Prymn Varieties I

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INTRODUCTION

This paper gives the first steps in a purely algebraic version (in all characteristics except two) of the Riemann–Prym–Wirtinger–Schottky–Jung theory of double coverings of one curve (or compact Riemann surface) over another. It also tries to incorporate some of the interesting generalizations of this theory in the thesis of Fay [4]. The basic idea is this:

$$\pi: \quad \tilde{C} \longrightarrow C$$

is a double covering, where C and $\tilde{C}$ are nonsingular complete curves with Jacobians J and $\tilde{J}$. The involution $\iota: \tilde{C} \longrightarrow \tilde{C}$ interchanging sheets extends to $\iota: \tilde{J} \longrightarrow \tilde{J}$, and up to some points of order two, $\tilde{J}$ splits into an even part J and an odd part P, the *Prym variety*. The Prym P has a natural polarization on it, but only in two cases—where π has zero or two branch points—do we get a unique *principal* polarization on P, hence a theta divisor $\Xi \subset P$. This is discussed in the first part of this paper (Sections 1–3).

The surprise comes, however, on a closer analysis of the relations between the theta divisors $\Theta \subset J$ and $\tilde{\Theta} \subset \tilde{J}$: It turns out that they are related in a much tighter way than would be expected from looking only at the configuration of Abelian varieties and homomorphisms present. In the case of zero or two branch points this leads finally to identities relating (J, Θ) and (P, Ξ) discovered by Schottky and Jung [15] (cf. also Riemann [13] and Farkas and Rauch [3]). The point is that the existence of *any* (P, Ξ) standing in this relation to (J, Θ) means that if $g \geqslant 4$, J is not the most general Abelian variety of dimension g! Unfortunately, an efficient method of translating this into an equivalent polynomial identity on the theta nulls of J is only known at present for $g = 4$. These matters are discussed in the second part of this paper (Sections 4 and 5).

In the other direction, the curves C and $\tilde{C}$ and their geometry can be used to compute things about P. The importance of this is that it is usually quite hard to make detailed computations on the geometry of the theta divisor in a general principally polarized n-dimensional Abelian variety [which has $\frac{1}{2}n(n+1)$ moduli]; those which are Jacobians of curves of genus n (with $3n - 3$ moduli) are much better understood. However, by taking the Pryms for unramified double coverings $\tilde{C} \longrightarrow C$, genus

$C = n + 1$, we get a bigger family of principally polarized n-dimensional Abelian varieties which can be closely studied (depending on $3n$ moduli). For instance, for $n = 2, 3$ a generic principally polarized Abelian variety is a Jacobian; and according to Wirtinger, for $n = 4, 5$ a generic principally polarized Abelian variety appears to be a Prym but not of course a Jacobian. Moreover Pryms occur sometimes as the Intermediate Jacobians of unirational but not rational 3-folds (cf. Clemens and Griffiths [2], and Murre [12]). In the final part of the paper (Sections 6 and 7) with these applications in mind we compute the dimension of singular locus of the theta divisor in a Prym using results of Martens [8].

In a sequel to this paper we would like to discuss (a) how close the Schottky–Jung identities come to characterizing Jacobians among all Abelian varieties, and (b) ways of utilizing the Schottky–Jung identities in the two-branch-point case.

NOTATIONS

k	the algebraically closed ground field: always of char. $\neq 2$
$\mathbb{R}(X)$	field of rational functions on a variety X
$\mathrm{Pic}(X)$	group of divisor classes, line bundles, or invertible sheaves on a variety X
$\mathrm{Pic}^0(X)$	connected component of $0 \in \mathrm{Pic}(X)$
$\hat{X}$	another notation for $\mathrm{Pic}^0(X)$ if X is an Abelian variety (called the "dual" Abelian variety)
$\lambda_D\colon X \longrightarrow \hat{X}$	the homomorphism $x \longmapsto$ [divisor class of $T_x^{-1}D - D$], where D is a divisor on an Abelian variety X

A *polarization* of an Abelian variety X is a homomorphism $\lambda\colon X \longrightarrow \hat{X}$ such that $\lambda = \lambda_D$ for some ample D: in this case D is determined modulo $\mathrm{Pic}^0(X)$; λ is a *principle polarization* if λ is also an isomorphism, in which case $\lambda = \lambda_D$ for a positive ample D, unique up to a translation. (See my book [10] for a general reference for the facts on Abelian varieties.)

