O}_C\} \approx H^1(C, \mathbf{F}_2) && \text{(semi periods)} \\ J'_2 &\approx \{L \in \text{Pic}^{g-1}(C) \mid L^2 \approx \omega_C\} && \text{(theta characteristics)} \\ q(L) &\equiv h^0(C, L) \pmod{2} && \text{(by Riemann-Kempf)} \end{aligned}$$

Explicitly, the theorem says in this case that for $\nu, \sigma \in J_2$ and $L \in J'_2$:

$$(1.3) \quad \langle \nu, \sigma \rangle \equiv h^0(L) + h^0(L \otimes \nu) + h^0(L \otimes \sigma) + h^0(L \otimes \nu \otimes \sigma) \pmod{2}.$$

We note that non-zero elements $\mu \in J_2$ correspond exactly to irreducible double covers $\pi : \tilde{C} \rightarrow C$. Let X be the Prym $P = \mathcal{P}(C, \tilde{C})$, which we also denote $P(C, \mu)$, $P(C, \tilde{C})$, $P(\tilde{C}/C)$ etc. Now the divisor $\Xi' \subset P_0$ of Theorem 1.1 gives a natural identification

$$P' \approx P_0 \subset \tilde{J}'.$$

The pullback

$$\pi^* : J \longrightarrow \tilde{J}$$

sends J_2 to \tilde{J}_2 . Since $\text{Nm} \circ \pi^* = 2$, we see that

$$\pi^*(J_2) \subset P_2 \cup P_2^-.$$

Let $(\mu)^\perp$ denote the subgroup of J_2 perpendicular to μ with respect to \langle , \rangle .

Theorem 1.4 [M2]

- (1) For $\tau \in J_2$, $\pi^*\tau \in P_2$ iff $\tau \in (\mu)^\perp$.
- (2) This gives an exact sequence

$$0 \rightarrow (\mu) \rightarrow (\mu)^\perp \xrightarrow{\pi^*} P_2 \rightarrow 0.$$

- (3) In (2), π^* is symplectic, i.e.

$$\langle \nu, \sigma \rangle_J = \langle \pi^*\nu, \pi^*\sigma \rangle_P, \quad \nu, \sigma \in (\mu)^\perp \subset J_2.$$

This equality of bilinear forms can be refined to an equality of quadratic functions. The identifications

$$J' \approx \text{Pic}^{g-1}(C), \quad \tilde{J}' \approx \text{Pic}^{2g-2}(\tilde{C})$$

convert the pullback

$$\pi^* : \text{Pic}^{g-1}(C) \rightarrow \text{Pic}^{2g-2}(\tilde{C})$$

into a map of torsers

$$\pi^{*'} : J' \rightarrow \tilde{J}'$$

over the group homomorphism

$$\pi^* : J \rightarrow \tilde{J}.$$

Let

$$(\mu)^{\perp'} := (\pi^{*'})^{-1}(P'_2).$$

the refinement is:

Theorem 1.5 [D4]

- (1) $(\mu)^{\perp'}$ is contained in J'_2 and is a $(\mu)^{\perp}$ -coset there.
- (2) $\pi^{*'} : (\mu)^{\perp'} \rightarrow P'_2$ is a map of torsers over $\pi^* : (\mu)^{\perp} \rightarrow P_2$.
- (3) In (2), $\pi^{*'}$ is orthogonal, i.e.

$$q_J(\nu) = q_P(\pi^{*'}\nu), \quad \nu \in (\mu)^{\perp'}.$$

§1.3 The Prym Maps.

Let \mathcal{R}_g be the moduli space of irreducible double covers $\pi : \tilde{C} \rightarrow C$ of non-singular curves $C \in \mathcal{M}_g$. Equivalently, \mathcal{R}_g parametrizes pairs (C, μ) with $\mu \in J_2(C) \setminus (0)$, a semiperiod on C . The assignment of the Prym variety to a double cover gives a morphism

$$\mathcal{P} : \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}.$$

Let ι be the involution on \tilde{C} over C . The Abel- Jacobi map

$$\varphi : \tilde{C} \rightarrow J(\tilde{C})$$

induces the Abel-Prym map

$$\psi : \tilde{C} \rightarrow \text{Ker}(\text{Nm})$$

$$x \longmapsto \varphi(x) - \varphi(\iota x).$$

The image actually lands in the wrong component, P^- , but at least ψ is well-defined up to translation (by a point of order 2). In particular, its derivative is well-defined; it factors through C , yielding the Prym-canonical map

$$\Psi : C \rightarrow \mathbf{P}^{g-2}$$

given by the complete linear system $|\omega_C \otimes \mu|$. Beauville computed the codifferential of the Prym map:

Theorem 1.6 [B1] The codifferential

$$d\mathcal{P} : T_P^* \mathcal{A}_{g-1} \rightarrow T_{(C,\mu)}^* \mathcal{R}_g$$

can be naturally identified with restriction

$$\Psi^* : S^2 H^0(\omega_C \otimes \mu) \rightarrow H^0(\omega_C^2).$$

In particular, $\text{Ker}(d\mathcal{P})$ is given by quadrics through the Prym-canonical curve $\Psi(C) \subset \mathbf{P}^{g-2}$.

Let $\bar{\mathcal{A}}_g$ denote a toroidal compactification of \mathcal{A}_g . Its boundary $\partial \bar{\mathcal{A}}_g$ maps to $\bar{\mathcal{A}}_{g-1}$, and the fiber over generic $A \in \mathcal{A}_{g-1} \subset \bar{\mathcal{A}}_{g-1}$ is the Kummer variety $K(A) := A/(\pm 1)$. In codimension 1, this picture is independent of the toroidal compactification used.

Let $\mathcal{R}\mathcal{A}_g$ denote the level moduli space parametrizing pairs (A, μ) with $A \in \mathcal{A}_g$, $\mu \in \mathcal{A}_2 \setminus (0)$, and let $\overline{\mathcal{R}\mathcal{A}}_g$ be a toroidal compactification. In [D3] we noted that its boundary has 3 irreducible components, distinguished by the relation of the vanishing cycle (mod. 2), λ , to the semiperiod μ :

$$(1.7) \quad \begin{aligned} \partial^I & : \lambda = \mu \\ \partial^{II} & : \lambda \neq \mu, \langle \lambda, \mu \rangle = 0 \in \mathbf{F}_2 \\ \partial^{III} & : \langle \lambda, \mu \rangle \neq 0. \end{aligned}$$

Let $\bar{\mathcal{M}}_g, \overline{\overline{\mathcal{R}}}_g$ denote the Deligne-Mumford stable-curve compactifications of \mathcal{M}_g and \mathcal{R}_g . At least in codimension one, the Jacobi map extends:

$$\bar{\mathcal{M}}_g \rightarrow \bar{\mathcal{A}}_g, \overline{\overline{\mathcal{R}}}_g \rightarrow \overline{\mathcal{R}\mathcal{A}}_g.$$

We use $\partial \bar{\mathcal{M}}_g$, $\partial^i \overline{\overline{\mathcal{R}}}_g$ ($i = \text{I, II, III}$) to denote the intersections of $\bar{\mathcal{M}}_g, \overline{\overline{\mathcal{R}}}_g$ with the corresponding boundary divisors in $\bar{\mathcal{A}}_g, \overline{\mathcal{R}\mathcal{A}}_g$.

In [B1], Beauville introduced the notion of an allowable double cover. This leads to the construction ([DS] I, 1.1) of a proper version of the Prym map,

$$\bar{\mathcal{P}} : \overline{\mathcal{R}}_g \rightarrow \mathcal{A}_{g-1}.$$

Roughly, one extends \mathcal{P} to

then restricts to the open subset $\bar{\mathcal{R}}_g \subset \overline{\mathcal{R}}_g$ of covers which are allowable, in the sense that their Prym is in \mathcal{A}_{g-1} . This condition can be made more explicit:

Theorem 1.8 [B1] A stable curve \tilde{C} with involution ι , quotient C , is allowable if and only if all the fixed points of ι are nodes of \tilde{C} where the branches are not exchanged, and the number of nodes exchanged under ι equals the number of irreducible components exchanged under ι .

We illustrate the possibilities in codimension 1:

Examples 1.9

- (I) $X \in \mathcal{M}_{g-1}$, $p, q \in X$, $p \neq q$; let X_0, X_1 be isomorphic copies of X . Then $C := X/(p \sim q)$ is a point of $\partial\bar{\mathcal{M}}_g$. The Wirtinger cover

$$\tilde{C} := (X_0 \amalg X_1)/(p_0 \sim q_1, p_1 \sim q_0)$$

gives a point

$$(C, \tilde{C}) \in \partial^I \bar{\mathcal{R}}_g$$

which is allowable. The Prym is

$$\mathcal{P}(C, \tilde{C}) \approx J(X) \in \mathcal{A}_{g-1}.$$

- (II) Start with $(\tilde{X} \rightarrow X) \in \mathcal{R}_{g-1}$, choose distinct points $p, q \in X$, let $p_i, q_i (i = 0, 1)$ be their inverse images in \tilde{X} , and set

$$C := X/(p \sim q), \quad \tilde{C} := \tilde{X}/(p_0 \sim q_0, p_1 \sim q_1).$$

Then

$$(C, \tilde{C}) \in \partial^{II} \bar{\mathcal{R}}_g$$

is an unallowable cover. Its Prym is a \mathbf{C}^* -extension of $\mathcal{P}(X, \tilde{X})$; the extension datum defining this extension is given by

$$\psi(p_0) - \psi(q_0) \in \mathcal{P}(X, \tilde{X}),$$

which is well defined modulo ± 1 (i.e. in the Kummer), as it should be.

- (III) X, p, q as before, but now $\tilde{X} \rightarrow X$ is a double cover branched at p, q ; consider Beauville's cover

$$C := X/(p \sim q), \quad \tilde{C} := \tilde{X}/(\tilde{p} \sim \tilde{q})$$

where \tilde{p}, \tilde{q} are the ramification points in \tilde{X} above p, q . Then $(C, \tilde{C}) \in \partial^{III} \bar{\mathcal{R}}_g$ is allowable.

In [M1], Mumford lists all covers $(C, \tilde{C}) \in \mathcal{R}_g$ whose Pryms are in the Andreotti-Mayer locus (i.e. have theta divisors singular in codimension 4). A major result in [B1] (Theorem (4.10)) is the extension of this list to allowable covers in $\bar{\mathcal{R}}_g$. We do not copy Beauville's list here, but we will refer to it when needed.

§2 Polygonal constructions

§2.1 The n -gonal constructions

Let

$$f : C \rightarrow K$$

be a map of non singular algebraic curves, of degree n , and

$$\pi : \tilde{C} \rightarrow C$$

a branched double cover. These two determine a 2^n -sheeted branched cover of K ,

$$f_*\tilde{C} \rightarrow K,$$

whose fiber over a general point $k \in K$ consists of the 2^n sections s of π over k :

$$s : f^{-1}(k) \rightarrow \pi^{-1}f^{-1}(k), \quad \pi \circ s = id.$$

The curve $f_*\tilde{C}$ can be realized, for instance, as sitting in $\text{Pic}^n(\tilde{C})$ or $S^n\tilde{C}$:

$$(2.1) \quad f_*\tilde{C} = \{D \in S^n\tilde{C} \mid \text{Nm}(D) = f^{-1}(k), \text{ some } k \in K\}.$$

(If we think of \tilde{C} as a local system on an open subset of C , this is just the direct image local system on K , hence our notation $f_*\tilde{C}$.) On $f_*\tilde{C}$ we have two structures: an involution

$$\iota : f_*\tilde{C} \rightarrow f_*\tilde{C}$$

obtained by changing all n choices in the section s via the involution (also denoted ι) of \tilde{C} , and an equivalence relation

$$f_*\tilde{C} \rightarrow \tilde{K} \rightarrow K$$

where \tilde{K} is a branched double cover of K : two sections

$$s_1, s_2 : f^{-1}(k) \rightarrow \pi^{-1}f^{-1}(k)$$

are equivalent if they differ by an even number of changes.

For even n , the involution ι respects equivalence, so we have a sequence of maps

$$(2.1.1) \quad f_*\tilde{C} \rightarrow f_*\tilde{C}/\iota \rightarrow \tilde{K} \rightarrow K$$

of degrees $2, 2^{n-2}, 2$ respectively. For odd n the equivalence classes are exchanged by ι , so we have instead a Cartesian diagram:

$$(2.1.2) \quad \begin{array}{ccc} & f_*\tilde{C} & \\ & \swarrow \quad \searrow & \\ f_*\tilde{C}/\iota & & \tilde{K} \\ & \searrow \quad \swarrow & \\ & K & \end{array}$$

Remark 2.1.3 In practice we will often want to allow C to acquire some nodes, over which π may be étale (as in (1.9 II)) or ramified (as in (1.9 III)). We will always consider this as a limiting case of the non-singular situation, and interpret the n -gonal construction in the limit so as to make it depend continuously on the parameters, whenever possible. We will see various examples of this below.

§2.2 Orientation

We observe that the branched cover $\tilde{K} \rightarrow K$ depends on $f \circ \pi : \tilde{C} \rightarrow K$, but not on f, π or C directly. More generally, to an m -sheeted branched cover

$$g : M \rightarrow K$$

we can associate an $m!$ -sheeted branched cover (the Galois closure of M)

$$g! : M! \rightarrow K,$$

with an action of the symmetric group S_m ; the quotient by the alternating group A_m gives a branched double cover

$$O(g) : O(M) \rightarrow K$$

which we call the orientation cover of M . We say M is orientable (over K) if the double cover $O(M)$ is trivial. One verifies easily that the double cover $\tilde{K} \rightarrow K$ (obtained in §2.1 from the maps $\tilde{C} \xrightarrow{\pi} C \xrightarrow{f} K$ as quotient of $f_*\tilde{C}$) is the orientation cover $O(f \circ \pi)$ of \tilde{C} .

Corollary 2.2 If \tilde{C} is orientable over K then $f_*\tilde{C} = \tilde{C}_0 \cup \tilde{C}_1$ is reducible:

- (i) For n even, the involution ι acts on each \tilde{C}_i with quotient C_i of degree 2^{n-2} over K , $i = 0, 1$.

(ii) For n odd, ι exchanges \tilde{C}_0, \tilde{C}_1 . Each \tilde{C}_i has degree 2^{n-1} over K .

Lemma 2.3 Branch $(\tilde{K}/K) = f_*(\text{Branch}(\tilde{C}/C))$.

This means: if one point of $f^{-1}(k)$ is a branch point of $\tilde{C} \rightarrow C$, then k is a branch point of $\tilde{K} \rightarrow K$; if two points of $f^{-1}(k)$ are branch points of $\tilde{C} \rightarrow C$, then k is not a branch point of (the normalization of) $\tilde{K} \rightarrow K$, but the two sheets of \tilde{K} there intersect; etc. In particular, the ramification behavior of $f : C \rightarrow K$ does not affect the ramification of \tilde{K} .

Corollary 2.4 Let $f : C \rightarrow \mathbf{P}^1$ be a branched cover, $\pi : \tilde{C} \rightarrow C$ an (unramified) double cover. Then \tilde{C} is orientable over \mathbf{P}^1 .

(More generally, the conclusion holds whenever

$$f_*(\text{Branch}(\pi)) = 2D$$

for some divisor D on \mathbf{P}^1 , since the normalization of $O(\tilde{C})$ is then an unramified double cover of the simply connected \mathbf{P}^1 , by (2.3). In this situation we say that π has cancelling ramification.)

Remark 2.5 Assume $K = \mathbf{P}^1$ and π unramified. The image of $f_*\tilde{C}$ in $\text{Pic}(\tilde{C})$ is:

$$\{L \in \text{Pic}^n(\tilde{C}) \mid \text{Nm}(L) = f^*\mathcal{O}_{\mathbf{P}^1}(1), \quad h^0(L) > 0\}.$$

This is contained in a translate of

$$\text{Nm}^{-1}(\omega_C) = P_0 \cup P_1,$$

and the splitting (2.2) of $f_*\tilde{C}$ is “explained”, in this case, by the splitting (1.1) of $\text{Ker}(\text{Nm})$, i.e. after translation:

$$\tilde{C}_i \subset P_i, \quad i = 0, 1,$$

cf. [D1, §6], [B2].

Remark 2.6 The splitting of $f_*\tilde{C}$ can also be explained group theoretically. Let WC_n be the group of signed permutations of n letters, i.e. the subgroup of S_{2n} centralizing a fixed-point-free involution of the $2n$ letters. Let WD_n be its subgroup of index 2 consisting of even signed permutations, i.e. permutations of n letters followed by an even number of sign changes. (These are the Weyl groups of the Dynkin diagrams C_n, D_n .) Over an arbitrary space X , we have equivalences:

$$\begin{aligned} \{ n\text{-sheeted cover } Y \rightarrow X \} &\longleftrightarrow \{ \text{Representation } \pi_1(X) \rightarrow S_n \} \\ \left\{ \begin{array}{l} n\text{-sheeted cover } Y \rightarrow X \\ \text{with a double cover } \tilde{Y} \rightarrow Y \end{array} \right\} &\longleftrightarrow \{ \text{Representation } \pi_1(X) \rightarrow WC_n \} \\ \left\{ \begin{array}{l} n\text{-sheeted cover } Y \rightarrow X \\ \text{with an orientable double} \\ \text{cover } \tilde{Y} \rightarrow Y \end{array} \right\} &\longleftrightarrow \{ \text{Representation } \pi_1(X) \rightarrow WD_n \} \end{aligned}$$

The basic construction of $f_*\tilde{C}$ then corresponds to the standard representation

$$\rho: WC_n \hookrightarrow S_{2^n}.$$

The existence of the involution ι on $f_*\tilde{C}$ corresponds to the factoring of ρ through

$$WC_{2^{n-1}} \subset S_{2^n}.$$

The restriction $\bar{\rho}$ of ρ to WD_n factors through

$$S_{2^{n-1}} \times S_{2^{n-1}},$$

explaining the splitting when \tilde{C} is orientable.

§2.3 The bigonal construction

The case $n = 2$ of our construction (“bigonal”) takes a pair of maps of degree 2:

$$\tilde{C} \xrightarrow{g} C \xrightarrow{f} K$$

and produces another such pair

$$f_*\tilde{C} \xrightarrow{g'} \tilde{K} \xrightarrow{f'} K.$$

Above any given point $k \in K$, the possibilities are:

- (i) If f, g are etale then so are f' and g' .
- (ii) If f is etale and g is branched at one of the two points $f^{-1}(k)$, then f' is branched at k and g' is etale there.
- (iii) Vise versa, if f is branched and g is etale then f' is etale and g' is branched at one point of $f'^{-1}(k)$.
- (iv) If both f and g are branched over k then so are f', g' .

- (v) If f is étale and g is branched at both points $f^{-1}(k)$, then \widetilde{K} will have a node over k , and $g' : f_*\widetilde{C} \rightarrow \widetilde{K}$ will be a ∂^{III} degeneration, i.e. will look like (1.9 III).
- (vi) Vice versa, we can extend the bigonal construction by continuity, as in (2.1.3), to allow $g : \widetilde{C} \rightarrow C$ to degenerate to a ∂^{III} -cover. This leads to f' which is étale and g' which is branched at both points of $f'^{-1}(k)$.

The following general properties are immediately verified:

Lemma 2.7

- (1) The bigonal construction is symmetric, i.e. if it takes $\widetilde{C} \xrightarrow{g} C \xrightarrow{f} K$ to $\widetilde{C}' \xrightarrow{g'} C' \xrightarrow{f'} K$ then it takes $\widetilde{C}' \rightarrow C' \rightarrow K$ to $\widetilde{C} \rightarrow C \rightarrow K$.
- (2) The bigonal construction exchanges branch loci:

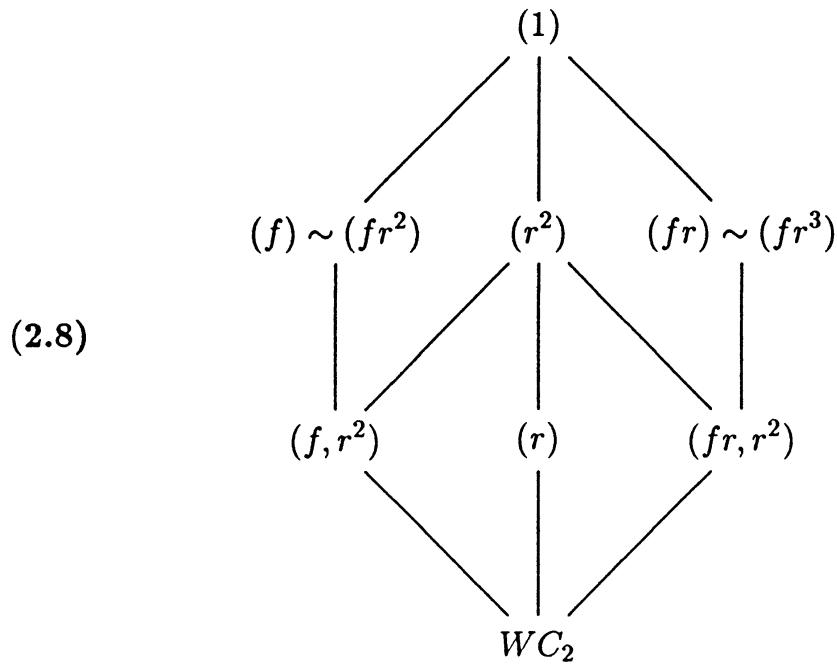
$$\text{Branch}(g') = f_*(\text{Branch}(g)), \quad \text{Branch}(f) = g'_*(\text{Branch}(f')).$$

(As in lemma (2.3), this requires the following convention in case (vi) above: the local contribution to $\text{Branch}(f)$ is $2k$, and the contribution to $\text{Branch}(g)$ is 0).