I

1. DOUBLE COVERINGS OF CURVES

The main object of our study is a morphism

$$\pi\colon \quad \tilde{C} \longrightarrow C,$$

where C and $\tilde{C}$ are nonsingular complete curves and π is of degree two, i.e., π is surjective and via π^*, $\mathbb{R}(\tilde{C})$ is a quadratic extension of $\mathbb{R}(C)$. In fact, in this case C has an open covering by affines $U_\alpha = \mathrm{Spec}\ R_\alpha$ such that $\pi^{-1}(U_\alpha) = \mathrm{Spec}\ S_\alpha$, where S_α is an R_α-algebra of the form

$$S_\alpha \cong R_\alpha[t_\alpha]/(t_\alpha{}^2 - \beta_\alpha), \qquad \beta_\alpha \in R_\alpha.$$

Or, sheaf-theoretically, we may put this in the equivalent form $\tilde{C} = \mathbf{Spec}(\mathscr{S})$, where $\mathscr{S}$ is a sheaf of $\mathcal{O}_C$ algebras of the form

$$\mathscr{S} \cong \mathcal{O}_C \oplus L$$

with L an invertible sheaf of $\mathcal{O}_C$ modules. Multiplication is given by

$$(a + l) \cdot (b + m) = (a \cdot b + \phi(l \otimes m), a \cdot m + b \cdot l),$$

$C = n + 1$, we get a bigger family of principally polarized n-dimensional Abelian varieties which can be closely studied (depending on $3n$ moduli). For instance, for $n = 2, 3$ a generic principally polarized Abelian variety is a Jacobian; and according to Wirtinger, for $n = 4, 5$ a generic principally polarized Abelian variety appears to be a Prym but not of course a Jacobian. Moreover Pryms occur sometimes as the Intermediate Jacobians of unirational but not rational 3-folds (cf. Clemens and Griffiths [2], and Murre [12]). In the final part of the paper (Sections 6 and 7) with these applications in mind we compute the dimension of singular locus of the theta divisor in a Prym using results of Martens [8].

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$$(a + l) \cdot (b + m) = (a \cdot b + \phi(l \otimes m), a \cdot m + b \cdot l),$$

a, b sections of $\mathcal{O}_C$, l, m sections of L, for some

$$\phi\colon\ L^2 \xrightarrow{\approx} \mathcal{O}_C\Big(-\sum_{i=1}^{m} P_i\Big) \subset \mathcal{O}_C.$$

Then the zeros of β_α, or equivalently the points P_i where $\phi(L^2) \neq \mathcal{O}_C$, are the branch points of π. Since $\tilde{C}$ is nonsingular, they are all simple zeros (equivalently, $\sum P_i$ has no multiple points); and because

$$\#\ \text{branch points} = -\deg L^2 = 2(-\deg L),$$

there are an even number $m = 2n$ of them.

Let J and $\tilde{J}$ be the Jacobians of C and $\tilde{C}$: By definition, we take this to mean

$$J = \mathrm{Pic}^0(C), \qquad \tilde{J} = \mathrm{Pic}^0(\tilde{C}).$$

Now fix base points $x_0 \in C$, and $\tilde{x}_0 \in \tilde{C}$ such that $\pi(\tilde{x}_0) = x_0$. Then we get the Albanese mappings:

$$t\colon\ C \longrightarrow J \qquad \text{via} \qquad x \longmapsto \text{divisor class } (x - x_0)$$

and

$$\tilde{t}\colon\ \tilde{C} \longrightarrow \tilde{J} \qquad \text{via} \qquad x \longmapsto \text{divisor class } (\tilde{x} - \tilde{x}_0).$$

Moreover, define

$$\mathrm{Nm}\colon\ \tilde{J} \longrightarrow J$$

by either (a) the restriction of the map Nm,

$$\begin{array}{ccc} \tilde{J} \subset H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}^*) \cong & H^1(C, (\pi_* \mathcal{O}_{\tilde{C}})^*) \\ & \downarrow \mathrm{Nm} \\ J \hookrightarrow & H^1(C, \mathcal{O}_C^*) \end{array}$$

or (b) the induced map on divisor classes given on divisors by $\mathfrak{A} \longmapsto \pi(\mathfrak{A})$ ($\mathfrak{A}$ a divisor on $\tilde{C}$). Then we get a commutative diagram:

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\tilde{t}} & \tilde{J} \\ \pi \downarrow & & \downarrow \mathrm{Nm} \\ C & \xrightarrow{t} & J. \end{array}$$

Now this diagram defines a second by applying the functor Pic^0:

$$\begin{array}{ccc} \tilde{J} = \mathrm{Pic}^0(\tilde{C}) & \xleftarrow{\tilde{t}^*} & \mathrm{Pic}^0(\tilde{J}) \underset{\mathrm{def}}{=} \hat{\tilde{J}} \\ \uparrow \pi^* & & \uparrow \mathrm{Nm}^* \text{ or } \widehat{\mathrm{Nm}} \\ J = \mathrm{Pic}^0(C) & \xleftarrow{t^*} & \mathrm{Pic}^0(J) \underset{\mathrm{def}}{=} \hat{J} \end{array}$$

where $\hat{\tilde{J}}$ and $\hat{J}$ are the "duals" of $\tilde{J}$ and J, respectively. By the standard theory of Jacobians, t^* and $\tilde{t}^*$ are isomorphisms and, in fact, if $\Theta \subset J$, $\tilde{\Theta} \subset \tilde{J}$ are the theta divisors, then

$$(t^*)^{-1} = -\lambda_{\Theta}, \qquad (\tilde{t}^*)^{-1} = -\lambda_{\tilde{\Theta}}$$

(where for any divisor D on an Abelian variety X, $\lambda_D: X \longrightarrow \hat{X}$ is the homomorphism given by $x \longrightarrow$ [divisor class $T_x^{-1}(D) - D$]). Thus the principally polarized Abelian varieties (J, Θ) and $(\tilde{J}, \tilde{\Theta})$ are related by two maps:

$$\pi^*: \ J \longrightarrow \tilde{J}, \qquad \text{Nm}: \ \tilde{J} \longrightarrow J$$

and the main result is that these have two properties:

(i) π^* and Nm are dual to each other:

$$\widehat{\text{Nm}} = \lambda_{\tilde{\Theta}} \cdot \pi^* \cdot \lambda_{\Theta}^{-1}, \qquad \hat{\pi}^* = \lambda_{\Theta} \cdot \text{Nm} \cdot \lambda_{\tilde{\Theta}}^{-1}.$$

(ii) $\text{Nm} \cdot \pi^*: J \longrightarrow J$ is multiplication by two.

Proof of (ii). If $\mathfrak{A}$ is a divisor class of degree zero on C, and α is the corresponding point of J, then $\pi^{-1}(\mathfrak{A})$ represents $\pi^*\alpha \in \tilde{J}$ and $\pi(\pi^{-1}\mathfrak{A})$ represents $\text{Nm}(\pi^*\alpha)$. But $\pi(\pi^{-1}\mathfrak{A}) = 2\mathfrak{A}$. Q.E.D.

Rather than studying in detail the implications of (i) and (ii) in this special case, it seems easier at this point to study such a situation in general, and afterward to specialize the study to the case of Jacobians.

2. A CONFIGURATION OF ABELIAN VARIETIES

Suppose (X, θ_X) and (Y, θ_Y) are two principally polarized Abelian varieties: Thus θ_X and θ_Y are positive divisors on X and Y, given only up to translations, however, such that λ_{θ_X} and λ_{θ_Y} are isomorphisms. (It is well known then that θ_X and θ_Y are ample and are the only positive divisors D such that $\lambda_D = \lambda_{\theta_X}$ or λ_{θ_Y}.)

DATA I. Suppose $\phi: X \longrightarrow Y$ is a homomorphism and assume that

$$\phi^*(\theta_Y) \ \text{ algebraically equivalent to } \ 2\theta_X \tag{2.1}$$

i.e., $\phi^*\theta_Y - 2\theta_X \in \text{Pic}^0(X)$. This is *equivalent* to saying

$$\lambda_{\phi^*(\theta_Y)} = 2\lambda_{\theta_X}, \tag{2.2}$$

hence (since $\lambda_{\phi^*D} = \hat{\phi} \cdot \lambda_D \cdot \phi$), it is equivalent to having the following diagram commute:

$$\begin{array}{ccc} Y & \xrightarrow{\lambda_{\theta_Y}} & \hat{Y} \\ {\scriptstyle\phi}\uparrow & & \downarrow{\scriptstyle\hat{\phi}.} \\ X & \xrightarrow[2\lambda_{\theta_X}]{} & \hat{X} \end{array} \tag{2.3}$$

Thus if we define $\psi\colon Y \longrightarrow X$ to be the dual $\lambda_{\theta_X}^{-1} \cdot \hat{\phi} \cdot \lambda_{\theta_Y}$ of ϕ, we get $\psi \cdot \phi =$ mult. by two: exactly the situation of Section 1.