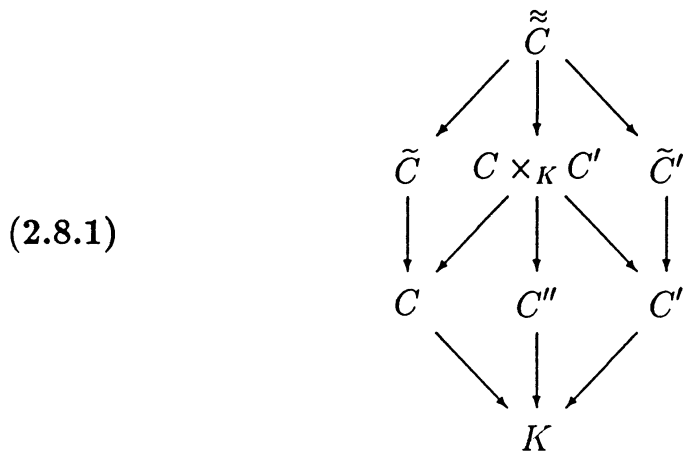
The symmetry group of this situation, WC_2 , is the dihedral group of the square:

$$WC_2 = \langle r, f \mid f^2 = r^4 = (rf)^2 = 1 \rangle.$$

($r = 90^\circ$ rotation, $f = \text{flip around } x\text{-axis}$, in the 2-dimensional representation.) It has a non-trivial outer automorphism (=conjugation by a 45° rotation), which explains why conjugacy classes of representations (of $\pi_1(X)$) in WC_2 come in (bigonally related) pairs. We list all conjugacy classes of subgroups of WC_2 in the following diagram (\sim denotes conjugate subgroups):



Correspondingly, we obtain the diagram of curves and maps of degree 2:

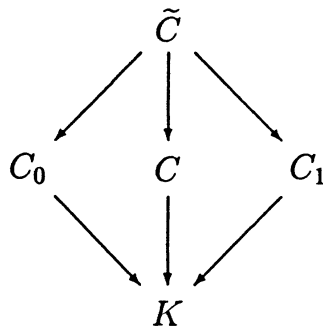


Here the two sides are bigonally related.

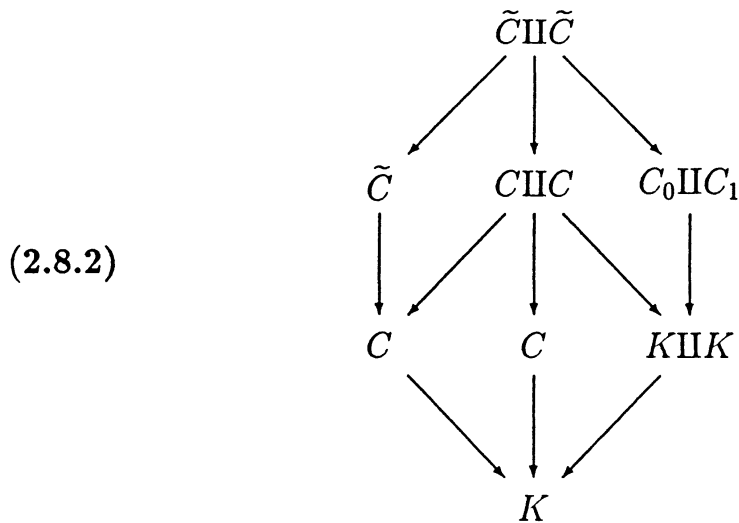
Note that C' is $O(\tilde{C})$; so if \tilde{C} is orientable (e.g. if $K = \mathbf{P}^1$ and g is unramified) then everything splits:

$$\tilde{C}' = C_0 \amalg C_1 \rightarrow K \amalg K = C',$$

\tilde{C} is Galois over K with group $(\mathbf{Z}/2\mathbf{Z})^2$ and quotients



(cf. [M1]), and (2.8.1) simplifies to:



Given an arbitrary branched double cover $\tilde{C} \rightarrow C$, we form its Prym variety

$$P(\tilde{C}/C) := \text{Ker}^0(\text{Nm} : J(\tilde{C}) \rightarrow J(C)).$$

It is an abelian variety (for C, \tilde{C} non-singular), but in general not a principally polarized one. Nevertheless, there is a simple relationship between the bigonally-related Pryms $P(\tilde{C}/C)$ and $P(\tilde{C}'/C')$: in the case $K = \mathbf{P}^1$, Pantazis [P] showed that these abelian varieties are dual to each other.

§2.4 The trigonal construction.

The case $n = 3$ of our construction was discovered by Recillas [R]. Start with a tower

$$\tilde{C} \xrightarrow{\pi} C \xrightarrow{f} \mathbf{P}^1$$

where f has degree 3, and $\tilde{C} \rightarrow C$ is an unramified double cover. By Corollaries (2.4) and (2.2), $f_*\tilde{C}$ consists of two copies of a tetragonal

curve $g : X \rightarrow \mathbf{P}^1$. Since f and g have the same branch locus by Lemma (2.3), we find from Hurwitz' formula:

$$\text{genus}(X) = \text{genus}(C) - 1.$$

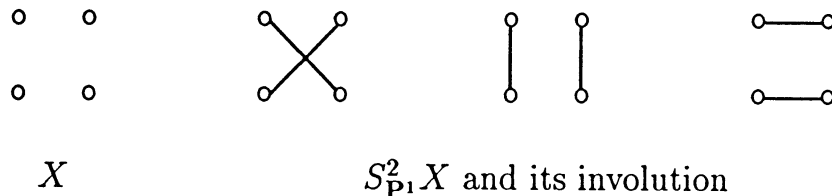
All in all, we have constructed a map:

$$T : \left\{ \begin{array}{l} \text{trigonal curves } C \text{ of} \\ \text{genus } g \text{ with a double cover } \tilde{C} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{tetragonal curves} \\ X \text{ of genus } g - 1 \end{array} \right\}.$$

We claim that this map is a bijection (except that sometimes a nonsingular object on one side may correspond to a singular one on the other): given $g : X \rightarrow \mathbf{P}^1$, we recover \tilde{C} as the relative second symmetric product of X over \mathbf{P}^1 ,

$$\tilde{C} := S_{\mathbf{P}^1}^2 X \rightarrow \mathbf{P}^1,$$

whose fiber over $p \in \mathbf{P}^1$ consists of all unordered pairs in $g^{-1}(p)$; this has an involution ι (=complementation of pairs), giving the quotient $C := \tilde{C}/\iota$.

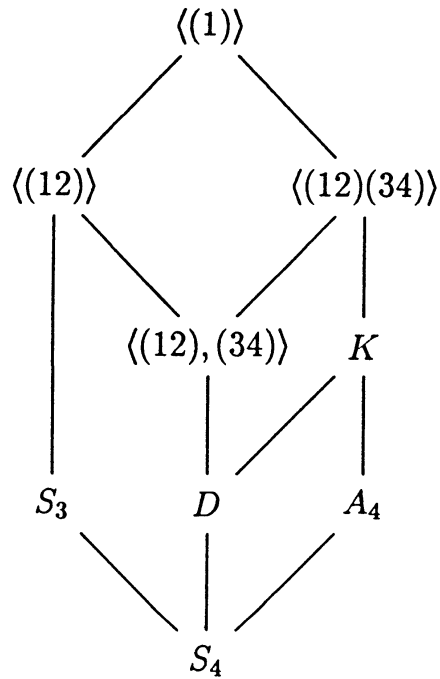


In the group-theoretic setup of Remark (2.6), $\bar{\rho}$ induces an isomorphism

$$WD_3 \xrightarrow{\sim} S_4.$$

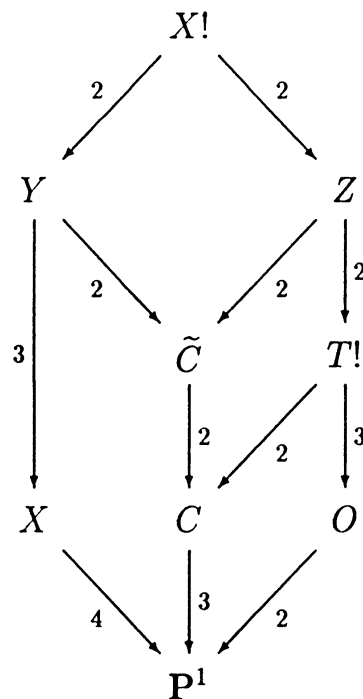
(This is the standard isomorphism, reflecting the isomorphism of the Dynkin diagrams D_3, A_3 .) Recillas' map T then corresponds to composition of a representation with this isomorphism.

We list a few of the subgroups of S_4 :



D : The dihedral group $\langle\langle 12 \rangle, (1324) \rangle$
 $K = D \cap A_4$: The Klein group $\langle\langle 12 \rangle(34), (13)(24) \rangle$.

The corresponding curves are:



$O \approx O(X) \approx O(C)$: The orientation
 $Y \approx (X \times_{P^1} X) \setminus (\text{diagonal})$
 $Z \approx \tilde{C} \times_{P^1} O$.

Using either of these constructions, we can easily describe the behavior of X, C, \tilde{C} around various types of branch points. Keeping X non-singular, there are the following five possible local pictures, cf. [DS, III 1.4].

	(i)	(ii)	(iii)	(iv)	(v)
X					
\tilde{C}					
C					

Legend

- unramified sheet
- simple ramification
- node (two unramified sheets glued together)
- two ramified sheets glued together
- ramification point of index 2
- ramification point of index 3
- glueing of two sheets of different ramification indices

- (i) f, π, g are étale.
- (ii) f and g have simple ramification points, π is étale.
- (iii) f and g each have a ramification point of index 2, π is étale.
- (iv) g has two simple ramification points, π is a Beauville cover:
 $\bar{f} : N \rightarrow \mathbb{P}^1$ is trigonal with a fiber $\{p, q, r\}$; $\bar{\pi} : \tilde{N} \rightarrow N$ is

branched at $p, q : \bar{\pi}^{-1}(p) = \tilde{p}, \bar{\pi}^{-1}(q) = \tilde{q}$; and we have $C = N/(p \sim q), \tilde{C} = \tilde{N}/(\tilde{p} \sim \tilde{q}),$ and $\pi : \tilde{C} \rightarrow C, f : C \rightarrow \mathbf{P}^1$ are induced by $\bar{\pi}, \bar{f}$.

- (v) g has a ramification point of index 3, π is Beauville, f is ramified at one of the two branches of the node of C .

Considering first the first three cases, then all five, we conclude:

Theorem 2.9 The trigonal construction gives isomorphisms

$$T^0 : \mathcal{R}_g^{\text{Trig}} \xrightarrow{\sim} \mathcal{M}_{g-1}^{\text{Tet},0}$$

and

$$T : \bar{\mathcal{R}}_g^{\text{Trig}} \xrightarrow{\sim} \mathcal{M}_{g-1}^{\text{Tet}},$$

where:

$\mathcal{M}_{g-1}^{\text{Tet}}$ is the moduli space of (non-singular) curves of genus $g - 1$ with a tetragonal line bundle.

$\mathcal{M}_{g-1}^{\text{Tet},0}$ is the open subset of tetragonal curves X with the property that above each point of \mathbf{P}^1 there is at least one etale point of X .

$\mathcal{R}_g^{\text{Trig}}$ is the moduli space of etale double covers of non-singular curves of genus g with a trigonal bundle.

$\bar{\mathcal{R}}_g^{\text{Trig}}$ is the partial compactification of $\mathcal{R}_g^{\text{Trig}}$ using allowable covers in $\bar{\mathcal{R}}_g$ of type ∂^{III} (cf (1.9.III)).

Examples 2.10

- (i) \tilde{C} is the trivial cover, $\tilde{C} = C_0 \amalg C_1,$ iff X is disconnected, $X = \mathbf{P}^1 \amalg C,$ with $f = g|_C, id_{\mathbf{P}^1} = g|_{\mathbf{P}^1}.$
- (ii) Wirtinger covers $(C_0 \amalg C_1) / (p_0 \sim q_1, q_0 \sim p_1) \rightarrow C/(p \sim q),$ where $\{p, q, r\}$ form a trigonal fiber in $C,$ correspond to reducible $X = \mathbf{P}^1 \cup_r C,$ the two components meeting at $r \in C.$
- (iii) C is reducible: $C = H \cup \mathbf{P}^1,$ with H hyperelliptic, and $\tilde{C} = \tilde{H} \cup \mathbf{P}^1$ with $\tilde{H} \rightarrow H$ and $\tilde{\mathbf{P}}^1 \rightarrow \mathbf{P}^1$ branched over $B := H \cap \mathbf{P}^1.$ This situation corresponds to $g : X \rightarrow \mathbf{P}^1$ factoring through a hyperelliptic $H'.$ Indeed, such a pair (C, \tilde{C}) is uniquely determined by the tower $\tilde{H} \rightarrow H \rightarrow \mathbf{P}^1.$ The trigonal construction for C is reduced to the bigonal construction for $H,$ which then gives $X = \tilde{H}' \rightarrow H' \rightarrow \mathbf{P}^1.$ In particular:

(iv) $C = H \amalg \mathbf{P}^1$ is disconnected iff $X = H_0 \amalg H_1$ is disconnected with hyperelliptic pieces, and then $\tilde{C} = \tilde{H} \amalg \mathbf{P}^1 \amalg \mathbf{P}^1$, where \tilde{H} is the Cartesian cover:

$$\tilde{H} = H_0 \times_{\mathbf{P}^1} H_1.$$

So far, we have only used the fact that \tilde{C} is an orientable double cover of a triple cover. We now use our two assumptions, that π is unramified and that the base K equals \mathbf{P}^1 , to obtain an identity of abelian varieties. Namely, by Remark 2.5 we have a map, natural up to translation.

$$\alpha : X \rightarrow P(\tilde{C}/C).$$

The result, due to S. Recillas, is:

Theorem 2.11 [R] If X is trigonally related to (\tilde{C}, C) , then the above map α induces an isomorphism

$$\alpha_* : J(X) \xrightarrow{\sim} P(\tilde{C}/C).$$

Proof.

By naturality of α and irreducibility of $\mathcal{M}_{g-1}^{\text{Tet}}$, it suffices to prove this for any one convenient X . We take $\tilde{C} \rightarrow C$ to be a Wirtinger cover as in (2.10)(ii), so

$$X = \mathbf{P}^1 \cup_r C'.$$

where $p + q + r$ is a trigonal divisor on C' , and $C = C'/(p \sim q)$. We have natural identifications:

$$J(X) \approx J(C') \approx P(\tilde{C}/C),$$

in terms of which α becomes the Abel-Jacobi map φ on C' , and collapses \mathbf{P}^1 to a point. The induced α_* is therefore the identity.

QED

Corollary 2.12 All trigonal Pryms are Jacobians, and all tetragonal Jacobians are Pryms.

§2.5 The tetragonal construction

Consider now a tower

$$\tilde{C} \rightarrow C \xrightarrow{f} \mathbf{P}^1$$

where f has degree 4 and \tilde{C} is a double cover (unramified) of C . The general construction yields a sequence of maps of degrees 2, 4, 2:

By (2.2) and (2.4) again, $\tilde{\mathbf{P}}^1$ is unramified, hence we have splittings:

$$\begin{aligned}\tilde{\mathbf{P}}^1 &= \mathbf{P}_0^1 \amalg \mathbf{P}_1^1 \\ f_*\tilde{C} &= \tilde{C}_0 \amalg \tilde{C}_1 \\ f_*\tilde{C}/\iota &= C_0 \amalg C_1.\end{aligned}$$

The tetragonal construction thus associates to a tower

$$\tilde{C} \xrightarrow{2} C \xrightarrow{4} \mathbf{P}^1$$

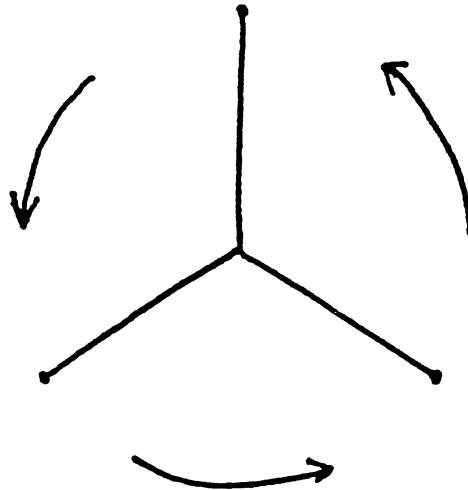
two other towers

$$\tilde{C}_i \rightarrow C_i \rightarrow \mathbf{P}^1, \quad i = 0, 1$$

of the same type.

Lemma 2.13 The tetragonal construction is a triality, i.e. starting with $\tilde{C}_0 \rightarrow C_0 \rightarrow \mathbf{P}^1$ it returns $\tilde{C} \rightarrow C \rightarrow \mathbf{P}^1$ and $\tilde{C}_1 \rightarrow C_1 \rightarrow \mathbf{P}^1$.

On the group level, the point is this: Our tower $\tilde{C} \rightarrow C \rightarrow \mathbf{P}^1$ corresponds to a representation (of $\pi_1(\mathbf{P}^1 \setminus (\text{branch locus}))$) in WD_4 . Now the Dynkin diagram D_4 has an automorphism of order 3:

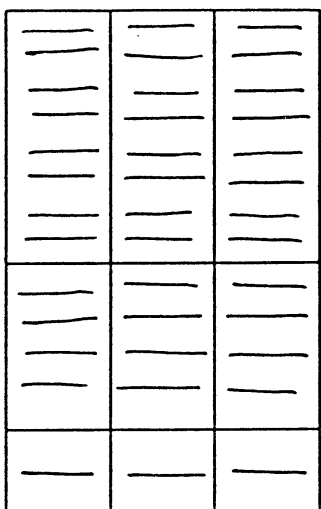


This corresponds to an outer automorphism of WD_4 , of order 3. Hence representations in WD_4 come in packets of three. The various groups involved are described in some detail in the proof of Lemma (5.5), below.

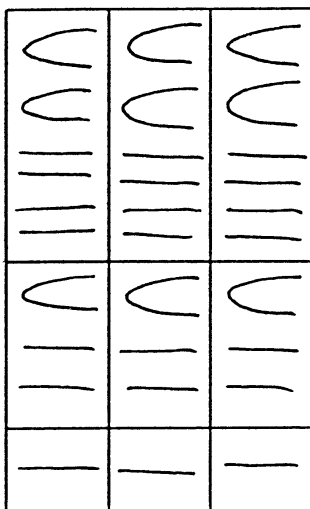
Local pictures 2.14 Given the local behavior of C and \tilde{C} over a point of \mathbf{P}^1 , it is quite straightforward to compute $f_*\tilde{C}$ and hence \tilde{C}_i, C_i ($i = 0, 1$) over the same point. Since these local pictures are needed quite frequently, we record the simplest ones here.

- (1) C, \tilde{C} unramified $\Rightarrow C_i, \tilde{C}_i$ are also unramified.
- (2) C has one simple ramification point (and two unramified sheets), $\tilde{C} \rightarrow C$ unramified $\Rightarrow C_i, \tilde{C}_i$ have the same local picture as C, \tilde{C} respectively.
- (3) C has two distinct simple ramification points, $\tilde{C} \rightarrow C$ unramified \Rightarrow One pair, say C_0, \tilde{C}_0 , has the same local pictures as C, \tilde{C} , while the other is a Beauville degeneration: C_1 is unramified but two of its four sheets are glued, $\tilde{C}_1 \rightarrow C_1$ is ramified over these two sheets (and the ramification points are glued) while the other sheets are unramified.
- (4) C is unramified but two of its sheets are glued, $\tilde{C} \rightarrow C$ is ramified over these two sheets $\Rightarrow C_i$ has two distinct ramification points, $\tilde{C}_i \rightarrow C_i$ is unramified ($i = 0, 1$). (This is the same triple as in (3).)
- (5) C has a simple ramification point and the other two sheets are glued, \tilde{C} is ramified over the glued sheets $\Rightarrow C_i, \tilde{C}_i$ have the same local pictures as C, \tilde{C} .
- (6) C has a ramification point of index 2 (i.e. 3 of its sheets are permuted by the local monodromy), $\tilde{C} \rightarrow C$ unramified \Rightarrow same local picture for $\tilde{C}_i \rightarrow C_i$.
- (7) C has a ramification point of index 3 (all 4 sheets permuted), $\tilde{C} \rightarrow C$ unramified $\Rightarrow C_0, \tilde{C}_0$ have the same local picture as C, \tilde{C} , but C_1 has a simple ramification point glued to an unramified point, so \tilde{C}_1 must be simply ramified over each. (I.e. \tilde{C}_1 has a point which is simply ramified over \mathbf{P}^1 , glued to a point which has ramification index 3 over \mathbf{P}^1 !)