I claim that all triples $((X, \theta_X), (Y, \theta_Y), \phi)$ satisfying (2.1)—call these Data I—and only such triples arise in the following way.

DATA II.

(i) (X, θ_X) is a principally polarized Abelian variety.

(ii) P and $\rho\colon P \longrightarrow \hat{P}$ is some Abelian variety and a polarization of P.

(iii) $H_0 \subset H_1 \subset X_2$ are subgroups of points of order two, and $\psi\colon H_1/H_0 \longrightarrow \ker \rho$ is an isomorphism.

These data should satisfy:

(iv) With respect to the skew-symmetric multiplicative pairings induced by the Riemann forms of θ_X and ρ

$$e_{2,X}\colon\ X_2 \times X_2 \longrightarrow \{\pm 1\}, \qquad e_\rho\colon\ \ker \rho \times \ker \rho \longrightarrow \{\pm 1\},$$

we have the following:

(a) $e_{2,X}(\alpha, \beta) = 1$, all $\alpha, \beta \in H_0$.

(b) $H_1 = H_0^{\perp}$, where $H_0^{\perp} = \{\alpha \in X_2 \mid e_{2,X}(\alpha, \beta) = 1, \text{ all } \beta \in H_0\}$.

(c) $e_\rho(\psi\alpha, \psi\beta) = e_{2,X}(\alpha, \beta)$, all $\alpha, \beta \in H_1$.

In this case we set $Y = X \times P/H$, where

$$H = \{(\alpha, \psi\alpha) \mid \alpha \in H_1\}$$

and let ϕ be the composition of canonical maps:

$$X \longrightarrow X \times P \longrightarrow Y.$$

Moreover, if $\sigma\colon X \times P \longrightarrow Y$ is the canonical map, then the polarization λ_{θ_Y} is determined by the requirement that the diagram

$$\begin{array}{ccc} X \times P & \xrightarrow{2\lambda_{\theta_X} \times \rho} & \hat{X} \times \hat{P} \\ \sigma \downarrow & & \uparrow \hat{\sigma} \\ Y & \xrightarrow{\lambda_{\theta_Y}} & \hat{Y} \end{array}$$

commutes.

In other words, we find that whenever one has such a ϕ, then up to a small group H of points of order two, Y and its polarization split into a product of two natural blocks, one being X and the other we call P—which in the case of curves will be the "Prym variety." Moreover, to tie the two types of data together, I claim that:

(v) $H_0 = \ker \phi$.

(vi) There is an involution ι on Y such that

$$P = \operatorname{Im}(1_Y - \iota) = \ker(1_Y + \iota)^0, \qquad \phi(X) = \operatorname{Im}(1_Y + \iota) = \ker(1_Y - \iota)^0.$$

In fact, if $\psi = \lambda_{\theta_X}^{-1} \cdot \hat{\phi} \cdot \lambda_{\theta_Y}\colon Y \longrightarrow X$, then

$$\ker(1_Y - \iota) = X \times P_2/H \cong \phi(X) \times (\mathbb{Z}/2\mathbb{Z})^{2b-2c}, \qquad \ker \psi = X_2 \times P/H \cong P \times (\mathbb{Z}/2\mathbb{Z})^{a-c}$$

[for a,b,c see (viii)].

(vii) If $\sigma\colon X \times P \longrightarrow Y$ is the canonical map and $\tau\colon Y \longrightarrow X \times P$ is the map $\tau(x) = (\psi x, x - \iota x)$, then

$$\sigma \cdot \tau = 2_Y, \qquad \tau \cdot \sigma = 2_{X \times P}.$$

(viii) If $\dim X = a$, $\dim Y = a + b$, $\# \ker \phi = 2^{a-c}$, then $\dim P = b$, $\# H_0 = 2^{a-c}$, $\# H_1 = 2^{a+c}$, $\# \ker \rho = 2^{2c}$, and $0 \leqslant c \leqslant \min(a, b)$.

Much of the verification of the equivalence here is straightforward, so we will run through only the first part.