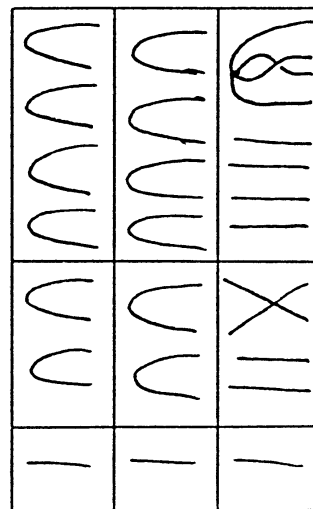
We note that in examples (3) and (7), the tetragonal construction applied to $(\tilde{C} \rightarrow C) \in \mathcal{RM}_g$ produces an (allowable) degenerate cover, $(\tilde{C}_1 \rightarrow C_1) \in \partial^{\text{III}}(\mathcal{RM}_g)$.



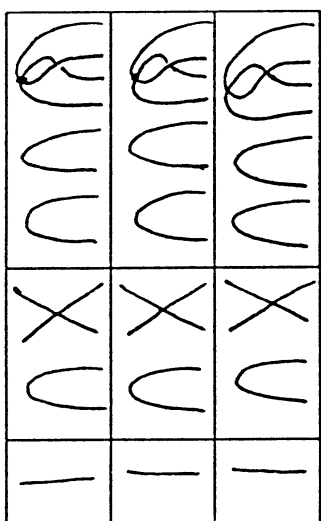
(1)



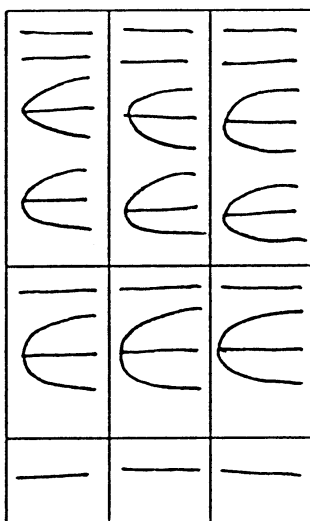
(2)



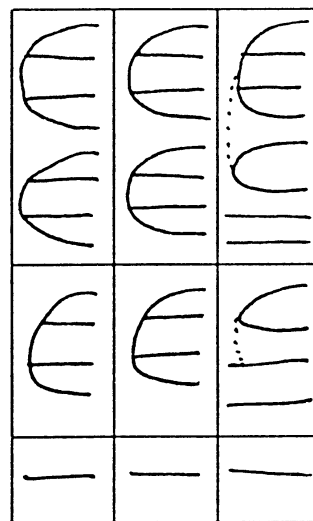
(3,4)



(5)



(6)



(7)

\tilde{C}	\tilde{C}_0	\tilde{C}_1
C	C_0	C_1
P^1	P^1	P^1

(pattern)

Examples 2.15

- (1) It is perhaps not terribly surprising that the trigonal construction is a degenerate case of the tetragonal construction. Start with $\tilde{C} \rightarrow C$ the split double cover of the curve C with the tetragonal map $f : C \xrightarrow{4} \mathbf{P}^1$. Then $f_*\tilde{C}$ splits into 5 components, of degrees 1, 4, 6, 4, 1 respectively over \mathbf{P}^1 . The components of degree 4 make up $\tilde{C}_1 \rightarrow C_1$, which is isomorphic to $\tilde{C} \rightarrow C$. The remaining components give

$$\mathbf{P}^1 \amalg \tilde{T} \amalg \mathbf{P}^1 \rightarrow T \amalg \mathbf{P}^1$$

where (\tilde{T}, T) is associated to C by the trigonal construction. Vice versa, starting with an (unramified) double cover

$\mathbf{P}^1 \amalg \tilde{T} \amalg \mathbf{P}^1$ of $T \amalg \mathbf{P}^1$, the tetragonal construction produces $C \amalg C \rightarrow C$, twice.

- (2) Let $p + q + r + s$ be a tetragonal divisor on C . Then $C/(p \sim q)$ is still tetragonal. Tacking a node onto the previous example, we see that the Wirtinger cover

$$(C' \amalg C'')/(p' \sim q', q' \sim p'') \rightarrow C/(p \sim q)$$

is taken by the tetragonal construction to :

- Another Wirtinger Cover,

$$(C' \amalg C'')/(r' \sim s', s' \sim r'') \rightarrow C/(r \sim s),$$

and to:

- $\mathbf{P}^1 \cup_t \tilde{T} \cup_{t''} \mathbf{P}^1 \rightarrow T \cup_t \mathbf{P}^1$, where (\tilde{T}, T) is associated by the trigonal construction to C . (Each copy of \mathbf{P}^1 meets \tilde{T} or T in the unique point indicated. $t \in T$ corresponds to the partition $\{\{p, q\}, \{r, s\}\}$.)

- (3) We will see in Lemma (3.5) that if $C \rightarrow \mathbf{P}^1$ factors through a hyperelliptic curve, so do C_0, C_1 . An interesting subcase occurs when $C = H^0 \cup H^1$ has two hyperelliptic components, cf. Proposition (3.6).

- (4) Let X be a non-singular cubic hypersurface in \mathbf{P}^4 , $\ell \subset X$ a line, and \tilde{X} the blowup of X along ℓ , with projection from ℓ :

$$\pi : \tilde{X} \rightarrow \mathbf{P}^2.$$

This is a conic bundle [CG] whose discriminant is a plane quintic curve $Q \subset \mathbf{P}^2$. The set of lines $\ell' \subset X$ meeting ℓ is a double cover \tilde{Q} of Q . Now choose a plane $A \subset \mathbf{P}^4$ meeting X in 3 lines ℓ, ℓ', ℓ'' ;

we get 3 plane quinties Q, Q', Q'' , with double covers $\tilde{Q}, \tilde{Q}', \tilde{Q}''$. Note that ℓ', ℓ'' map to a point $p \in Q$, hence determine a tetragonal map $f : Q \rightarrow \mathbf{P}^1$, given by $\mathcal{O}_Q(1)(-p)$, and similarly for Q', Q'' . Our observation is that the 3 objects

$$(\tilde{Q}, Q, f) ; (\tilde{Q}', Q', f') ; (\tilde{Q}'', Q'', f'')$$

are tetragonally related. Indeed, the 3 maps can be realized simultaneously via the pencil of hyperplanes $S \subset \mathbf{P}^4$ containing A . Such an S meets X in a (generally non-singular) cubic surface Y . A line in Y (and not in A) which meets ℓ' , also meets 4 of the 8 lines (in Y , not in A) meeting ℓ , one in each of 4 coplanar pairs. this gives the desired injection $\tilde{Q}' \hookrightarrow f_*\tilde{Q}$.

Our main interest in the tetragonal construction stems from:

Theorem 2.16 The tetragonal construction commutes with the Prym map,

$$P(\tilde{C}/C) \approx P(\tilde{C}_0/C_0) \approx P(\tilde{C}_1/C_1).$$

Proof

As in Remark (2.5), we have a map

$$\alpha : \tilde{C}_i \hookrightarrow f_*\tilde{C} \rightarrow \text{Pic}(\tilde{C}), \quad i = 0, 1.$$

The image sits in a translate of $P(\tilde{C}/C)$, so we get induced maps

$$\alpha_* : J(\tilde{C}_i) \rightarrow P(\tilde{C}/C)$$

and by restriction

$$\beta : P(\tilde{C}_i/C_i) \rightarrow P(\tilde{C}/C).$$

By Masiewicki's criterion [Ma], β will be an isomorphism if we can show:

- (1) The image $\alpha(\tilde{C}_i)$ of \tilde{C}_i in $P(\tilde{C}/C)$ is symmetric;
- (2) The fundamental class in $P(\tilde{C}/C)$ of $\alpha(\tilde{C}_i)$ is twice the minimal class, $\frac{2}{(g-1)!}[\Theta]^{g-1}$.

Now (1) is clear, since the involution on \tilde{C}_i commutes with -1 in $P(\tilde{C}/C)$. The fundamental class in (2) can be computed directly, as is done in [B2]. Instead, we find it here by a degeneration argument: it varies continuously with $(C, \tilde{C}) \in \mathcal{RM}_g^{\text{Tet}}$, which is an irreducible

parameter space, so it suffices to do the computation for a single (C, \tilde{C}) . We take

$$C = T \cup_t \mathbf{P}^1, \quad \tilde{C} = \mathbf{P}^1 \cup_{t'} \tilde{T} \cup_{t''} \mathbf{P}^1,$$

as in Example (2.15)(2). Then (C_i, \tilde{C}_i) is a Wirtinger cover, $i = 0, 1$, and the normalization of C_i is the tetragonal curve N associated to (T, \tilde{T}) by the trigonal construction. We have identifications

$$J(N) \approx P(\tilde{T}/T) \approx P(\tilde{C}/C)$$

(Theorem (2.11)), in terms of which $\alpha(\tilde{C}_i)$ consists of the Abel-Jacobi image $\varphi(N) \subset J(N)$ and of its image under the involution. Thus the fundamental class is twice that of $\varphi(N)$, as required.

(Note: since this argument works for any double cover $\tilde{T} \rightarrow T$, and since any semiperiod on a nearby non-singular C specializes to a semiperiod on $T \cup_t \mathbf{P}^1$ which is supported on T , we need only the irreducibility of $\mathcal{M}_g^{\text{Tet}}$, instead of $\mathcal{RM}_g^{\text{Tet}}$.)

QED

§3 Bielliptic Pryms.

As a first application of the tetragonal construction, we show that some remarkable coincidences occur among the various loci in Beauville's list [B1]. The central role here is played by Pryms of bielliptic curves. We see in (3.7), (3.8) that the bielliptic loci can be tetragonally related to various other components in Beauville's list, and therefore give the same Pryms. As suggested in [D1], this leads to a complete, short list of the irreducible components of the Andreotti-Mayer locus in genus ≤ 5 , and of its intersection with the image of the proper Prym map for arbitrary g . We do not include here the complete analysis of the Andreotti-Mayer locus itself, since this has already appeared in [Deb1] and [D5] (together with some corrections to the original list in [D1]). Nevertheless, we could not resist describing explicitly the operation of the tetragonal construction on Beauville's list, as it is such a pretty and straightforward application of the results of §2.

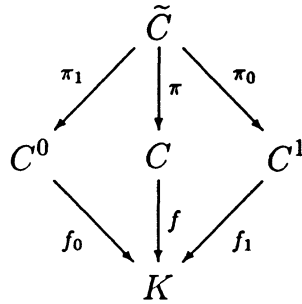
We recall Mumford's results on hyperelliptic Pryms. Let

$$f_i : C^i \rightarrow K, \quad i = 0, 1$$

be two ramified double covers of a curve K . The fiber product

$$\tilde{C} := C^0 \times_K C^1$$

has 3 natural involutions: $\tau_i (i = 0, 1)$, with quotient C^i , and $\tau := \tau_0 \circ \tau_1$, with a new quotient, C . This all fits in a Cartesian diagram:



If the branch loci of f_0, f_1 are disjoint, then

$$\pi : \tilde{C} \rightarrow C$$

is unramified. We say that a double cover obtained this way is Cartesian.

Lemma 3.1 Let $f : C \rightarrow K$ be a ramified double cover. A double cover

$$\pi : \tilde{C} \rightarrow C,$$

given by a semiperiod $\eta \in J_2(C)$, is Cartesian if and only if $f_*(\eta) = 0 \in J_2(K)$.

Proof: apply the bigonal construction.

QED

Proposition 3.2 [M1]

- (1) Any double cover \tilde{C} of a hyperelliptic C is Cartesian.
- (2) Any hyperelliptic Prym is a product of 2 hyperelliptic Jacobians (one of which may vanish): If \tilde{C} arises as $C^0 \times_{\mathbf{P}^1} C^1$ then

$$P(\tilde{C}/C) \approx J(C^0) \times J(C^1).$$

Proof: (2) follows from (1), (1) follows from lemma (3.1) with $K = \mathbf{P}^1$.

QED

A bielliptic curve (aka elliptic-hyperelliptic, superelliptic, ...) is a branched double cover of an elliptic curve. In this section we apply the tetragonal construction to find various identities between bielliptic Pryms and Pryms of other, usually degenerate, curves. Some of the results extend to bihyperelliptic curves, i.e. branched double covers of hyperelliptic curves. To warm up, we consider Jacobians of bihyperelliptic curves. Example (2.10)(iii) can be restated:

Lemma 3.3 The trigonal construction gives a bijection between:

- Bihyperelliptic, non singular curves C :

$$C \xrightarrow{f} H \xrightarrow{g} \mathbf{P}^1;$$

- Reducible trigonal double covers $\tilde{X} \rightarrow X$:

$$\begin{array}{ccccc} \tilde{X} & = & C' & \cup & H \\ \downarrow & & \downarrow & & \downarrow \\ X & = & H' & \cup & \mathbf{P}^1 \end{array}$$

where

$X = H' \cup \mathbf{P}^1$ is reducible

$\tau : X \rightarrow \mathbf{P}^1$, the trigonal map, has degree 2 on H' and 1 on \mathbf{P}^1 .

$\tau(H' \cap \mathbf{P}^1) = \text{Branch}(g)$

$\tilde{X} \rightarrow X$ is allowable of type ∂^{III} at each point of $H' \cap \mathbf{P}^1$.

We note that $C' \rightarrow H' \rightarrow \mathbf{P}^1$ is bigonally related to $C \rightarrow H \rightarrow \mathbf{P}^1$.

Corollary 3.4 The Jacobian of a bihyperelliptic curve C ,

$$C \xrightarrow{f} H \xrightarrow{g} \mathbf{P}^1,$$

is isogenous to the product

$$J(H) \times P(g_*C, \iota)$$

of a hyperelliptic Jacobian and a bihyperelliptic (branched) Prym.

We move to the Pryms of bihyperelliptic curves. First we note that this class is closed under the tetragonal construction:

Lemma 3.5 Let (\tilde{C}_i, C_i) be tetragonally related to (\tilde{C}, C) , with C non-singular. If $C \rightarrow \mathbf{P}^1$ factors through a (possibly reducible) hyperelliptic H , so do the C_i :

$$C_i \xrightarrow{f_i} H_i \xrightarrow{g_i} \mathbf{P}^1, \quad i = 0, 1.$$

Proof.

The bigonal construction applied to

$$\tilde{C} \xrightarrow{\pi} C \xrightarrow{f} H$$

yields

$$f_*\tilde{C} \rightarrow \tilde{H} \rightarrow H,$$

and when applied again to

$$\tilde{H} \rightarrow H \xrightarrow{g} \mathbf{P}^1$$

yields

$$g_*\tilde{H} \rightarrow \tilde{\mathbf{P}}^1 \rightarrow \mathbf{P}^1.$$

Since π is unramified, so are $\tilde{H} \rightarrow H$ and $\tilde{\mathbf{P}}^1 \rightarrow \mathbf{P}^1$. Hence $\tilde{\mathbf{P}}^1$ splits:

$$\tilde{\mathbf{P}}^1 = \mathbf{P}_0^1 \amalg \mathbf{P}_1^1,$$

and this splitting climbs its way up the tower:

$$\begin{aligned} (g \circ f)_*\tilde{C} &= \tilde{C}_0 \amalg \tilde{C}_1 \\ &\downarrow \\ &C_0 \amalg C_1 \\ &\downarrow \\ g_*\tilde{H} &= H_0 \amalg H_1 \\ &\downarrow \\ \tilde{\mathbf{P}}^1 &= \mathbf{P}_0^1 \amalg \mathbf{P}_1^1 \\ &\downarrow \\ &\mathbf{P}^1 \end{aligned}$$

QED

Remark 3.5.1 The rational map $f_i : C_i \rightarrow H_i$ can, in a couple of cases, fail to be a morphism; this is easily remedied by identifying a pair of points in H_i . Among the local pictures (2.14), the ones that can occur here are (1), (2), (7) and (3) :

- In cases (1), (2), the hyperelliptic maps g, g_0, g_1 are all unramified

- In case (7), g and g_0 are ramified, g_1 is not, f and f_0 are (ramified) morphisms, but f_1 is not, since C_1 is singular above a point where H_1 , as constructed above, is nonsingular. To make f_1 into a morphism, we must glue the two points of $g_1^{-1}(k)$.
- In case (3) we find two possibilities:
 - (3a) g is etale, f is ramified at both points of $g^{-1}(k)$; then g_0, g_1 are also etale, f_0 is ramified at both points of $g_0^{-1}(k)$, C_1 has a node but f_1 is still a morphism.
 - (3b) g is ramified, f is etale; then g_0 is ramified, f_0 is etale, g_1 is etale, but the two branches of the node of C_1 are sent by f_1 to opposite sheets of H_1 , so f_1 is again not a morphism.

Proposition 3.6 Let $\tilde{C} \rightarrow C$ be a Cartesian double cover of a bihyperelliptic C :

$$C \xrightarrow{f} H \xrightarrow{g} \mathbf{P}^1, \quad \tilde{C} = C^0 \times_H C^1.$$

The tetragonal construction applied to $\tilde{C} \rightarrow C \rightarrow \mathbf{P}^1$ yields:

- A similar Cartesian tower $\tilde{C}_0 \rightarrow C_0 \xrightarrow{f_0} H \xrightarrow{g_0} \mathbf{P}^1$, same H .
- A tower $\tilde{C}_1 \rightarrow C_1 \rightarrow \mathbf{P}^1$ where:

$$\begin{aligned} C_1 &\text{ is reducible, } C_1 = H^0 \cup H^1, \\ H^0, H^1 &\text{ are hyperelliptic,} \\ H^0 \cap H^1 &\text{ maps onto } B := \text{Branch}(g) \subset \mathbf{P}^1, \\ \tilde{C}_1 = \tilde{H}^0 \cup \tilde{H}^1 &\text{ is allowable over } C_1, \\ C^i \rightarrow H \rightarrow \mathbf{P}^1 &\text{ is bigonally related to } \tilde{H}^i \rightarrow H^i \rightarrow \mathbf{P}^1, \quad i = 1, 2. \end{aligned}$$

Vice versa, the tetragonal construction takes any tower $\tilde{C}_1 \rightarrow C_1 \rightarrow \mathbf{P}^1$ as above to two Cartesian bihyperelliptic towers

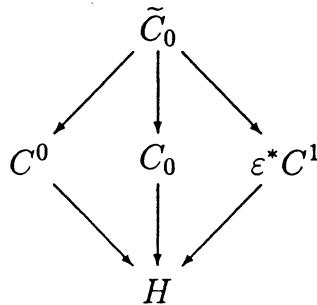
$$\tilde{C} \rightarrow C \rightarrow H \rightarrow \mathbf{P}^1 \quad \text{and} \quad \tilde{C}_0 \rightarrow C_0 \rightarrow H \rightarrow \mathbf{P}^1.$$

The proof is quite straightforward, and we will simply write down a few of the relationships involved, using the notation of the previous proof:

- \tilde{H} splits into two copies of H , by (3.1). Hence:
- $g_*\tilde{H} \approx H \cup \mathbf{P}^1 \cup \mathbf{P}^1$, say $H_0 \approx H$, $H_1 = R^0 \cup R^1$, $R^i \approx \mathbf{P}^1$, $i = 0, 1$.
- Let H^i, \tilde{H}^i be the inverse image of R^i in C_1, \tilde{C}_1 respectively. Then $\tilde{H}^i \rightarrow H^i \rightarrow \mathbf{P}^1$ is bigonally related to $C^i \rightarrow H \rightarrow \mathbf{P}^1$.
- The intersection properties of the H^i (or \tilde{H}^i) can be read off the

local pictures (2.14.3).

- Finally, let $\varepsilon : H \rightarrow H$ be the hyperelliptic involution. A cover $C^1 \rightarrow H$ determines a mirror-image ε^*C^1 . The remaining tower $\tilde{C}_0 \rightarrow C_0 \rightarrow H \rightarrow \mathbf{P}^1$ is given by the Cartesian diagram:



QED

Remarks

(3.6.1) Since the branch points of $C^i \rightarrow H$ map to the branch points of $H^i \rightarrow \mathbf{P}^1$, we have the relation between the genera:

$$g(H^i) = g(C^i) - 2 \cdot g(H).$$

(3.6.2) The possible local pictures are exactly the same as in (3.5.1). (The use of C_0, C_1 in (3.6) is consistent with that of (2.14).)

(3.6.3) Another way of proving both lemma (3.5) and proposition (3.6) is based on lemma (5.5), which says that the three tetragonal curves C, C_0, C_1 which are tetragonally related are obtained, via the trigonal construction, from one and the same trigonal curve X (with three distinct double covers). Lemma (3.3) characterizes the possible curves X , hence proves that the locus of bihyperelliptics is closed under the tetragonal construction, lemma (3.5). To complete the proof of proposition (3.6), one simply needs to characterize the double covers \tilde{X} which correspond to Cartesian covers of C .

For the rest of this section, we specialize to the case where the hyperelliptic H is an elliptic curve E , i.e. C is bielliptic. First, we write out explicitly the content of Proposition (3.6) in this case:

Corollary 3.7 The Pryms of double covers $\pi : \tilde{C} \rightarrow C$ where

- C is bielliptic, $C \xrightarrow{f} E \xrightarrow{g} \mathbf{P}^1$,
- $\tilde{C} \rightarrow C$ is Cartesian, $\tilde{C} = C^0 \times_E C^1$, C^0 is of genus n ,

are precisely (via the tetragonal construction) the Pryms of the following allowable double covers $\tilde{X} \rightarrow X$:

- $n = 1$: X is obtained from a hyperelliptic curve by identifying two pairs of points, $X = H/(p \sim q, r \sim s)$.
- $n = 2$: $X = X_0 \cup X_1$, X_0 rational, X_1 hyperelliptic, $\#(X_0 \cap X_1) = 4$.
- $n \geq 3$: $X = X_0 \cup X_1$, each X_i hyperelliptic, $g(X_0) = n - 2$, $g(X_1) = g(C) - n - 1$, $\#(X_0 \cap X_1) = 4$, and both hyperelliptic maps are restrictions of the same tetragonal map on X (i.e. they agree on $X_0 \cap X_1$).

Everything here follows directly from the proposition, except that for $n = 1$ we need to use (twice) the following observation of Beauville. Let $\pi : \tilde{X} \rightarrow X$ be an allowable double cover where

$$X = Y \cup R, \quad R \text{ rational}, \quad Y \cap R = \{a, b\}$$

$$\tilde{X} = \tilde{Y} \cup \tilde{R}, \quad \tilde{R} = \pi^{-1}(R) \text{ rational}, \quad \tilde{Y} \cap \tilde{R} = \{\tilde{a}, \tilde{b}\}$$

and π is ramified at \tilde{a}, \tilde{b} , which map to a, b . Construct a new cover $\tilde{Z} \rightarrow Z$ where

$$\tilde{Z} := \tilde{Y}/(\tilde{a} \sim \tilde{b})$$

$$Z := Y/(a \sim b).$$

Then this is still allowable, and

$$P(\tilde{Z}/Z) \approx P(\tilde{X}/X).$$

(Indeed, there are natural isomorphisms of generalized Jacobians

$$J(\tilde{Z}) \approx J(\tilde{X}), \quad J(Z) \approx J(X)$$

commuting with π_* and inducing the desired isomorphisms.)

QED

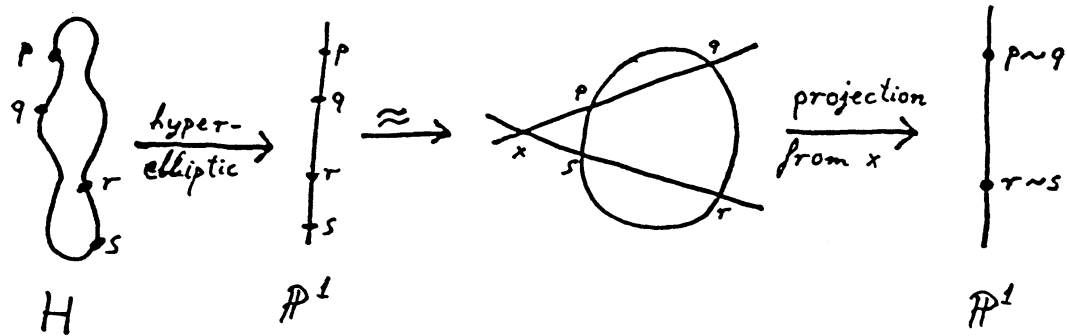
We are left with the Pryms of non-Cartesian double covers of bielliptic curves. The result here may be somewhat surprising:

Proposition 3.8 Pryms of non-Cartesian double covers of bielliptic curves are precisely the Pryms of Cartesian covers (of bielliptic curves) with $n(= g(C_0)) = 1$. (The isomorphism is obtained through a sequence of 2 tetragonal moves.)

The point is that if $X = H/(p \sim q, r \sim s)$ with H hyperelliptic, and $\tilde{X} \rightarrow X$ is an allowable double cover, then $P(\tilde{X}/X)$ is the Prym of a Cartesian cover (with $n = 1$) of a bielliptic curve, as we've just seen;

but X has another g_4^1 , and applying the tetragonal construction to it yields a non-Cartesian double cover of a bielliptic curve.

The g_4^1 is obtained as follows: map H to a conic in \mathbf{P}^2 (by the hyperelliptic map), then project the conic to \mathbf{P}^1 from the unique point x in \mathbf{P}^2 (and not on the conic) on the intersection of the lines \overline{pq} and \overline{rs} .



We should now check that the tetragonal construction yields a non-Cartesian cover of a bielliptic curve, and that all covers arise this way. We leave the former to the reader, and do the latter.

Let $\tilde{C} \rightarrow C$ be a non-Cartesian cover of C , which is bielliptic:

$$C \xrightarrow{f} E \xrightarrow{g} \mathbf{P}^1.$$

Let (\tilde{C}_i, C_i) , $i = 0, 1$, be the tetragonally related covers. By lemma (3.5), C_i is bihyperelliptic:

$$C_i \xrightarrow{f_i} H_i \xrightarrow{g_i} \mathbf{P}^1.$$

By the local pictures (2.14),

$$B := \text{Branch}(g) = B_0 \amalg B_1, \quad B_i := \text{Branch}(g_i).$$

(As we saw in Remark (3.5.1), the possible pictures are (1), (2), (7), (3a) and (3b). Of these, (7) and (3b) contribute to B , and each contributes also to one of the B_i .) Since $\#B = 4$ (E is elliptic), and $\#B_i$ is even and > 0 (non-Cartesian!), we find

$$\#B_i = 2, \quad i = 0, 1,$$

hence H_i is rational and C_i is hyperelliptic. Again by the local pictures, C_i will have two nodes, at points lying over B_{1-i} .

QED

We observe that the last argument works not only for bielliptics but also for branched double covers of hyperelliptic curves of genus 2, since now

$$\begin{aligned} \#B_0 > 0, \#B_1 > 0, \#B_0 + \#B_1 = 6, \#B_i \text{ even} \Rightarrow \\ \text{either } \#B_0 = 2 \text{ or } \#B_1 = 2. \end{aligned}$$

However, the resulting hyperelliptic curve with 4 nodes does not carry other g_4^1 's and is not necessarily related to any other covers.

We leave one more corollary of proposition (3.6) to the reader.

Corollary 3.9 Let K be hyperelliptic, $\widetilde{K} \rightarrow K$ a double cover with 2 branch points. Then $P(\widetilde{K}/K)$ is a hyperelliptic Jacobian.

(Hint: take both H and C^0 in proposition (3.6) to be rational, show $P(\widetilde{C}_1/C_1) \approx J(C^1)$ and $C_1 = K \cup_{(2 \text{ points})} \mathbf{P}^1$, K hyperelliptic.)

§4 Fibers of $\mathcal{P} : \mathcal{R}_6 \rightarrow \mathcal{A}_5$.

§4.1 The structure

We recall the main result of [DS]:

Theorem 4.1 [DS] $\mathcal{P} : \mathcal{R}_6 \rightarrow \mathcal{A}_5$ is generically finite, of degree 27.

Recall that $\mathcal{M}_6^{\text{Tet}}$ denotes the moduli space of curves of genus 6 with a g_4^1 . The forgetful map $\mathcal{M}_6^{\text{Tet}} \rightarrow \mathcal{M}_6$ is generically finite, of degree 5 [ACGH]. By base change we get a corresponding object $\mathcal{R}_6^{\text{Tet}}$, with map

$$\mathcal{R}_6^{\text{Tet}} \rightarrow \mathcal{R}_6$$

of degree 5. The tetragonal construction gives a triality, or (2,2)-correspondence, on $\mathcal{R}_6^{\text{Tet}}$. The image in \mathcal{R}_6 is then a (10,10)-correspondence:

$$(4.1.1) \quad \text{Tet} \subset \mathcal{R}_6 \times \mathcal{R}_6.$$

Theorem 4.2 The correspondence Tet induced by the tetragonal construction on the fiber $\mathcal{P}^{-1}(A)$, for generic $A \in \mathcal{A}_4$, is isomorphic to the incidence correspondence on the 27 lines on a non-singular cubic surface. The monodromy group of \mathcal{R}_6 over \mathcal{A}_5 (i.e. the Galois group of its Galois closure) is the Weyl group WE_6 , the symmetry group of the incidence of the 27 lines on the cubic surface.

This was conjectured in [DS] and announced in [D1]. The proof will be given below. For the symmetry group of the line incidence on a cubic surface, or other del Pezzo surfaces, we refer to [Dem].

(4.3) The blowup map

Let $\mathcal{Q} \subset \mathcal{M}_6$ denote the moduli space of non-singular plane quintic curves, $\mathcal{R}\mathcal{Q}$ its inverse image in \mathcal{R}_6 . By Theorem (1.2), it splits:

$$\mathcal{R}\mathcal{Q} = \mathcal{R}\mathcal{Q}^+ \cup \mathcal{R}\mathcal{Q}^-$$

with $(Q, \mu) \in \mathcal{R}\mathcal{Q}^+$ (respectively, $\mathcal{R}\mathcal{Q}^-$) iff $h^0(\mu \otimes \mathcal{O}_Q(1))$ is even (respectively, odd). The point is that $\mathcal{O}_Q(1)$ gives a uniform choice of theta characteristics over \mathcal{Q} , hence the spaces of theta characteristics and semiperiods over \mathcal{Q} are identified.

Let \mathcal{J} be the closure in \mathcal{A}_5 of the locus of Jacobians of curves, and let \mathcal{C} denote the moduli space of non-singular cubic threefolds. Via the intermediate Jacobian map, we identify \mathcal{C} with its image in \mathcal{A}_5 .

The Prym map sends $\mathcal{R}\mathcal{Q}^+$ to \mathcal{J} and $\mathcal{R}\mathcal{Q}^-$ to \mathcal{C} . Since the fiber dimensions can be positive, it is useful to consider the blowup Prym map

$$\tilde{\mathcal{P}} : \tilde{\mathcal{R}}_6 \rightarrow \tilde{\mathcal{A}}_5$$

where \mathcal{J}, \mathcal{C} on the right are blown up to divisors $\tilde{\mathcal{J}}, \tilde{\mathcal{C}}$, while on the left we blow up $\mathcal{R}\mathcal{Q}^+, \mathcal{R}\mathcal{Q}^-$, as well as the locus \mathcal{R}_6^{Trig} of double covers of trigonal curves. The result is a morphism which is generically finite over $\tilde{\mathcal{J}}$ and $\tilde{\mathcal{C}}$. We recall the geometric description of points of the various loci, and give the map in these geometric terms. This is taken from [CG], [T] and [DS].

(4.3.1) A point of \mathcal{C} is given by a non-singular cubic threefold $X \subset \mathbf{P}^4$. A point of $\tilde{\mathcal{C}}$ is given by a pair (X, H) , $H \in (\mathbf{P}^4)^*$ a hyperplane.

(4.3.2) A point of $\tilde{\mathcal{R}}\mathcal{Q}$ is given by (Q, μ, L) , or (Q, \tilde{Q}, L) , where $Q \subset \mathbf{P}^2$ is a plane quintic, $L \in (\mathbf{P}^2)^*$ a line, and μ a semiperiod on Q (or \tilde{Q} the corresponding double cover).

(4.3.3) The fiber $\mathcal{P}^{-1}(J(X)) \subset \mathcal{R}\mathcal{Q}^-$ over a cubic threefold X can be identified with the Fano surface $F(X)$ of lines $\ell \subset X$. (Projection from ℓ puts a conic bundle structure $\pi : X \dashrightarrow \mathbf{P}^2 = \mathbf{P}^4/\ell$ on X ; the corresponding point of $\mathcal{R}\mathcal{Q}^-$ is (Q, \tilde{Q}) , where the plane quintic Q is the discriminant locus of π , and its double cover \tilde{Q} parametrizes lines $\ell' \in F(X)$ meeting ℓ .)

(4.3.4) The fiber $\tilde{\mathcal{P}}^{-1}(X, H)$ corresponds to the lines ℓ in the cubic surface $X \cap H$. For general X, H , there are 27 of these. The corresponding objects are of the form (Q, \tilde{Q}, L) where (Q, \tilde{Q}) are as above, and $L \in \mathbf{P}^2$ is the projection, $L = \pi^{-1}(H)$.

(4.3.5) A point of \mathcal{R}_6^{Trig} is given by a curve $T \in \mathcal{M}_6$ with a trigonal line bundle $\mathcal{L} \in \text{Pic}^3(T)$, $h^0(\mathcal{L}) = 2$, and a double cover $\tilde{T} \rightarrow T$. The fiber of $\widetilde{\mathcal{R}}_6^{Trig}$ above it is given by the linear system $|\omega_T \otimes \mathcal{L}^{-2}|$, a \mathbf{P}^1 .

(4.3.6) A point of \mathcal{J} is given by the Jacobian of a curve $C \in \mathcal{M}_5$. The canonical curve $\Phi(C) \subset \mathbf{P}^4$, for general C , is the base locus of a net of quadrics:

$$A_p \subset \mathbf{P}^4, \quad p \in \mathbf{P}^2 = \mathbf{P}^2(C).$$

A point of $\widetilde{\mathcal{J}}$ above C is then given by a pair (C, L) , where L is a line in $\mathbf{P}^2(C)$. (Choosing such a line is the same as choosing a quartic del Pezzo surface

$$S = S_L = \bigcap_{p \in L} A_p$$

containing $\Phi(C)$.)

(4.3.7) Consider the map

$$\alpha : \mathcal{M}_5 \rightarrow \mathcal{RQ}^+$$

sending $C \in \mathcal{M}_5$ to $\alpha(C) = (Q, \tilde{Q})$, where:

$$Q := \{p \in \mathbf{P}^2(C) \mid A_p \text{ is singular}\} \subset \mathbf{P}^2(C),$$

and \tilde{Q} is the double cover whose fiber over a general $p \in Q$ corresponds to the two rulings on the rank-4 quadric A_p . This α is a birational isomorphism; its inverse is the restriction to \mathcal{RQ}^+ of \mathcal{P} .

The fiber $\tilde{\mathcal{P}}^{-1}(C, L)$ over generic $(C, L) \in \widetilde{\mathcal{J}}$ is given by the following 27 objects:

- The quintic object $(Q, \tilde{Q}, L) \in \widetilde{\mathcal{RQ}}^+$, where $(Q, \tilde{Q}) = \alpha(C)$ and L is the given line in $\mathbf{P}^2(C)$.
- Ten trigonals T_i^ε , $1 \leq i \leq 5$, $\varepsilon = 0, 1$, each with a double cover \tilde{T}_i^ε : each of the 5 points $p_i \in Q \cap L$ determines two g_4^1 's on C , cut out by the rulings R_i^ε on A_{p_i} , and the $(T_i^\varepsilon, \tilde{T}_i^\varepsilon)$ are associated to these by the trigonal construction.
- Sixteen Wirtinger covers $(X_j, \tilde{X}_j) \in \partial^I \mathcal{R}_6$: the quartic del Pezzo surface S_L contains 16 lines ℓ_j [Dem], each meeting $\Phi(C)$ in two points, say p_j, q_j , and then

$$X_j = C/(p_j \sim q_j)$$

and \tilde{X}_j is its unique Wirtinger cover (1.9.I).

(4.3.8) We observe that the generically finite map $\mathcal{R}_6^{\text{Tet}} \rightarrow \mathcal{R}_6$ has 1-dimensional fibers over both \mathcal{RQ} and $\mathcal{R}_6^{\text{Trig}}$. After blowing up and

normalizing, we obtain finite fibers generically over the exceptional loci. In the limit:

- Over (Q, L) , the 5 g_4^1 's correspond to projections of the plane quintic Q from one of the 5 points $p_i \in Q \cap L$.
- Over (T, D) , with T a trigonal curve, \mathcal{L} the trigonal bundle, and $D \in |\omega_T \otimes \mathcal{L}^{-2}|$, four of the g_4^1 's are of the form $\mathcal{L}(q)$ with $q \in D$ (i.e. they are the trigonal \mathcal{L} with base point q); the fifth g_4^1 is $\omega_T \otimes \mathcal{L}^{-2}$.
- Given $X = C/(p \sim q) \in \partial\bar{\mathcal{M}}_5$, there is a pencil $L \subset \mathbf{P}^2(C)$ of quadrics A_p , $p \in L$, which contain both $\Phi(C)$ and its chord \overline{pq} . Among these there are 5 quadrics A_{p_i} which are singular, generically of rank 4. Each of these has a single ruling R_i containing a plane containing \overline{pq} . These R_i cut the 5 g_4^1 's on X .

We conclude that the tetragonal correspondence Tet of (4.1.1) lifts to

$$\widetilde{\text{Tet}} \subset \widetilde{\mathcal{R}}_6 \times \widetilde{\mathcal{R}}_6$$

which is generically finite, of type (10,10), over each of our special loci.

Theorem 4.4 Structure of the blowup Prym map.

Over each of the following loci, the blowup Prym map $\widetilde{\mathcal{P}}$ has the listed monodromy group, and the lifted tetragonal correspondence $\widetilde{\text{Tet}}$ induces the listed structure.

- (1) $\widetilde{\mathcal{C}}$: The group is WE_6 , the structure is that of lines on a general non-singular cubic surface.
- (2) $\widetilde{\mathcal{J}}$: The group is WD_5 , the symmetry group of the incidence of lines on a quartic del Pezzo surface, or stabilizer in WE_6 of a line. The structure is that of lines on a non-singular cubic surface, one of which is marked.
- (3) \mathcal{B} = the locus of intermediate Jacobians of Clemens' quartic double solids of genus 5 [C1]: The group is $WA_5 = S_6$, the structure is that of lines on a nodal cubic surface.

[Note: \mathcal{B} is contained in the branch locus of \mathcal{P} [DS, V.4] and in fact ([D6], and compare also [SV], [I]) equals the branch locus. The monodromy along \mathcal{B} acting on a nearby, unramified fiber is $(\mathbf{Z}/2\mathbf{Z}) \times S_6$, or the symmetry group of a double-six, which is a subgroup of WE_6 . The group S_6 thus occurs as a subquotient of WE_6 .]

- (4) (cf. [I]) $\widetilde{\mathcal{P}}$ extends naturally to the boundary $\partial = \partial\mathcal{A}_5$; the monodromy is WE_6 and the structure is that of lines on a general cubic surface.

We will prove parts (1), (2) and (3) in §4.2. In the rest of this section we show that theorems (4.1) and (4.2) follow from (4.4).

(4.5) Proofs of Theorem (4.2).

By Theorem (2.16), Tet commutes with \mathcal{P} , therefore $\widetilde{\text{Tet}}$ commutes with $\widetilde{\mathcal{P}}$. To identify this structure over a generic point, it suffices to do so over any point over which $\widetilde{\mathcal{P}}$ and $\widetilde{\text{Tet}}$ are etale. These conditions hold, e.g., over a generic $(X, H) \in \widetilde{\mathcal{C}}$, where (4.4.1) identifies the structure. This implies that the monodromy is contained in WE_6 , but we get all of WE_6 already over $\widetilde{\mathcal{C}}$ (by (4.4.1) again), so we are done.

We can work instead over $\widetilde{\mathcal{J}}$: again, $\widetilde{\mathcal{P}}$ and $\widetilde{\text{Tet}}$ are etale over generic $(C, L) \in \widetilde{\mathcal{J}}$, and $\widetilde{\text{Tet}}$ has the right structure there by (4.4.2). This shows

$$WD_5 \subset \text{Monodromy} \subset WE_6.$$

As there are no intermediate groups, the monodromy must equal WD_5 or WE_6 . But if it were the former, $\widetilde{\mathcal{R}}_6$ would be reducible (since WD_5 is the stabilizer in WE_6 of one of the 27 lines), contradiction.

QED

Remark 4.5.1 Along the same lines, we can also reprove Theorem (4.1). Let $\widetilde{\text{Tet}}^i$ denote the i -th iterate of the correspondence $\widetilde{\text{Tet}}$. On \mathcal{RQ}^- we have:

$$\begin{aligned} \widetilde{\text{Tet}}^2 & \text{ has degree } 27, \\ \widetilde{\text{Tet}}^i & = \widetilde{\text{Tet}}^2 \quad \text{for } i \geq 2. \end{aligned}$$

Since $\widetilde{\text{Tet}}$ is etale there, these properties persist generically on $\widetilde{\mathcal{R}}_6$. Let \sim be the equivalence relation generated by $\widetilde{\text{Tet}}$. We conclude that \sim has degree 27, and that $\widetilde{\mathcal{P}}$ factors through a proper quotient:

$$\mathcal{P}' : \widetilde{\mathcal{R}}_6 / \sim \longrightarrow \widetilde{\mathcal{A}}_5.$$

We still need to verify that $\text{deg}(\mathcal{P}') = 1$. There are several possibilities:

- We can work over $\widetilde{\mathcal{J}}$; as we will see in (4.7), the fiber of $\widetilde{\mathcal{P}}$ there consists of a unique \sim -equivalence class; so we need to check that \mathcal{P}' is unramified at that equivalence class. This reduces to seeing that $\widetilde{\mathcal{P}}$ is unramified at least at one point of the fiber; this is trivial at the plane-quintic point. (This argument avoids some of the detailed computations of the codifferential on the boundary, [DS, Ch., IV], but is still very close in spirit to [DS].)
- We could instead work over any other point of $\widetilde{\mathcal{A}}_5$ over which we know the complete fiber, e.g. over Andreotti-Mayer points, coming

from bielliptic Pryms, as in §3. (This was proposed in [D1], as a way to avoid the boundary computations.)