Start with Data I. Define

$$P = \lambda_{\theta_Y}^{-1}(\ker \hat{\phi})^0,$$

and ν the number of components of $\lambda_{\theta_Y}^{-1}(\ker \hat{\phi})$. Via ϕ and the inclusion of P in Y, we get $\sigma\colon X \times P \longrightarrow Y$. Let $H = \ker \sigma$. Note that

$$(x, y) \in H \Longrightarrow \phi(x) + y = 0 \Longrightarrow \hat{\phi}(\lambda_{\theta_Y}(\phi(x))) = 0 \Longrightarrow 2x = 0 \Longrightarrow 2y = 0;$$

hence $H \subset X_2 \times P_2$. Since $H \cap (0) \times P_2 = (0) \times (0)$, there is a subgroup $H_1 \subset X_2$ and a homomorphism $\psi\colon H_1 \longrightarrow P_2$ such that

$$H = \{(\alpha, \psi\alpha) \mid \alpha \in H_1\}.$$

Also, if $H_0 = \ker \phi$, then $H_0 \subset H_1$, and ψ factors as $H_1/H_0 \hookrightarrow P_2$. Moreover, for all $y \in Y$, let

$$x = \lambda_{\theta_X}^{-1}(\hat{\phi}(\lambda_{\theta_Y}(y))).$$

Then

$$2y = \phi(x) + (2y - \phi(x))$$

and

$$\hat{\phi}(\lambda_{\theta_Y}(2y - \phi(x))) = 2\lambda_{\theta_X}(x) - \hat{\phi}(\lambda_{\theta_Y}(\phi(x))) = 0;$$

hence $\nu \cdot (2y - \phi(x)) \in P$. Therefore

$$2\nu \cdot y \in \phi(x) + P \subset \operatorname{Im} \sigma$$

and since Y is a divisible group, this implies that σ is surjective. Next, the polarization λ_{θ_Y} of Y "pulls back" to a polarization of $X \times P$ given by the composition:

$$X \times P \xrightarrow{\ \sigma\ } Y \xrightarrow{\ \lambda_{\theta_Y}\ } \hat{Y} \xrightarrow{\ \hat{\sigma}\ } \hat{X} \times \hat{P},$$

which may be considered as being given by a 2×2 matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \begin{array}{ll} \alpha\colon X \longrightarrow \hat{X}, & \beta\colon P \longrightarrow \hat{X}, \\ \gamma\colon X \longrightarrow \hat{P}, & \delta\colon P \longrightarrow \hat{P}. \end{array}$$

Also, because any polarization is symmetric, $\gamma = \hat{\beta}$. But by the very definition of P, the coefficient β is zero. So $\gamma = 0$, too, and the polarization splits. Note that by assumption (2.1) on Data I, $\alpha = 2\lambda_{\theta_X}$. Define ρ to be δ. Next, the fact that the polarization $(2\gamma_{\theta_X}, \rho)$ of $X \times P$ is a pullback of a principal polarization with respect to the isogeny σ is equivalent to the condition that $\ker \sigma$, as a subgroup of $\ker(2\lambda_{\theta_X}, \rho)$, is maximal isotropic for the skew-symmetric form of this polarization (cf. Mumford [10, Section 23]). Hence

$$H \subset X_2 \times \ker \rho$$

and if $(\alpha, \psi\alpha), (\beta, \psi\beta) \in H$, then

$$e_{2,X}(\alpha, \beta) \cdot e_\rho(\psi\alpha, \psi\beta) = 1.$$

This means that $\psi(H_1/H_0) \subset \ker \rho$ and ψ is "symplectic" in the sense of (iv)c. Moreover, counting orders, the maximality of H implies

$$(\# H_1)^2 = (\# H)^2 = \#(X_2 \times \ker \rho);$$

hence

$$\# H_1^{\perp} = \frac{\# X_2}{\# H_1} = \frac{\# H_1}{\# \ker \rho} \leqslant \frac{\# H_1}{\# \operatorname{Im} \psi} \leqslant \# H_0 .$$

Since $\ker \rho \subseteq H_1^{\perp}$, this implies that ψ maps H_1 onto $\ker \rho$ and that $H_0 = H_1^{\perp}$, hence $H_1 = H_0^{\perp}$. Thus we have Data II.

We leave it to the reader to check now that one can go backward from Data II to Data I and that for corresponding data, (v)–(viii) hold.