- Izadi [I] applies a similar argument over boundary points, in $\partial\mathcal{A}_5$. This lets her reduce the degree computation over \mathcal{A}_5 to her results on \mathcal{A}_4 , cf. (4.9).

§4.2 Special Fibers

In this section we exhibit the cubic surface of theorem (4.2) explicitly over three special loci in \mathcal{A}_5 . We do not know how to do this at the generic point of \mathcal{A}_5 .

(4.6) Cubic threefolds

From (4.3.4) we have an identification of $\tilde{\mathcal{P}}^{-1}(X, H)$, where $X \subset \mathbf{P}^4$ is a cubic threefold and $H \subset \mathbf{P}^4$ a hyperplane, with the set of lines ℓ on the cubic surface $X \cap H$. For Theorem (4.4.1) we need to check that two of these, say $\ell, \ell' \in F(X)$, intersect each other if and only if the corresponding objects $(Q, \tilde{Q}, L), (Q', \tilde{Q}', L')$ correspond under $\widetilde{\text{Tet}}$. If the lines ℓ, ℓ' intersect, we are in the situation of (2.15.4), so the corresponding objects

$$(Q, \tilde{Q}, f), (Q', \tilde{Q}', f')$$

(notation of (2.15.4)) are tetragonally related. Since f, f' are both cut out by hyperplanes through the span A of ℓ, ℓ' , we find points

$$p \in Q \cap L, \quad p' \in Q' \cap L'$$

(namely, the projection of A from ℓ, ℓ' respectively) such that f, f' are the projections of Q from p and of Q' from p' , respectively. The description of $\widetilde{\mathcal{R}}_6^{\text{Tet}}$ in (4.3.8) then shows that

$$((Q, \tilde{Q}, L), (Q', \tilde{Q}', L')) \in \widetilde{\text{Tet}},$$

as required. Since both the line incidence and $\widetilde{\text{Tet}}$ are of bidegree (10,10), and we have an inclusion, it must be an equality.

This shows that $\widetilde{\text{Tet}}$ induces on $\tilde{\mathcal{P}}^{-1}(X, H)$ the structure of line incidence on the cubic surface $X \cap H$. Fix the ambient \mathbf{P}^4 and the hyperplane H , and let the cubic X vary. We clearly get all cubic surfaces in H as intersections $X \cap H$; therefore the monodromy group is the full symmetry group of the line configuration. This completes the proof of (4.4.1), hence also of Theorem (4.2).

(4.7) Jacobians

Start with $(C, L) \in \widetilde{\mathcal{J}}$. The fiber $\widetilde{\mathcal{P}}^{-1}(C, L)$ consists of the 27 objects listed in (4.3.7). Each of these comes with the 5 g_4^1 's given in (4.3.8). These give the correspondence $\widetilde{\text{Tet}}$, which we claim is equivalent to the line incidence on a cubic surface.

Let $S = S_L$ be the quartic del Pezzo surface determined by (C, L) , as in (4.3.6). Let S' be its blowup at a generic point $r \in S$. Then S' is a cubic surface; its lines correspond to:

- ℓ_Q , the exceptional divisor over r .
- 10 conics through r in S ; these correspond naturally to the 10 rulings $\mathcal{R}_i^\varepsilon$ (as in (4.3.7)). [Each $\mathcal{R}_i^\varepsilon$ contains a unique plane through r , which meets S in a conic through r .]
- The 16 lines ℓ_j in S .

There is thus a natural bijection between the lines of S' and $\widetilde{\mathcal{P}}^{-1}(C, L)$. We need to check that this correspondence takes incident lines to covers which are tetragonally related to each other through the g_4^1 's of (4.3.8). To that end, we list the effects of the tetragonal constructions on our curves. The details are straightforward, and are omitted.

(4.7.1) The quintic (Q, \widetilde{Q}) , with the $g_4^1 : \mathcal{O}_Q(1)(-p_i)$, $p_i \in Q \cap L$, is taken to the two trigonals

$$(T_i^\varepsilon, \widetilde{T}_i^\varepsilon), \quad \varepsilon = 0, 1,$$

each with its unique base-point-free g_4^1 , $\omega_T \otimes \mathcal{L}^{-2}$.

(4.7.2) The trigonal $(T_i^\varepsilon, \widetilde{T}_i^\varepsilon)$ with its base-point-free g_4^1 goes to (Q, \widetilde{Q}) with $\mathcal{O}_Q(1)(-p_i)$, and to $(T_i^{1-\varepsilon}, \widetilde{T}_i^{1-\varepsilon})$ with the base-point-free g_4^1 .

Consider $(T_i^\varepsilon, \widetilde{T}_i^\varepsilon)$ with the $g_4^1 : \mathcal{L}_i^\varepsilon(p)$. The actual 4-sheeted cover of \mathbf{P}^1 in this case is reducible, consisting of the trigonal T_i^ε together with a copy of \mathbf{P}^1 glued to it at p . We are thus precisely in the situation of Example (2.15.2): both tetragonally related objects are Wirtinger covers (X_j, \widetilde{X}_j) .

(4.7.3) A Wirtinger cover (X_j, \widetilde{X}_j) with the g_4^1 cut out by the ruling $\mathcal{R}_i^\varepsilon$ on the singular quadric A_{p_i} , is taken to the trigonal $(T_i^\varepsilon, \widetilde{T}_i^\varepsilon)$ and to another Wirtinger cover.

(4.8) Quartic double solids and the branch locus of \mathcal{P} .

The fiber of \mathcal{P} over the Jacobian $J(X) \in \mathcal{B}$ of a quartic double solid X of genus 5 is described in [DS, V.4], following ideas of Clemens. It consists of 6 objects (C_i, \tilde{C}_i) , $0 \leq i \leq 5$, each with multiplicity 2, and 15 objects (C_{ij}, \tilde{C}_{ij}) , $0 \leq i < j \leq 5$. The monodromy group S_6 permutes the six values of i : clearly the two sets $\{C_i\}$ and $\{C_{ij}\}$ must be separately permuted, and any permutation of the C_i induces a unique permutation of the C_{ij} . The situation is precisely that of lines on a nodal cubic surface: the C_i correspond to lines ℓ_i through the node; and the plane through ℓ_i, ℓ_j meets the cubic residually in a line $\ell_{i,j}$.

The best way to see the symmetry is to consider Segre's cubic threefold $Y \subset \mathbf{P}^4$, image of \mathbf{P}^3 by the linear system of quadrics through 5 points p_i , $1 \leq i \leq 5$, in general position in \mathbf{P}^3 . (cf. [SR] for the details.) Y contains six irreducible, two-dimensional families of lines, which we call the "rulings" R_i , $0 \leq i \leq 5$: For $1 \leq i \leq 5$, R_i consists of proper transforms of lines through p_i ; while R_0 parametrizes twisted cubics through p_1, \dots, p_5 . Y also contains 15 planes Π_{ij} , $0 \leq i < j \leq 5$ (= the 5 exceptional divisors and the proper transforms of the 10 planes $\overline{p_i p_j p_k}$); the ruling R_i is characterized as the set of lines in \mathbf{P}^4 meeting the 5 planes Π_{ij} , $j \neq i$. The symmetric group S_6 acts linearly on \mathbf{P}^4 , preserving Y , permuting the R_i and correspondingly the Π_{ij} .

The quartic double solids in question are essentially the double covers

$$\zeta : X \rightarrow Y$$

branched along the intersection of Y with a quadric $Q \subset \mathbf{P}^4$. The Prym fiber is obtained as follows:

- $C_i := \{ \text{lines } \ell \in R_i, \text{ tangent to } Q \}$
 $\tilde{C}_i := \{ \text{irreducible curves } \ell' \subset X \text{ such that } \zeta(\ell') = \ell \in C_i \}$

Thus (C_i, \tilde{C}_i) is the discriminant of a conic-bundle structure on X given by $\zeta^{-1}(R_i)$. The Prym canonical curve $\Psi(C_i) \subset \mathbf{P}^4$ is traced by the tangency points of ℓ and Q ; in particular, $\Psi(C_i) \subset Q$, so (C_i, \tilde{C}_i) is a ramification point of \mathcal{P} , by (1.6).

- (C_{ij}, \tilde{C}_{ij}) is similarly obtained as discriminant of a conic bundle structure on X given by projection from Π_{ij} , cf. [DS, V4.5].

(4.9) Boundary behavior

In [I], Izadi uses results on the structure of $\mathcal{P} : \mathcal{R}_5 \rightarrow \mathcal{A}_4$ to find the incidence structure on the fibers of the compactified map $\overline{\mathcal{P}} : \overline{\mathcal{R}}_6 \rightarrow \overline{\mathcal{A}}_5$

over boundary points of the toroidal compactification $\bar{\mathcal{A}}_5$. The picture is as follows:

$$\begin{array}{ccc} \bar{\mathcal{P}} & : & \bar{\mathcal{R}}_6 \longrightarrow \bar{\mathcal{A}}_5 \\ & & \cup \qquad \qquad \cup \\ \partial\bar{\mathcal{P}} & : & \partial^{\text{II}}\bar{\mathcal{R}}_6 \longrightarrow \partial\bar{\mathcal{A}}_5 \\ & & \alpha \downarrow \qquad \qquad \downarrow \beta \\ \mathcal{P} & : & \mathcal{R}_5 \longrightarrow \mathcal{A}_4 \end{array}$$

Over general $A \in \mathcal{A}_4$, the fiber $\beta^{-1}(A)$ is isomorphic to the Kummer variety $A/(\pm 1)$. Over $(\tilde{C}, C) \in \mathcal{R}_5$, the fiber of α is $S^2\tilde{C}/\iota$, and $\partial\mathcal{P}$ becomes (cf. [D3, (4.6)]) the map

$$\begin{array}{ccc} x + y \mapsto \psi(x) + \psi(y) \\ S^2\tilde{C} & \longrightarrow & A \\ \downarrow & & \downarrow \\ S^2\tilde{C}/\iota & \longrightarrow & A/(\pm 1) \end{array}$$

where ψ is the Abel-Prym map $\tilde{C} \rightarrow A$. All in all then, we are considering the map

$$\partial\mathcal{P} : \cup_{(\tilde{C}, C) \in \mathcal{P}^{-1}(A)} S^2\tilde{C} =: E \longrightarrow A.$$

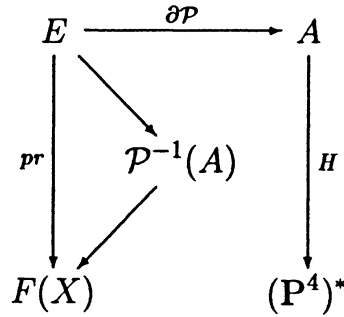
Theorem (4.1) says that its degree is 27, and Theorem (4.2) predicts an incidence structure on its fibers, i.e. a way of associating a cubic surface to each point $a \in A$.

In §5 we associate to $A \in \mathcal{A}_4$ a cubic threefold $X = \kappa(A) \subset \mathbf{P}^4$ such that $\mathcal{P}^{-1}(A)$ is a double cover of the Fano surface $F(X)$ of lines in X . For generic $a \in A$, we are looking for a cubic surface; it is reasonable to hope that this should be of the form $H(a) \cap X$, where $H(a)$ is an appropriate hyperplane in \mathbf{P}^4 . We thus want a map

$$H : A \rightarrow (\mathbf{P}^4)^*$$

such that

$$pr((\partial\mathcal{P})^{-1}(a)) = \{\text{lines in } H(a) \cap X\}.$$



Izadi’s beautiful observation is that such an H is given by the linear system Γ_{00} (sections of $|2\Theta|$ vanishing to order ≥ 4 at 0). The identification of Γ_{00} with the ambient \mathbf{P}^4 of X uses a construction of Clemens relating his double solids to Γ_{00} , and the interpretation of (a cover of) X as parametrizing double solids with intermediate Jacobians isomorphic to A , cf. [D6] or [I].

§5 Fibers of $P : \mathcal{R}_5 \rightarrow A_4$.

§5.1 The general fiber.

Our main result in this section is:

Theorem 5.1 For generic $A \in A_4$, the fiber $\overline{\mathcal{P}}^{-1}(A)$ is isomorphic to a double cover of the Fano surface $F = F(X)$ of lines on some cubic threefold X .

Let \mathcal{RC} denote the inverse image in \mathcal{RA}_5 of the locus \mathcal{C} of (intermediate Jacobians $J(X)$ of) cubic threefolds X . We recall from [D4] that it splits into even and odd components:

$$(5.1.1) \quad \mathcal{RC} = \mathcal{RC}^+ \amalg \mathcal{RC}^-,$$

distinguished by a parity function. This follows from the existence of a natural theta divisor $\Xi \subset J(X)$, characterized (cf. [CG]) by having a triple point at 0 : Ξ translates the parity function q of (1.2), on theta characteristics, to a parity on semiperiods. More explicitly, pick $(Q, \sigma) \in \mathcal{P}^{-1}(J(X)) \subset \mathcal{RQ}^-$; Mumford’s exact sequence (Theorem (1.4)(2)) says that any $\delta \in J_2(X)$ is $\pi^*\nu$ for some $\nu \in (\sigma)^\perp \subset J_2(Q)$. The compatibility result, theorem (1.5), then gives (cf. [D4], Proposition (5.1)):

$$(5.1.2) \quad q_X(\delta) = q_Q(\nu) = q_Q(\nu\sigma).$$

In case δ is even, we end up with an isotropic subgroup $(\nu, \sigma) \subset J_2(Q)$, with σ odd and $\nu, \nu\sigma$ even. The Pryms of the latter are therefore Jacobians of curves:

(5.1.3) $P(Q, \nu) \approx J(C), \quad P(Q, \nu\sigma) \approx J(C'),$
and the image of σ gives semiperiods $\mu \in J_2(C), \mu' \in J_2(C')$.

Reversing direction, we can construct an involution

$$\lambda : \mathcal{R}_5 \longrightarrow \mathcal{R}_5$$

and a map

$$\kappa : \mathcal{R}_5 \longrightarrow \mathcal{RC}^+,$$

as follows: Start with $(C, \mu) \in \mathcal{R}_5$, pick the unique (Q, ν) in $\mathcal{P}^{-1}(C) \cap \mathcal{RQ}^+$, and let $\sigma, \nu\sigma \in J_2(Q)$ map to $\mu \in J_2(C)$. Then formula (1.3) reads:

$$(5.1.4) \quad 0 \equiv 3 + \text{even} + q(\sigma) + q(\nu\sigma) \pmod{2},$$

so after possibly relabeling, we may assume

$$(Q, \sigma) \in \mathcal{RQ}^-, \quad (Q, \nu\sigma) \in \mathcal{RQ}^+$$

so that there is a well-defined curve $C' \in \mathcal{M}_5$ and a cubic threefold $X \in \mathcal{C}$ such that

$$P(Q, \sigma) \approx J(X)$$

$$(5.1.5) \quad P(Q, \nu\sigma) \approx J(C').$$

We can thus define λ and κ by:

$$\lambda(C, \mu) := (C', \mu')$$

$$(5.1.6) \quad \kappa(C, \mu) := (X, \delta),$$

where $\mu' \in J_2(C'), \delta \in J_2(X)$ are the images of $\nu \in J_2(Q)$. The precise version of our results is in terms of λ and κ :

Theorem 5.2

(1) (C, μ) is related to $\lambda(C, \mu)$ by a sequence of two tetragonal constructions. Hence λ commutes with the Prym map:

$$\mathcal{P} \circ \lambda = \mathcal{P}, \quad \lambda \circ \lambda = id.$$

(2) κ factors through the Prym map:

$$\kappa : \mathcal{R}_5 \xrightarrow{\mathcal{P}} A_4 \xrightarrow{\chi} \mathcal{RC}^+,$$

where χ is a birational map.

Recall the Abel-Jacobi map [CG],

$$AJ : F(X) \longrightarrow J(X),$$

which is well-defined up to translation in $J(X)$. (It can be identified with the Albanese map of the Fano Surface $F(X)$.) A point $\delta \in J_2(X)$ determines a double cover of $J(X)$, hence of $F(X)$.

Theorem 5.3 For generic $A \in \mathcal{A}_4$, set

$$(X, \delta) := \chi(A) = \kappa(\mathcal{P}^{-1}(A)) \in \mathcal{RC}^+.$$

Let $F(X)$ be the Fano surface of X , $\widetilde{F(X)}$ its double cover determined by δ via the Abel-Jacobi map.

(1) There is a natural isomorphism

$$\mathcal{P}^{-1}(A) \approx \widetilde{F(X)}.$$

(2) The action of λ on the left corresponds to the sheet interchange on the right.

(3) Two objects $(C, \mu), (C', \mu) \in \mathcal{P}^{-1}(A)$ are tetragonally related if and only if the lines $\ell, \ell' \in F(X)$ which they determine intersect.

Remark 5.4 Izadi has recently analyzed the birational map χ , in [I]. In particular, she shows that χ is an isomorphism on an explicitly described, large open subset of \mathcal{A}_4 .

§5.2 Isotropic subgroups.

By isotropic subgroup of rank r on a curve C we mean an r -dimensional \mathbb{F}_2 -subspace of $J_2(C)$ on which the intersection pairing $\langle \cdot, \cdot \rangle$ is identically zero. Choosing an isotropic subgroup of rank 1 is the same as choosing a non-zero semiperiod.

Start with a trigonal curve $T \in \mathcal{M}_{g+1}$, with a rank-2 isotropic subgroup $W \subset J_2(T)$ whose non-zero elements we denote ν_i , $i = 0, 1, 2$. The trigonal construction associates to (T, ν_i) the tetragonal curve $X_i \in \mathcal{M}_g$. Mumford's sequence (1.4)(2) sends W to an isotropic subgroup of rank 1 on X_i , whose non-zero element we denote μ_i .

Lemma 5.5 The construction above sets up a bijection between the following data:

- A trigonal curve $T \in \mathcal{M}_{g+1}$ with rank-2 isotropic subgroup.
- A tetragonally related triple $(X_i, \mu_i) \in \mathcal{R}_g$, $i = 0, 1, 2$.

Proof.

We think of WD_4 as the group of signed permutations of the 8 objects $\{x_i^\pm\}$, $1 \leq i \leq 4$. Start with a tetragonal double cover $\widetilde{X}_0 \rightarrow X_0 \rightarrow \mathbf{P}^1$. It determines a principal WD_4 -bundle over $\mathbf{P}^1 \setminus (\text{Branch})$. The original covers \widetilde{X}_0, X_0 are recovered as quotients by the following subgroups of WD_4 :

$$\begin{aligned}\widetilde{H}_0 &:= \text{Stab}(x_1^+), \\ H_0 &:= \text{Stab}(x_1^\pm),\end{aligned}$$

Consider also the subgroup

$$G := \text{Stab}\{\{x_1^+, x_2^+\}, \{x_1^-, x_2^-\}\}.$$

It has index 12 in WD_4 . Its normalizer is:

$$N(G) = \text{Stab}\{\{x_1^\pm, x_2^\pm\}, \{x_3^\pm, x_4^\pm\}\},$$

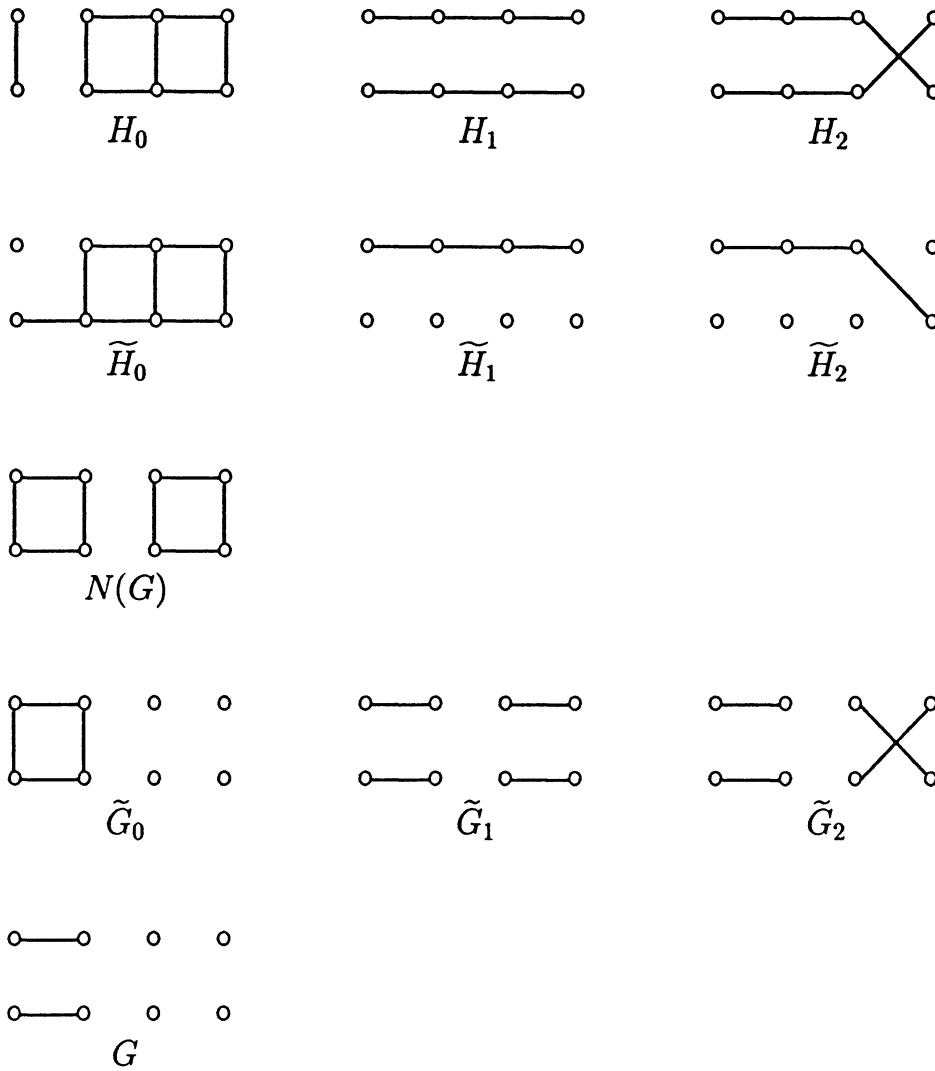
of index 3. The quotient is

$$N(G)/G \approx (\mathbf{Z}/2\mathbf{Z})^2,$$

so there are 3 intermediate groups \widetilde{G}_i , $i = 0, 1, 2$. We single out one of them:

$$\widetilde{G}_0 := \text{Stab}\{x_1^\pm, x_2^\pm\}.$$

The three subgroups \widetilde{G}_i are not conjugate to each other, but can be taken to each other by outer automorphisms of WD_4 . In fact, the action of $\text{Out}(WD_4) \approx S_3$ sends G , and hence also $N(G)$, to conjugate subgroups; it permutes the \widetilde{G}_i transitively, modulo conjugation; and it also takes H_0, \widetilde{H}_0 to non-conjugate subgroups H_i, \widetilde{H}_i , $i = 1, 2$. We illustrate each of these subgroups as the stabilizer in WD_4 of a corresponding partition of $\begin{pmatrix} x_1^+ & x_2^+ & x_3^+ & x_4^+ \\ x_1^- & x_2^- & x_3^- & x_4^- \end{pmatrix}$:



Let $X_i, \tilde{X}_i, T, \tilde{T}, \tilde{T}_i$ ($i = 0, 1, 2$) be the quotients of the principal WD_4 -bundle by the subgroups $H_i, \tilde{H}_i, N(G), G, \tilde{G}_i$ respectively, compactified to branched covers of \mathbf{P}^1 . We see immediately that:

- The trigonal construction takes $X_0 \rightarrow \mathbf{P}^1$ to $\tilde{T}_0 \rightarrow T \rightarrow \mathbf{P}^1$.
- The double cover $\tilde{X}_0 \rightarrow X_0$ corresponds via (1.4)(2) to the double cover $\tilde{\tilde{T}} \rightarrow \tilde{T}_0$.
- The tetragonal construction acts by outer automorphisms, hence exchanges the three tetragonal double covers $\tilde{X}_i \rightarrow X_i \rightarrow \mathbf{P}^1$.

Applying the same outer automorphisms, we see that the trigonal construction also takes $X_i \rightarrow \mathbf{P}^1$ to $\tilde{T}_i \rightarrow T \rightarrow \mathbf{P}^1$, $i = 1, 2$. To a tetragonally related triple $(\tilde{X}_i \rightarrow X_i \rightarrow \mathbf{P}^1)$ we can thus unambiguously associate the trigonal $T \rightarrow \mathbf{P}^1$ together with the rank-2, isotropic

subgroup corresponding to the covers \tilde{T}_i . This inverts the construction preceding the lemma.

QED.

Note 5.5.1 The basic fact in the above proof is that the 3 tetragonals X_i yield the same trigonal T . This can be explained more succinctly: outer automorphisms take the natural surjection $\alpha_0 : WD_4 \twoheadrightarrow S_4$ to homomorphisms α_1, α_2 which are not conjugate to it. But the composition $\beta \circ \alpha_i : WD_4 \twoheadrightarrow S_3$, where $\beta : S_4 \twoheadrightarrow S_3$ is the Klein map, are conjugate to each other.

Construction 5.6 Now let $T \in \mathcal{M}_{g+1}$ be a trigonal curve, together with an isotropic subgroup of rank 3,

$$V \subset J_2(T).$$

We think of V as a vector space over \mathbf{F}_2 ; the projective plane $\mathbf{P}(V)$ is identified with $V \setminus (0)$. For each $i \in \mathbf{P}(V)$, the trigonal construction gives a tetragonal curve $Y_i \in \mathcal{M}_g$. Mumford's sequence (1.4)(2) gives an isotropic subgroup of rank 2,

$$W_i \subset J_2(Y_i),$$

with a natural identification $W_i \approx V/(i)$.

Let $U \subset V$ be a rank-2 subgroup, so $\mathbf{P}(U) \subset \mathbf{P}(V)$ is a projective line. Lemma (5.5) shows that the 3 objects

$$(Y_i, U/(i)) \in \mathbf{R}_g, \quad i \in \mathbf{P}(U)$$

are tetragonally related. In particular, they have a common Prym variety

$$P_U \approx \mathcal{P}(Y_i, U/(i)) \in \mathcal{A}_{g-1}, \quad \forall i \in \mathbf{P}(U).$$

Applying (1.4) twice, we see that the original rank-3 subgroup V determines a rank-1 subgroup

$$V/U \subset (P_U)_2,$$

so we let $\mu_U \in (P_U)_2$ be its non-zero element. Altogether then, we have a map

$$\begin{aligned} \mathbf{P}(V)^* &\longrightarrow \mathcal{RA}_{g-1} \\ U &\longmapsto (P_U, \mu_U). \end{aligned}$$

(5.6.1) Assume now that one of the Y_i happens to be trigonal. (This can only happen if $g \leq 6$.) Whenever $U \ni i$, we find a tetragonal curve

$C_U \in \mathcal{M}_{g-1}$ such that $P_U \approx J(C_U)$. Lemma (5.5), applied to (Y_i, W_i) , shows that the 3 objects

$$(C_U, \mu_U) \in \mathcal{R}_{g-1}, \quad U \ni i$$

are tetragonally related, so they have a common Prym variety $A = P_V \in \mathcal{A}_{g-2}$.

(5.6.2) Assume instead that $g = 6$ and that P_U happens to be a Jacobian $J(C_U) \in \mathcal{J}_5$, for some $U \in \mathbf{P}(V)^*$. Of the three Y_i , $i \in U$, we claim two are trigonal and the third, a plane quintic. Indeed, by (4.7), the tetragonal triples above $J(C_U)$ consist either of a plane quintic and two trigonals, as claimed, or of a trigonal and two Wirtingers. The latter is excluded since the isomorphism

$$J(Y_i) \approx P(T, i)$$

implies that Y_i is non-singular for each $i \in \mathbf{P}(V)$.

Assume from now on that $g = 6$. Our data consists of:

- $T \in \mathcal{M}_7$, trigonal, with $V \subset J_2(T)$ isotropic of rank 3.
- For each $i \in \mathbf{P}(V)$, a curve $Y_i \in \mathcal{M}_6$, with a rank-2 isotropic subgroup $W_i \subset J_2(Y_i)$.
- For each $U \in \mathbf{P}(V)^*$, an object $(P_U, \mu_U) \in \mathcal{RA}_5$
- An abelian variety $A = P_V \in \mathcal{A}_4$.

We display $\mathbf{P}(V)$ as a graph with 7 vertices $i \in \mathbf{P}(V)$ and 7 edges $U \in \mathbf{P}(V)^*$, in (3,3)-correspondence. We write T (or Q) on a vertex corresponding to a trigonal (or quintic) curve, and C on an edge corresponding to a Jacobian. We restate our observations:

(5.6.3): Edges through a T -vertex are C -edges.

(5.6.4): On a C -edge, the vertices are T, T, Q .

It follows that only one configuration is possible:

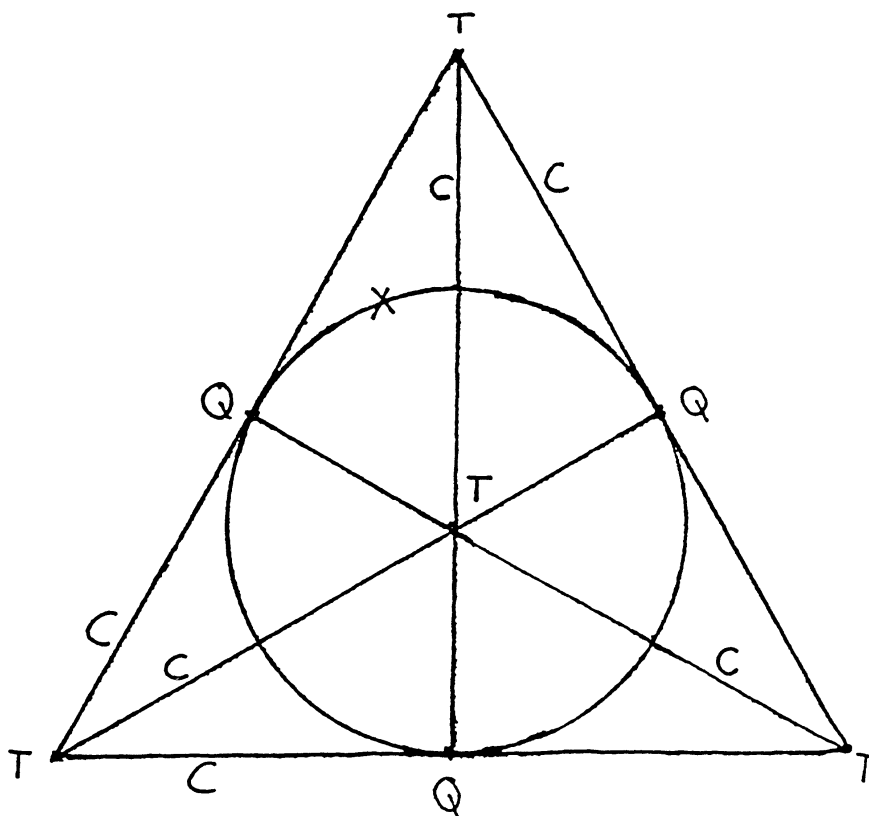


Figure 5.7

Thus four of the Y_i are trigonal, the other three are quintics, and six of the P_U , corresponding to the straight lines, are Jacobians of curves. Let $U_0 \in \mathbf{P}(V)^*$ correspond to the circle. For $i \in U_0$, Y_i is a quintic Q . Through Q pass two C edges and U_0 , and the semiperiods corresponding to the C -edges are even; by (1.3), the semiperiod $U_0/(i)$ corresponding to U_0 must be odd, so there is a cubic threefold $X \in \mathcal{C}$ such that

$$\mathbf{P}_{U_0} \approx J(X).$$

Finally, theorem (1.5), or formula (5.1.2), shows that the semiperiod $\delta := \mu_{U_0} \in J_2(X)$ is even.

We observe that the three tetragonally related quintics correspond to 3 lines on the cubic threefold which meet each other and thus form the intersection of X with a (tritangent) plane. We are thus exactly in the situation of (2.15.4).

§5.3 Proofs.

(5.8) Theorems (5.1),(5.2) and (5.3) all follow from the following statements:

- (1) (C, μ) is related to $\lambda(C, \mu)$ by a sequence of two tetragonal constructions.
- (2) κ is invariant under the tetragonal construction
- (3) For $(X, \delta) \in \mathcal{RC}^+$, $\kappa^{-1}(X, \delta) \approx \widetilde{F(X)}$, the isomorphism takes λ to the involution on $\widetilde{F(X)}$ over $F(X)$, and two objects on the left are tetragonally related iff the corresponding lines intersect.
- (4) Any two objects in $\mathcal{P}^{-1}(A)$, generic $A \in \mathcal{A}_4$, are connected by a sequence of (two) tetragonal constructions.

Indeed, (1) is (5.2)(1); (2) and (4) imply the existence of $\chi : \mathcal{A}_4 \rightarrow \mathcal{RC}^+$ such that $\kappa = \chi \circ \mathcal{P}$, while (3) shows that any two objects in a κ -fiber are also connected by a sequence of two tetragonal constructions, so χ must be birational, giving (5.2)(2). This gives an isomorphism $\mathcal{P}^{-1}(A) \approx \kappa^{-1}(X, \delta)$, so (5.3) follows.

(5.9) We let $\mathcal{R}^2Q^+, \mathcal{R}^2Q^-$ denote the moduli spaces of plane quintic curves Q together with:

- A rank-2, isotropic subgroup $W \subset J_2(Q)$, containing one odd and two even semiperiods, and
- a marked even (respectively odd) semiperiod in $W \setminus (0)$.

Exchanging the two even semiperiods gives an involution on \mathcal{R}^2Q^+ , with quotient \mathcal{R}^2Q^- . The birational map

$$\alpha : \mathcal{M}_5 \xrightarrow{\sim} \mathcal{R}Q^+,$$

of (4.3.7), lifts to a birational map

$$(5.9.1) \quad \mathcal{R}\alpha : \mathcal{R}_5 \xrightarrow{\sim} \mathcal{R}^2Q^+.$$

From the construction of λ in (5.1.6) it follows that the involution on the right hand side corresponds to λ on the left, so we have a commutative diagram:

$$(5.9.2) \quad \begin{array}{ccc} \mathcal{R}_5 & \xrightarrow[\sim]{\mathcal{R}\alpha} & \mathcal{R}^2Q^+ \\ \downarrow & & \downarrow \pi \text{ 2:1} \\ \mathcal{R}_5/\lambda & \xrightarrow{\sim} & \mathcal{R}^2Q^- \end{array}$$

Start with $(C, \mu) \in \mathcal{R}_5$ and any g_4^1 on C . The trigonal construction produces a trigonal $Y \in \mathcal{M}_6$ with rank-2, isotropic subgroup W_Y . On Y we have a natural g_4^1 , namely $w_Y \otimes L^{-2}$, where L is the trigonal bundle, so we bootstrap again, to a trigonal $T \in \mathcal{M}_7$ with rank-3

isotropic subgroup V . Applying construction (4.6) we obtain a diagram like (5.7), including an edge for (C, μ) and on it a vertex for $(Q, W_Q) := \pi \mathcal{R}\alpha(C, \mu)$. But then $\lambda(C, \mu)$ and $\kappa(C, \mu)$ also appear in the same diagram, as the two other edges (the line, respectively the circle) through Q ! Statement (5.8.1) now follows, since any two edges of (5.7) which meet in a trigonal vertex are tetragonally related. (5.8.2) also follows, since any (C', μ') tetragonally related to (C, μ) will appear in the same diagram with (C, μ) (for the obvious initial choice of g_4^1 on C), so they have the same κ .

From the restriction to $\mathcal{R}Q^-$ of the Prym map we obtain, by base change:

$$(5.9.3) \quad \begin{array}{ccc} \mathcal{R}^2Q^- & \longrightarrow & \mathcal{R}Q^- \\ \mathcal{R}\mathcal{P} \downarrow & & \downarrow \mathcal{P} \\ \mathcal{R}C^+ & \longrightarrow & C \end{array}$$

Combining with (5.8)(1),(2) and (5.9.2), we find that κ factors

$$(5.9.4) \quad \begin{array}{ccc} \mathcal{R}_5 & \xrightarrow{\sim \mathcal{R}\alpha} & \mathcal{R}^2Q^+ \\ \downarrow & & \downarrow \pi \\ \mathcal{R}_5/\lambda & \xrightarrow{\sim} & \mathcal{R}^2Q^- \\ & & \downarrow \mathcal{R}\mathcal{P} \\ & & \mathcal{R}C^+ \end{array}$$

We know $\mathcal{P}^{-1}(X)$ from (4.6), so by (5.9.3):

$$(5.9.5) \quad \mathcal{R}\mathcal{P}^{-1}(X, \delta) \approx \mathcal{P}^{-1}(X) \approx F(X),$$

and $\kappa^{-1}(X, \delta)$ is a double cover, which by the following lemma is identified with $\widetilde{F(X)}$. (The compatibility with λ follows from (5.9.4); line incidence in $F(X)$ corresponds by (4.6) to the tetragonal relation among the quintics, which by figure (5.7) corresponds, in turn, to the tetragonal relation in \mathcal{R}_5 , so the proof of (5.8)(3) is complete.)

Lemma 5.10 The Albanese double cover $\widetilde{F(X)}$ determined by $\delta \in J_2(X)$ is isomorphic to $\pi^{-1}\mathcal{R}\mathcal{P}^{-1}(X, \delta)$ (notation of (5.9.4)).

Proof.

The second isomorphism in (5.9.5) sends a line $\ell \in F(X)$ to the object $(\tilde{Q}_\ell, Q_\ell) \in \mathcal{P}^{-1}(X)$, where the curves \tilde{Q}_ℓ, Q_ℓ parametrize ordered (respectively, unordered) pairs $\ell', \ell'' \in F(X)$ satisfying:

$$\ell + \ell' + \ell'' = 0 \quad (\text{sum in } J(X)).$$

We may of course think of \tilde{Q}_ℓ as sitting in $F(X)$, since ℓ' uniquely determines ℓ'' : \tilde{Q}_ℓ is the closure in $F(X)$ of

$$(5.10.1) \quad \{\ell' \in F(X) \mid \ell' \cap \ell \neq \emptyset, \ell' \neq \ell\}.$$

The corresponding object of $\mathcal{RP}^{-1}(X, \delta)$ is $(\tilde{\tilde{Q}}_\ell, \tilde{Q}_\ell, Q_\ell)$, where $\tilde{\tilde{Q}}_\ell$ is the inverse image in $F(\tilde{X})$ of \tilde{Q}_ℓ embedded in $F(X)$ via (5.10.1). Now to specify a point in $\pi^{-1}\mathcal{RP}^{-1}(X, \delta)$ we need, additionally, a double cover $\tilde{Q}'_\ell \rightarrow Q_\ell$ satisfying:

$$(5.10.2) \quad \tilde{Q}_\ell \times_{Q_\ell} \tilde{Q}'_\ell \approx \tilde{\tilde{Q}}_\ell .$$

We need to show that a choice of $\tilde{\ell} \in F(\tilde{X})$ over $\ell \in F(X)$ determines such a \tilde{Q}'_ℓ . Recall that $F(\tilde{X}) \rightarrow F(X)$ is obtained by base change, via the Albanese map, from the double cover $J(\tilde{X}) \rightarrow J(X)$ determined by δ . \tilde{Q}'_ℓ can thus be taken to parametrize unordered pairs $\tilde{\ell}', \tilde{\ell}'' \in F(\tilde{X})$ satisfying:

$$\tilde{\ell} + \tilde{\ell}' + \tilde{\ell}'' = 0 \quad (\text{sum in } J(\tilde{X})) .$$

The fiber product in (5.10.2) then parametrizes such ordered pairs, so the required isomorphism to $\tilde{\tilde{Q}}_\ell$ simply sends

$$(\tilde{\ell}', \tilde{\ell}'') \mapsto \tilde{\ell}.$$

Q.E.D.

Finally, we prove (5.8)(4). Let $\bar{\mathcal{P}} : \bar{\mathcal{R}}_5 \rightarrow \mathcal{A}_4$ be the proper Prym map. By (5.8)(3) it factors

$$\bar{\mathcal{P}} = \iota \circ \kappa$$

where $\iota : \mathcal{RC}^+ \rightarrow \mathcal{A}_4$ is a rational map, which we are trying to show is birational. It suffices to find some $A \in \mathcal{A}_4$ such that:

- (1) Any two objects in $\bar{\mathcal{P}}^{-1}(A)$ can be related by a sequence of tetragonal constructions.
- (2) The differential $d\bar{\mathcal{P}}$ is surjective over A .

In §5.4 we see that (1) holds for various examples, including generic Jacobians $\in \mathcal{J}_4$: for generic $C \in \mathcal{M}_4$, $\bar{\mathcal{P}}^{-1}(J(C))$ consists of Wirtinger

covers $\tilde{C} \rightarrow C'$ (with normalization C) and of trigonals T , and the two types are exchanged by λ . It is easier to check surjectivity of $d\mathcal{P}$ at the Wirtingers: by theorem (1.6), this amounts to showing that the Prym-canonical curve $\Psi(X) \subset \mathbf{P}^3$ is contained in no quadrics. By [DS] IV, Propo. 3.4.1, $\Psi(X)$ consists of the canonical curve $\Phi(C)$ together with an (arbitrarily chosen) chord. Since $\Phi(C)$ is contained in a unique quadric Q , which does not contain the generic chord, we are done. [Another argument: it suffices to show that no one quadric contains $\Psi(T)$ for all trigonal T in $\mathcal{P}^{-1}(J(C))$. By [DS], III 2.3 we have

$$\cup_T \Psi(T) \supset \Phi(C),$$

so the only possible quadric would be Q . Consider the g_4^1 on C given by $\omega_C(-p-q)$, where $p, q \in C$ are such that the chord $\overline{\Phi(p), \Phi(q)}$ is not in Q . Let T be the trigonal curve associated to $(C, \omega_C(-p-q))$, and choose a plane $A \subset \mathbf{P}^3$ through $\Phi(p), \Phi(q)$, meeting Q and $\Phi(C)$ transversally, say

$$A \cap \Phi(C) = \Phi(p + q + \sum_{i=1}^4 x_i),$$

then by [DS], III 2.1, $\Psi(T)$ contains the point

$$\overline{\Phi(x_1), \Phi(x_2)} \cap \overline{\Phi(x_3), \Phi(x_4)}$$

which cannot be in Q .]

Q.E.D.

§5.4 Special fibers.

We want to illustrate the behavior of the Prym map over some special loci in $\overline{\mathcal{A}}_4$. The common feature to all of these examples is that the cubic threefold X given in Theorem (5.1) acquires a node. We thus begin with a review of some results, mostly from [CG], on nodal cubics.

(5.11) Nodal cubic threefolds

There is a natural correspondence between nodal cubic threefolds $X \subset \mathbf{P}^4$ and nonhyperelliptic curves B of genus 4. Either object can be described by a pair of homogeneous polynomials F_2, F_3 , of degrees 2 and 3 respectively, in 4 variables x_1, \dots, x_4 : X has homogeneous equation $0 = F_3 + x_0 F_2$ (in \mathbf{P}^4), and the canonical curve $\Phi(B)$ has equations $F_2 = F_3 = 0$ in \mathbf{P}^3 .

More geometrically, we express the Fano surface $F(X)$ in terms of B . Assume the two g_3^1 's on B , \mathcal{L}' and \mathcal{L}'' , are distinct. They give maps

$$\tau', \tau'' : B \hookrightarrow S^2B$$

sending $r \in B$ to $p + q$ if $p + q + r$ is a trigonal divisor in $|\mathcal{L}'|$, $|\mathcal{L}''|$ respectively. We then have the identification

$$(5.11.1) \quad F(X) \approx S^2B/(\tau'(B) \sim \tau''(B)).$$

Indeed, we have an embedding

$$\tau : B \hookrightarrow F(X),$$

identifying B with the family of lines through the node $n = (1, 0, 0, 0, 0)$. This gives a map $S^2B \rightarrow F(X)$ sending a pair ℓ_1, ℓ_2 of lines through n to the residual intersection with X of the plane (ℓ_1, ℓ_2) . this map identifies $\tau'(B)$ with $\tau''(B)$, and induces the isomorphism (5.11.1).

(5.11.2) A line $\ell \in F(X)$ determines a pair $(Q, \tilde{Q}) \in \overline{\mathcal{RQ}^-}$, which must be in $\partial^{\text{II}}\mathcal{RQ}^-$, i.e. for generic ℓ we obtain a nodal quintic Q with étale double cover \tilde{Q} . We can interpret (5.11.1) in terms of these nodal quintics: Start with a divisor $p + q \in S^2B$. Then $\omega_B(-p - q)$ is a g_4^1 on B , so the trigonal construction produces a double cover $\tilde{T} \rightarrow T$, where $T \in \mathcal{M}_5$ comes with a trigonal bundle \mathcal{L} . The linear system $|\omega_T \otimes \mathcal{L}^{-1}|$ maps T to a plane quintic Q , with a single node given by the divisor $|\omega_T \otimes \mathcal{L}^{-2}|$ on T .

(5.11.3) In the special case that there exists $r \in B$ such that $p + q = \tau''(r)$, i.e. $p + q + r \in |\mathcal{L}''|$ is a trigonal divisor, our g_4^1 acquires a base point:

$$\omega_B(-p - q) \approx \mathcal{L}'(r).$$

As seen in (2.10.ii), the trigonal construction produces the nodal trigonal curve

$$T := B/(p' \sim q')$$

with its Wirtinger double cover \tilde{T} , where $p', q' \in B$ are determined by:

$$p' + q' + r \in |\mathcal{L}'|,$$

i.e. $p' + q' = \tau'(r)$. In this case, the quintic Q is the projection of $\Phi(B)$ from $\Phi(r)$, with 2 nodes $p \sim q$, $p' \sim q'$, and \tilde{Q} is the reducible double cover with crossings over both nodes.

(5.12) Degenerations in \mathcal{RC}^+ .

We fix our notation as in §5.1. Thus we have:

$$\begin{aligned}
 X \in \mathcal{C} & \quad (X, \delta) \in \mathcal{RC}^+ \\
 (Q, \sigma) \in \mathcal{RQ}^-, & \quad (Q, \nu), (Q, \nu\sigma) \in \mathcal{RQ}^+ \\
 (C, \mu), (C', \mu') \in \mathcal{R}_5 & \\
 A \in \mathcal{A}_4 &
 \end{aligned}$$

and these objects satisfy:

$$\begin{aligned}
 \mathcal{P}(Q, \sigma) = J(X) , & \quad \nu, \nu\sigma \mapsto \delta \\
 \mathcal{P}(Q, \nu) = J(C) , & \quad \sigma, \nu\sigma \mapsto \mu \\
 \mathcal{P}(Q, \nu\sigma) = J(C') , & \quad \nu, \sigma \mapsto \mu'
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{P}(C, \mu) = \mathcal{P}(C', \mu') = A \\
 \lambda(C, \mu) = (C', \mu') \\
 \kappa(C, \mu) = (X, \delta) .
 \end{aligned}$$

Now let X degenerate, acquiring a node, with $\bar{\varepsilon} \in J_2(X) \setminus (0)$ the vanishing cycle mod. 2. From (5.11) we see that Q also degenerates, with a vanishing cycle ε which maps (via. (1.4)) to $\bar{\varepsilon}$. Lemma (5.9) of [D4] shows that ε , hence also $\bar{\varepsilon}$, must be even.

There are 3 types of degenerations of (X, δ) , distinguished as in (1.7) by the relationship of $\delta, \bar{\varepsilon}$. (A fourth type, where Q degenerates but X does not, is explained in (5.13).) The possibilities are summarized below:

- (I) If $\bar{\varepsilon} = \delta$ then either $\varepsilon = \nu$ or $\varepsilon = \nu\sigma$, which gives the same picture with C, C' exchanged. In case $\varepsilon = \nu$, (Q, ν) undergoes a ∂^I degeneration, while $(Q, \nu\sigma)$ is ∂^{II} . (The notation is that of (1.7).) Thus A is a Jacobian.

The double cover $F(\widetilde{X})$ is itself a ∂^I cover. In terms of the curve B of (5.11), we have

$$F(\widetilde{X}) = (S^2B)_0 \amalg (S^2B)_1 / (\tau'(B)_0 \sim \tau''(B)_1, \tau''(B)_0 \sim \tau'(B)_1) .$$

This is clear, either from the definition of $F(\widetilde{X})$ via the Albanese map, or by considering the restriction to $\mathcal{RP}^{-1}(X, \delta)$ of the double cover

$$\mathcal{R}^2Q^+ \xrightarrow{\pi} \mathcal{R}^2Q^-$$

of (5.9). One of the components parametrizes the trigonal objects (C, μ) , the other parametrizes the nodals (C', μ') .

- (II) σ is always perpendicular to ε, ν , and the condition $\langle \bar{\varepsilon}, \delta \rangle = 0$ implies $\langle \varepsilon, \nu \rangle = 0$ by (1.4.3). Both (Q, ν) and $(Q, \nu\sigma)$ then give ∂^{II} -covers, so C, C' are nodal. Again by (1.4.3), both (C, μ) and (C', μ') are ∂^{II} , so their common Prym A is in $\partial\bar{\mathcal{A}}_4$.

From the Albanese map we see that $F(\widetilde{X})$ is an etale cover of $F(X)$. Indeed, δ comes from a semiperiod δ' on B , giving a double cover \widetilde{B} with involution ι ; the normalization of $F(\widetilde{X})$ is then $S^2\widetilde{B}/\iota$, and $F(\widetilde{X})$ is obtained by glueing above $\tau(B)$.

- (III) In this case both (Q, ν) and $(Q, \nu\sigma)$ are ∂^{III} , so C, C' are non-singular. The node of Q represents a quadric of rank 3 through $\Phi(C)$, so \mathcal{L} is cut out by the unique ruling. By the Schottky-Jung relations [M2], the vanishing theta null on C descends to one on A .

The double cover $F(\widetilde{X})$ is again a ∂^{III} -cover, in the sense that its normalization is ramified over $\tau'(B), \tau''(B)$, the sheets being glued. Each of the quintics in (5.11.2) gives two points of $F(\widetilde{X})$, while the two-nodal quintics (5.11.3) land in the branch locus of π (5.9.4).

Degeneration type of (X, δ)	Degeneration type of $(Q, \sigma, \nu, \nu\sigma)$	(C, μ)	(C', μ')	A
I : $\bar{\varepsilon} = \delta$	$\varepsilon = \nu$	nonsingular trigonal	nodal, ∂^{I}	\mathcal{J}_4
II : $\bar{\varepsilon} \neq \delta, \langle \bar{\varepsilon}, \delta \rangle = 0$	$(\varepsilon, \sigma, \nu)$ rank 3 isotropic subgroup	nodal, ∂^{II}	nodal, ∂^{II}	$\partial\bar{\mathcal{A}}_4$
III : $\langle \bar{\varepsilon}, \delta \rangle \neq 0$	$\langle \varepsilon, \sigma \rangle = 0$ $\langle \varepsilon, \nu \rangle \neq 0$	nonsingular, has vanishing thetanull \mathcal{L} , $\mathcal{L}(\mu)$ even	nonsingular, has vanishing thetanull \mathcal{L}' , $\mathcal{L}'(\mu')$ even	θ_{null}
IV : nonsingular	$\langle \varepsilon, \nu \rangle = 0$ $\langle \varepsilon, \sigma \rangle \neq 0$	nodal, ∂^{II}	nonsingular, has vanishing thetanull \mathcal{L}' , $\mathcal{L}'(\mu')$ odd	\mathcal{A}_4

(5.13) Degenerations in \mathcal{R}^2Q^+ .

We have just described the universe as seen by a degenerating cubic threefold. From the point of view of a degenerating plane quintic, there are a few more possibilities though they lead to no new components. We retain the notation: $Q, \nu, \sigma, \varepsilon$ etc.

0. ε cannot equal σ , since ε is even, σ odd.

I. $\varepsilon = \nu$ reproduces case I of (5.12), as does:

I'. $\varepsilon = \nu\sigma$.

II. Excluding the above, ν, σ, ε generate a subgroup of rank 3. If it is isotropic, we are in case II above.

III. If $\langle \varepsilon, \sigma \rangle = 0$ but $\langle \varepsilon, \nu \rangle = \langle \varepsilon, \nu\sigma \rangle \neq 0$, we're in case III.

The only new cases are thus:

IV. $\langle \varepsilon, \nu \rangle = 0 \neq \langle \varepsilon, \sigma \rangle$, or :

IV.' $\langle \varepsilon, \nu\sigma \rangle = 0 \neq \langle \varepsilon, \sigma \rangle$,

which is the same as IV after exchanging C, C' .

In case IV, we find:

- X is non-singular, in fact any X can arise. What is special is the line $\ell \in F(X)$ corresponding to Q : it is contained in a plane which is tangent to X along another line, ℓ' .
- (Q, ν) is a ∂^{II} degeneration, so C is nodal, and (C, μ) is a ∂^{II} degeneration.
- On the other hand, $(Q, \nu\sigma)$ is ∂^{III} , so C' is non-singular, and has a vanishing theta null \mathcal{L}' (corresponding, as before, to the node of Q).
- This time though, $\mathcal{L}'(\mu')$ is odd, so $A \in \mathcal{A}_4$ does not inherit a vanishing theta null. In fact, any $A \in \mathcal{A}_4$ arises from a singular quintic with degeneration of type IV.

So far, we found three loci in $\bar{\mathcal{A}}_4$ which are related to nodal cubics:

$$\begin{aligned} \mathcal{P} \circ \kappa^{-1}(\partial^{\text{I}}\mathcal{RC}^+) &\subset \mathcal{J}_4 \\ \bar{\mathcal{P}} \circ \kappa^{-1}(\partial^{\text{II}}\mathcal{RC}^+) &\subset \partial\bar{\mathcal{A}}_4 \\ \mathcal{P} \circ \kappa^{-1}(\partial^{\text{III}}\mathcal{RC}^+) &\subset \theta_{\text{null}} \end{aligned}$$

We are now going to study, one at a time, the fibers of \mathcal{P} above generic points in these three loci. We note that related results have recently been obtained by Izadi. In a sense, her results are more precise: she knows (cf. Remark 5.4) that χ is an isomorphism on the open

complement \mathcal{U} of a certain 6-dimensional locus in \mathcal{A}_4 . In [I] she shows that for $A \in \mathcal{U}$, $\chi(A)$ is singular if and only if

$$A \in \mathcal{J}_4 \cup \theta_{\text{null}}.$$

Her description of the cubic threefold corresponding to $A \in \mathcal{J}_4$ complements the one we give below. In general her techniques, based on Γ_{00} , are very different than our degeneration arguments.

(5.14) Jacobians

Theorem 5.14 Let $B \in \mathcal{M}_4$ be a general curve of genus 4, and let $(X, \delta) = \chi(J(B))$.

- (1) X is the nodal cubic threefold corresponding to B (5.11).
- (2) $(X, \delta) \in \partial^I$, so $F(\widetilde{X})$ is reducible, each component is isomorphic to S^2B .
- (3) Let (Q, σ, ν) be the plane quintic with rank-2 isotropic subgroup corresponding to some $\ell \in F(X)$. Then Q is nodal, with trigonal normalization T , ν is the vanishing cycle, and $(Q, \sigma) = (Q, \nu\sigma) \in \partial^{II}$.
- (4) $\bar{\mathcal{P}}^{-1}(J(B))$ is isomorphic to $F(\widetilde{X})$. The component corresponding to ν (respectively $\nu\sigma$) consists of trigonal curves $T_{p,q}$ (respectively Wirtinger covers of singular curves $S_{p,q}$), $(p, q) \in S^2B$.
- (5) The tetragonal construction takes both $S_{p,q}$ and $T_{p,q}$ to $S_{r,s}$ and $T_{r,s}$ if and only if $p + q + r + s$ is a special divisor on B . The involution λ exchanges $S_{p,q}, T_{p,q}$.
- (6) Any two objects in $\bar{\mathcal{P}}^{-1}(J(B))$ can be connected by a sequence of two tetragonal moves (generally, in 10 ways).

Proof

Since at least some of these results are needed for the proof of (5.8)(4), we do not use Theorem (5.3). For $(p, q) \in S^2B$, we consider:

- $\widetilde{T}_{p,q} \rightarrow T_{p,q}$, the trigonal double cover associated by the trigonal construction to B with the g_4^1 given by $\omega_B(-p-q)$.
- $\widetilde{S}_{p,q} \rightarrow S_{p,q}$, the Wirtinger cover of $S_{p,q} := B/(p \sim q)$. (When $p = q$, this specializes to $B \cup_p R$, where R is a nodal rational curve in which p is a non singular point.)

These objects are clearly in $\bar{\mathcal{P}}^{-1}(J(B))$. Beauville's list ([B1], (4.10)) shows that they exhaust the fiber. This proves part (4). Now clearly κ , as defined in (5.1.6), takes any of these objects to our (X, δ) ; so the analysis in (5.12)(I) applies, proving (1)-(3). (Note: this already suffices to complete the proof of (5.8)(4)!)

Let $r + s + t + u$ be an arbitrary divisor in $|\omega_B(-p - q)|$. Projection of $\Phi(B)$ from the chord $\overline{\Phi(t), \Phi(u)}$ gives (the general) g_4^1 on $S_{p,q}$. The tetragonal construction takes this to the curves $T_{t,u}$ and $S_{r,s}$. (The situation is that of (2.15.2).)

On $T_{p,q}$ there are two types of g_4^1 's, of the form $\mathcal{L}(x)$ and $\omega \otimes \mathcal{L}^{-1}(-x)$, where \mathcal{L} is the trigonal bundle and $x \in T_{p,q}$. Now x corresponds to a (2,2) partition, say $\{\{r, s\}, \{t, u\}\}$, of some divisor in $|\omega_B(-p - q)|$. The tetragonal construction, applied to $\mathcal{L}(x)$, yields the curves $S_{r,s}$ and $S_{t,u}$; while when applied to $\omega \otimes \mathcal{L}^{-1}(-x)$, it gives $T_{r,s}$ and $T_{t,u}$. Altogether, this proves (5). We conclude with:

Lemma 5.14.7 Given any $p, q, r, s \in B$, there are points $t, u \in B$ (in general, 5 such pairs) such that both $p + q + t + u, r + s + t + u$ are special.

Proof

Let α, β be the maps of degree 4 from B to \mathbf{P}^1 given by $|\omega_B(-p - q)|, |\omega_B(-r - s)|$. Then

$$\alpha \times \beta : B \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

exhibits B as a curve of type (4,4) on a non-singular quadric surface, hence the image has arithmetic genus $(4 - 1)^2 = 9)4 = g(B)$, so there must be (in general, 5) singular points; these give the desired pairs (t, u) .

QED

(5.15) The Boundary.

The results in this case were obtained by Clemens [C2]. A general point A of the boundary $\partial \bar{\mathcal{A}}_4$ of a toroidal compactification $\bar{\mathcal{A}}_4$ is a \mathbf{C}^* -extension of some $A_0 \in \mathcal{A}_3$. The extension data is given by a point a in the Kummer variety $A_0/(\pm 1)$.

Given $a \in A_0$, consider the curve

$$\tilde{B} = \tilde{B}_a := \Theta \cap \Theta_a \subset A_0$$

(where $x \in \Theta_a \Leftrightarrow x+a \in \Theta$), and its quotient $B = B_a$ by the involution $x \mapsto -a - x$. We have

$$(B, \tilde{B}) \in \mathcal{R}_4$$

and

$$\mathcal{P}(B, \tilde{B}) \approx A_0.$$

The pair (B, \tilde{B}) does not change (up to isomorphism) when a is replaced by $-a$.

Theorem 5.15 ([C2]) Let $A \in \partial\tilde{\mathcal{A}}_4$ be the \mathbf{C}^* -extension of $A_0 \in \mathcal{A}_3$, a generic PPAV, determined by $\pm a \in A_0$. Let $(X, \delta) = \chi(A)$.

- (1) X is the nodal cubic threefold corresponding to $B = B_a$.
- (2) $(X, \delta) \in \partial^{\text{II}}$, so $F(\tilde{X})$ is the etale double cover of $F(X)$ with normalization $S^2\tilde{B}/\iota$, as in (5.12.II).
- (3) The corresponding quintics Q are nodal; all three of $\sigma, \nu, \nu\sigma$ are of type ∂^{II} .
- (4) $\overline{\overline{\mathcal{P}}}^{-1}(A)$ is isomorphic to $F(\tilde{X})$, and consists of ∂^{II} -covers (C, \tilde{C}) whose normalizations (at one point) are of the form (B_b, \tilde{B}_b) for $b = b_1 - b_2, b_1, b_2 \in \psi(\tilde{B})$.

Proof

Clearly $\overline{\overline{\mathcal{P}}}^{-1}(A) \subset \partial^{\text{II}}\overline{\overline{\mathcal{R}}}_5$, so consider a pair $(C, \tilde{C}) \in \partial^{\text{II}}$, say

$$C = N/(p \sim q), \quad \tilde{C} = \tilde{N}/(p' \sim q', p'' \sim q'')$$

with $(N, \tilde{N}) \in \tilde{\mathcal{R}}_4$. Then $\overline{\overline{\mathcal{P}}}(C, \tilde{C})$ is a \mathbf{C}^* -extension of $P(N, \tilde{N})$, with extension data

$$\pm(\psi(p') - \psi(q')) \in \mathcal{P}(N, \tilde{N})/(\pm 1).$$

We see that $\overline{\overline{\mathcal{P}}}(C, \tilde{C}) = A$ if and only if

$$(5.15.5) \quad (N, \tilde{N}) \in \overline{\overline{\mathcal{P}}}^{-1}(A_0),$$

and:

$$(5.15.6) \quad \psi(p') - \psi(q') = a, \quad p', q' \in \tilde{N}.$$

Now, (5.15.5) says that (N, \tilde{N}) is taken, by its Abel-Prym map ψ , to (B_b, \tilde{B}_b) for some $b \in A_0$, and then (5.15.6) translates to:

which is equivalent to

$$b = b_1 - b_2, \quad b_1, b_2 \in \Theta \cap \Theta_a$$

(take $b_1 = a_2 + b$, $b_2 = a_2$). This proves (4), and everything else follows from what we have already seen.

QED

(5.16) Theta nulls

Let $A \in \mathcal{A}_4$ be a generic *PPAV* with vanishing thetanull, and (C, \tilde{C}) a generic element of $\mathcal{P}^{-1}(A)$. By [B1], Proposition (7.3), C has a vanishing thetanull. This implies that the plane quintic Q parametrizing singular quadrics through $\Phi(C)$ has a node, corresponding to the thetanull. The corresponding cubic threefold X is thus also nodal, and we are again in the situation of (5.11.III). I do not see, however, a more direct way of describing the curve B (or the cubic X) in terms of A .

(5.17) Pentagons and wheels.

In [V], Varley exhibits a two dimensional family of double covers $(C, \tilde{C}) \in \mathcal{R}_5$ whose Prym is the unique non-hyperelliptic *PPAV* $A \in \mathcal{A}_4$ with 10 vanishing thetanulls. The curves C involved are Humbert curves, and each of these comes with a distinguished double cover \tilde{C} . As an illustration of our technique, we work out the fiber of $\overline{\mathcal{P}}$ over A and the tetragonal moves on this fiber. This is, of course, a very special case of (5.12)(III) or (5.16).

We recall the construction of Humbert curves and their double covers. Start by marking 5 points $p_1, \dots, p_5 \in \mathbf{P}^1$. Take 5 copies L_i of \mathbf{P}^1 , and let E_i be the double cover of L_i branched at the 4 points p_j , $j \neq i$. Let

$$(5.17.1) \quad A := \coprod_{i=1}^5 L_i, \quad B := \coprod_{i=1}^5 E_i.$$

The pentagonal construction applied to

$$(5.17.2) \quad B \xrightarrow{g} A \xrightarrow{f} \mathbf{P}^1$$

(f is the forgetful map, of degree 5), yields a 32-sheeted branched cover $f_*B \rightarrow \mathbf{P}^1$ which splits, by (2.1.1), into 2 copies of the Humbert curve C , of degree 16 over \mathbf{P}^1 .

Let β_I , $I \subset S := \{1, \dots, 5\}$, be the involution of (5.17.2) which fixes A and acts non-trivially on E_i , $i \in I$. It induces an involution α_I on f_*B , hence on its quotient C . Let

$$G := \{ \alpha_I \mid I \subset S \} / \langle \alpha_S \rangle.$$

Then C is Galois over \mathbf{P}^1 , with group $G \approx (\mathbf{Z}/2\mathbf{Z})^4$. Let G_i , $1 \leq i \leq 5$, be the image in G of

$$\{\alpha_I | i \notin I, \#(I) = \text{even}\}.$$

Then

$$C/\alpha_i \approx C/G_i \approx E_i,$$

and the quotient map

$$E_i \approx C/\alpha_i \rightarrow C/G_i \approx E_i$$

becomes multiplication by 2 on E_i . In particular, the Humbert curve C has 5 bielliptic maps $h_i : C \rightarrow E_i$. The branch locus of h_i consists of the 8 points $x \in E_i$ satisfying $g(2x) = p_i$.

For ease of notation, set $E := E_5$, $p = p_5 \in \mathbf{P}^1$,

$$C \xrightarrow{h} E \xrightarrow{g} \mathbf{P}^1,$$

and

$$\{p^0, p^1\} := g^{-1}(p) \subset E.$$

Then for $j = 0, 1$, E has a natural double cover C^j , branched at the four points $\frac{1}{2}p^j$ and given by the line bundle $\mathcal{O}_E(2p^j)$. The fiber product

$$(5.17.3) \quad \tilde{C} := C^0 \times_E C^1$$

gives a Cartesian double cover of C .

Replacing E_5 by another E_i , we get an isomorphic double cover \tilde{C} . Here is an invariant description of this cover:

Let $p_{i,j} := L_i \cap f^{-1}(p_j) \in A$, and consider the curve

$$Q := A/(p_{i,j} \sim p_{j,i}, \quad i \neq j).$$

Then Q can be embedded in \mathbf{P}^2 as a pentagon, or completely reducible plane quintic curve: embed \mathbf{P}^1 as a non-singular conic, and take L_i to be the tangent line of the conic at p_i . We have two natural branched double covers of Q :

$$(5.17.4) \quad \tilde{Q}_\sigma := (\coprod_{i=1, \epsilon=0}^5 L_i^\epsilon)/(p_{i,j}^0 \sim p_{j,i}^1, \quad i \neq j)$$

$$(5.17.5) \quad \tilde{Q}_\nu := B/(\tilde{p}_{i,j} \sim \tilde{p}_{j,i}, \quad i \neq j),$$

where $\tilde{p}_{i,j} \in E_i$ is the unique (ramification) point above $p_{i,j} \in L_i$. We may think of \tilde{Q}_σ as a "totally ∂^I " degeneration, and of \tilde{Q}_ν as a "totally ∂^{III} " degeneration. We then find:

(5.17.6) $(Q, \tilde{Q}_\nu) \in \overline{\mathcal{RQ}}^+$ is the quintic double cover corresponding to the

Humbert curve $C \in \mathcal{M}_5$.

(5.17.7) The double cover \tilde{Q}_σ of Q corresponds, via (1.4.2), to the double cover \tilde{C} of C .

We note that \tilde{Q}_σ is itself an odd cover, so it corresponds to some (singular) cubic threefold. A moment's reflection shows that this must be Segre's cubic threefold Y which we have already met in (4.8). Indeed, the Fano surface $F(Y)$ consists of the six rulings R_i , $0 \leq i \leq 5$, plus the 15 dual planes $\Pi_{i,j}^*$ of lines in $\Pi_{i,j}$ (notation of (4.8)). We see that:

(5.17.8) The discriminant of projection of Y from a line $\ell \in R_i$ is a plane pentagon Q , with its double cover \tilde{Q}_σ as above.

The other covers, \tilde{Q}_σ , fit together to determine a point $(Y, \delta) \in \overline{\mathcal{RC}}^+$:

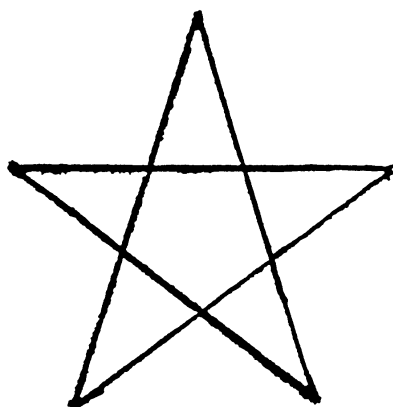
(5.17.9) $(Y, \delta) = \kappa(C, \tilde{C}),$

for any Humbert cover (C, \tilde{C}) . The tetragonal construction takes any (Q, \tilde{Q}_σ) to any other (in two steps), so we recover Varley's theorem:

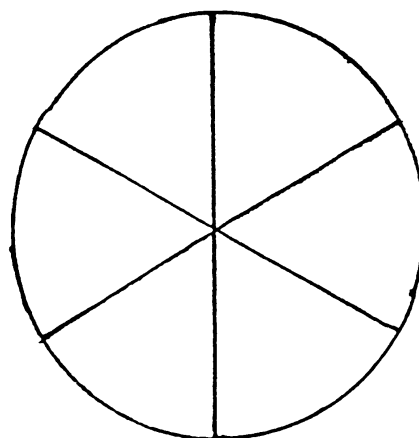
(5.17.10) $A := \mathcal{P}(C, \tilde{C}) \in \mathcal{A}_4$ is independent of the Humbert cover (C, \tilde{C}) .

But this is not the complete fiber: we have only used one of the two component types of $F(Y)$. We note:

(5.17.11) The discriminant of projection of Y from a line $\ell \subset \Pi_{ij}$ consists of a conic plus three lines meeting at a point; the double cover is split.



pentagon



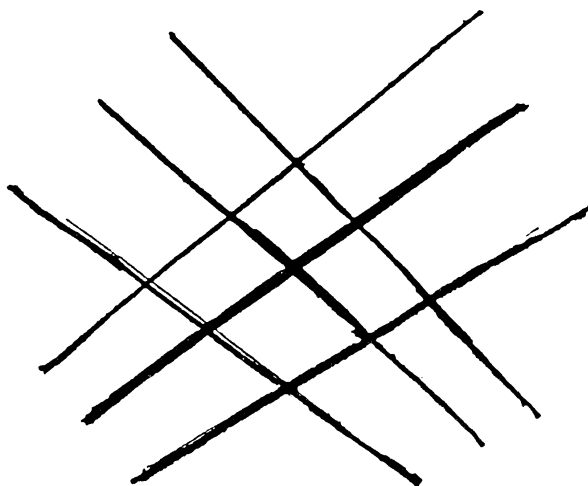
wheel

Consider a tritangent plane, meeting Y in lines $\ell_i \in R_i$, $\ell_j \in R_j$, and $\ell_{ij} \in \Pi_{i,j}^*$. It corresponds to a tetragonal construction involving two pentagons and a wheel. The other kind of tritangent plane intersects

Y in lines $\ell_{ij} \in \Pi_{ij}^*$, $\ell_{kl} \in \Pi_{kl}^*$, $\ell_{mn} \in \Pi_{mn}^*$, where $\{i, j, k, l, m, n\} = \{0, 1, 2, 3, 4, 5\}$; the tetragonal construction then relates three wheels.

Theorem 5.18 Let $A \in \mathcal{A}_4$ be the non-hyperelliptic PPAV with 10 vanishing thetanulls.

- (1) $\chi(A)$ consists of the Segre cubic threefold Y , with its degenerate semi-period δ (5.17.9).
- (2) The corresponding curve $B \in \bar{\mathcal{M}}_4$ (5.11) consists of six \mathbf{P}^1 's:



- (3) The Fano surface $F(Y)$ consists of the 6 rulings R_i ($0 \leq i \leq 5$) and the 15 dual planes $\Pi_{i,j}^*$. The plane quintics are pentagons, for $\ell \in R_i$, and wheels, for $\ell \in \Pi_{i,j}^*$, all with split covers σ (5.17.4, 5.17.11). (The ν covers are branched over all the double points.)
- (4) The fiber $\bar{\mathcal{P}}^{-1}(A)$ is contained in the fixed locus of the involution $\lambda: \bar{\mathcal{R}}_5 \rightarrow \bar{\mathcal{R}}_5$ (5.1.6), so it is a quotient of $F(Y)$.
- (5) $\bar{\mathcal{P}}^{-1}(A)$ consists of two components:
 - Humbert double covers $\tilde{C} \rightarrow C$ (5.17.3).
 - Allowable covers $\tilde{X}_0 \cup \tilde{X}_1 \rightarrow X_0 \cup X_1$, where X_0, X_1 are elliptic, meeting at their 4 points of order 2.

All of this follows from our previous analysis, except (5). The new, allowable, covers are obtained by applying Corollary (3.7), with $n = 3$, to the Cartesian cover $\tilde{C} \rightarrow C$ in (5.17.3). It is also easy to see that the plane quintic parametrizing singular quadrics through the canonical curve $\Phi(X_0 \cup X_1)$ is a wheel, and vice versa, that the generalized Prym of any wheel (with its ∂^{III} -cover) is the generalized Jacobian $J(X_0 \cup X_1)$ of such a curve. Thus every line in $F(Y)$ is accounted for, so we have the complete fiber $\bar{\mathcal{P}}^{-1}(A)$.

QED

§6 Other genera

For $g \leq 4$, it is relatively easy to describe the fibers of $\mathcal{P} : \bar{\mathcal{R}}_g \rightarrow \mathcal{A}_{g-1}$. Indeed, every curve in \mathcal{M}_g is trigonal, and every $A \in \mathcal{A}_{g-1}$ is a Jacobian (of a possibly reducible curve), so the situation is completely controlled by Recillas' trigonal construction. Similar results can be obtained, for $g \leq 3$, by using Masiewicki's criterion [Ma].

(6.1) $g = 1$. Here $\bar{\mathcal{P}}$ sends $\bar{\mathcal{R}}_1 \approx \mathbf{P}^1$ to \mathcal{A}_0 (= a point). The of $\bar{\mathcal{P}}, \mathcal{P}$ are then $\mathbf{P}^1, \mathbf{C}^*$ respectively.

(6.2) $g = 2$. All curves of genus 2 are hyperelliptic, and all covers are Cartesian (3.2). An element of \mathcal{R}_2 is thus given by 6 points in \mathbf{P}^1 , with 4 of them marked, modulo $\mathbf{PGL}(2)$; an element E of \mathcal{A}_1 is given by 4 points of \mathbf{P}^1 modulo $\mathbf{PGL}(2)$; and \mathcal{P} forgets the 2 unmarked points. The fiber of \mathcal{P} is thus rational; it can be described as S/G where

$$S := S^2 \left(\mathbf{P}^1 \setminus (4 \text{ points}) \right) \setminus (\text{diagonal})$$

and $G \approx (\mathbf{Z}/2\mathbf{Z})^2$ is the Klein group, whose action on S is induced from its action on \mathbf{P}^1 permuting the 4 marked points.

We note that S is \mathbf{P}^2 minus a conic C and four lines L_i tangent to it. To compactify it we add:

- a ∂^I cover for each point of $C \setminus \cup L_i$,
- a ∂^{III} cover for each point of $L_i \setminus C$, and
- an "elliptic tail" cover [DS, IV 1.3] for each point in the exceptional divisor obtained by blowing up one of the points $L_i \cap C$. (The limiting double cover obtained is

$$(E_0 \amalg E_1)/ \approx \longrightarrow E/ \sim$$

where \sim places a cusp at one of the four marked points p_i on E and \approx places a tacnode above it. These curves are unstable, and the family of elliptic-tail covers gives their stable models, each elliptic tail being blown down to the cusp.)

The resulting \bar{S} is \mathbf{P}^2 with 4 points in general position blown up, and the compactified fiber is \bar{S}/G , or \mathbf{P}^2/G with one point blown up.

(6.3) $g = 3$. Fix $A \in \mathcal{A}_2$. The Abel-Prym map sends pairs $(C, \tilde{C}) \in \mathcal{P}^{-1}(A)$ to curves $\psi(\tilde{C})$ in the linear system $|2\Theta|$ on A , uniquely defined modulo translation by the group $G = A_2 \approx (\mathbf{Z}/2\mathbf{Z})^4$. The fiber is therefore, birationally, the quotient \mathbf{P}^3/G . Since some curves in $|2\Theta|$

are not stable, some blowing up is required to obtain the biregular model of $\bar{\mathcal{P}}^{-1}(A)$. This is carried out in [Ve]. The quotient \mathbf{P}^3/G is identified with Siegel's modular threefold, or the minimal compactification $\bar{\mathcal{A}}_2^{(2)}$ of the moduli space of $PPAV$'s with level-2 structure. To obtain $\bar{\mathcal{P}}^{-1}(A)$, Verra shows that we need to blow $\bar{\mathcal{A}}_2^{(2)}$ up at a point A' , corresponding to a level-2 structure on A itself, and along a rational curve. The 2 exceptional divisors then parametrize hyperelliptic and elliptic-tail covers, respectively.

(6.4) $g = 4$.

As we noted in (5.15), the fiber $\mathcal{P}^{-1}(A)$, $A \in \mathcal{A}_3$, consists of covers (B_a, \tilde{B}_a) , $a \in A/(\pm 1)$:

$$\tilde{B}_a = \Theta \cap \Theta_a, \quad B_a = \tilde{B}_a/(x \sim (-a - x)).$$

The fiber is thus (birationally) the Kummer variety $A/(\pm 1)$.

(6.5) $g \geq 7$.

In this case, it was proved in [FS], [K], and [W], that \mathcal{P} is generically injective. The results in §3 show that it is never injective: on the hyperelliptic loci there are positive-dimensional fibers, and various coincidences occur on the bielliptic loci. In [D1] we conjectured:

Conjecture 6.5.1 Any two objects in a fiber of \mathcal{P} are connected by a sequence of tetragonal constructions.

We state this for \mathcal{P} , rather than $\bar{\mathcal{P}}$, since various other phenomena can contribute to non-trivial fibers at the boundary. For example, all fibers of $\bar{\mathcal{P}}$ on ∂^I are two-dimensional. On the other hand, from the local pictures (2.14) it is clear that the tetragonal construction can take a nonsingular curve to a singular one. In fact proposition (3.8) shows that it is possible for two objects in \mathcal{R}_g to be tetragonally related through an intermediate object of $\partial\mathcal{R}_g$, so some care must be taken in clarifying which class of tetragonal covers should be allowed. The conjecture is consistent with our results for $g \leq 6$. For $g \geq 13$, Debarre [Deb2] proved it for curves which are neither hyperelliptic, trigonal, or bielliptic. Naranjo [N] extended this to generic bielliptics, $g \geq 10$. The following result was communicated to me by Radionov:

Theorem 6.5.2 [Ra] For $g \geq 7$, $\mathcal{R}_g^{\text{Tet}}$ is an irreducible component of the noninjectivity locus of the Prym map, and for generic $(C, \tilde{C}) \in \mathcal{R}_g^{\text{Tet}}$, $\mathcal{P}^{-1}(\mathcal{P}(C, \tilde{C}))$ consists precisely of three tetragonally related objects.

REFERENCES

- [ACGH] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, *Geometry of algebraic curves*, Vol. I, Springer-Verlag, New York (1985).
- [B1] A. Beauville, *Prym varieties and the Schottky problem*, Inv. Math. 41 (1977), 149-196.
- [B2] A. Beauville, *Sous-variétés spéciales des variétés de Prym*, Compos. Math. 45, 357-383 (1982).
- [C1] H. Clemens, *Double Solids*, Advances in Math. 47 (1983) pp. 107-230.
- [C2] H. Clemens, *The fiber of the Prym map and the period map for double solids, as given by Ron Donagi*, U. of Utah, preprint.
- [CG] H. Clemens, P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. Math. 95 (1972), 281-356.
- [D1] R. Donagi, *The tetragonal construction*, AMS Bull. 4 (1981), 181-185.
- [D2] R. Donagi, *The unirationality of \mathcal{A}_5* , Ann. Math. 119 (1984), 269-307.
- [D3] R. Donagi, *Big Schottky*, Inv. Math. 89 (1987), 569-599.
- [D4] R. Donagi, *Non-Jacobians in the Schottky loci*, Ann. of Math. 126 (1987), 193-217.
- [D5] R. Donagi, *The Schottky problem*, in: *Theory of Moduli*, LNM 1337, Springer-Verlag (1988), 84-137.
- [D6] R. Donagi, *On the period map for Clemens' double solids*, preprint.
- [DS] R. Donagi, R. Smith, *The structure of the Prym map*, Acta Math. 146 (1981) 25-102.
- [Deb1] O. Debarre, *Variétés de Prym, conjecture de la trisécante et ensembles d'Andreotti et Mayer*, Univ. Paris Sud, Thesis, Orsay (1987).
- [Deb2] O. Debarre, *Sur les variétés de Prym des courbes tétraogonales*, Ann. Sci. E.N.S. 21 (1988), 545-559.
- [Dem] M. Demazure, *Seminaire sur les singularités des surfaces*. LNM 777, Springer-Verlag (1980), 23-69.
- [FS] R. Friedman, R. Smith, *The generic Torelli Theorem for the Prym map*, Inv. Math. 67 (1982), 473-490.

- [vG] B. van Geemen, *Siegel modular forms vanishing on the moduli space of curves*, Inv. Math. 78 (1984), 329-349.
- [vGvdG] B. van Geemen, G. van der Geer, *Kummer varieties and the moduli space of curves*, Am. J. of Math. 108 (1986), 615-642.
- [vGP] B. van Geemen, E. Previato, *Prym varieties and the Verlinde formula*, MSRI preprint, May 1991.
- [I] E. Izadi, *On the moduli space of four dimensional principally polarized abelian varieties*, Univ. of Utah Thesis, June 1991.
- [K] V. Kanev, *The global Torelli theorem for Prym varieties at a generic point*, Math. USSR Izvestija 20 (1983), 235-258.
- [M1] D. Mumford, *Theta characteristics on an algebraic curve*, Ann. Sci. E.N.S. 4 (1971), 181-192.
- [M2] D. Mumford, *Prym varieties I. Contributions to Analysis*, 325-350, New York, Acad. Press, 1974.
- [Ma] L. Masiewicki, *Universal properties of Prym varieties with an application to algebraic curves of genus five*, Trans. Amer. Math. Soc. 222 (1976), 221-240.
- [N] J. C. Naranjo, *Prym varieties of bi-elliptic curves*, Univ. de Barcelona preprint no. 65, June 1989.
- [P] S. Pantazis, *Prym varieties and the geodesic flow on $SO(n)$* , Math. Ann. 273 (1986), 297-315.
- [R] S. Recillas, *Jacobians of curves with a g_4^1 are Prym varieties of trigonal curves*, Bol. Soc. Math. Mexicana 19 (1974), 9-13.
- [Ra] D. Radionov, *letter*.
- [SR] J.G. Semple, L. Roth, *Introduction to Algebraic Geometry*, Oxford U. Press, 1949.
- [SV] R. Smith, R. Varley, *Components of the locus of singular theta divisors of genus 5*, LNM 1124, Springer-Verlag (1983), 338-416.
- [T] A. Tjurin, *Five lectures on three dimensional varieties*, Russ. Math. Surv. 27 (1972).
- [V] R. Varley, *Weddle's surfaces, Humbert's curves, and a certain 4-dimensional abelian variety*, Amer. J. Math. 108 (1986), 931-952.
- [Ve] A. Verra, *The fibre of the Prym map in genus three*, Math. Ann. 276 (1987), 433-448.
- [W] G. Welters, *Recovering the curve data from a general Prym variety*, Amer. J. of Math. 109 (1987), 165-182.