3. DEFINITION OF THE PRYM VARIETY

Returning to a covering $\pi: \tilde{C} \longrightarrow C$ and their Jacobians related by $\pi^*: J \longrightarrow \tilde{J}$, we see that $\tilde{J} \cong J \times P/H$. In this case there is an involution $\iota: \tilde{C} \longrightarrow \tilde{C}$ interchanging the two sheets above any point, which induces an involution $\iota: \tilde{J} \longrightarrow \tilde{J}$. Since for any divisor $\mathfrak{A}$ on $\tilde{C}$,

$$\pi^{-1}(\pi\mathfrak{A}) = \mathfrak{A} + \iota(\mathfrak{A}),$$

it follows that

$$\pi^*(\operatorname{Nm} x) = x + \iota(x), \qquad \text{all} \quad x \in \tilde{J}.$$

And since Nm is surjective, this also shows that

$$\iota(\pi^* y) = \pi^* y, \qquad \text{all} \quad y \in J.$$

Therefore $\iota = +1$ on $\pi^* J$ and $\iota = -1$ on ker Nm. Thus ι is precisely the involution introduced in (vi) of Section 2, and we find that

$$P \underset{\text{def}}{=} (\ker \operatorname{Nm})^0 = \ker(1_{\tilde{J}} + \iota)^0 = \operatorname{Im}(1_{\tilde{J}} - \iota),$$

i.e., P is the "odd" part of $\tilde{J}$, which we call the *Prym variety of* $\tilde{C}$ over C.

Let g = genus of C and let $2n$ = # of branch points. Then by Hurwitz's formula

$$\text{genus } \tilde{g} \text{ of } \tilde{C} = 2g + n - 1.$$

Therefore

$$\dim J = g, \qquad \dim \tilde{J} = 2g + n - 1, \qquad \dim P = g + n - 1.$$

To apply fully the theory of Section 2, we need only compute $\ker(\pi^*) \cong \{$div. classes $\mathfrak{A}$ on $C \mid \pi^{-1}\mathfrak{A} \equiv 0\}$. But

$$\pi^{-1}\mathfrak{A} \equiv 0 \Longrightarrow 2\mathfrak{A} = \pi\pi^{-1}\mathfrak{A} \equiv 0,$$

i.e., $\ker(\pi^*) \subset J_2$. If $\mathfrak{A}$ is any such divisor class, then $\mathfrak{A}$ defines an unramified double covering $\pi_{\mathfrak{A}}\colon C_{\mathfrak{A}} \longrightarrow C$ by "Kummer theory," i.e., $C_{\mathfrak{A}}$ is the normalization of C in $\mathbb{R}(C)(\sqrt{f})$, where $2\mathfrak{A} = (f)$, or

$$C_{\mathfrak{A}} = \mathbf{Spec}(\mathcal{O}_C \oplus \mathcal{O}_C(\mathfrak{A})), \qquad \text{mult. given by } \mathcal{O}_C(\mathfrak{A}) \times \mathcal{O}_C(\mathfrak{A}) \longrightarrow \mathcal{O}_C(2\mathfrak{A}) \cong \mathcal{O}_C.$$

Then

$$\pi^{-1}\mathfrak{A} \equiv 0 \iff \text{the double covering } C_{\mathfrak{A}} \times_C \tilde{C} \text{ of } \tilde{C} \text{ splits into two copies of } \tilde{C}$$

$$\iff \text{there is a morphism } f\colon \begin{array}{ccc} \tilde{C} & \xrightarrow{f} & C_{\mathfrak{A}} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_{\mathfrak{A}}} & \\ & C & \end{array}$$

and hence

$$\pi^{-1}\mathfrak{A} \equiv 0,\quad \mathfrak{A} \not\equiv 0 \iff \text{there is an isomorphism } f\colon \begin{array}{ccc} \tilde{C} & \xrightarrow{f} & C_{\mathfrak{A}} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_{\mathfrak{A}}} & \\ & C & \end{array}$$

This proves

Lemma. If π is ramified, $\ker \pi^* = (0)$. If π is unramified, hence $\tilde{C} = C_{\mathfrak{A}}$ for some $\mathfrak{A}$, then $\ker \pi^* = \{0, \mathfrak{A}\}$.

Combining this with the results of Section 2, we deduce the following.

Corollary 1. If π is ramified, we get a symplectic injection $\psi\colon J_2 \hookrightarrow P_2$ such that
(a) $\operatorname{Im}\psi = \ker \rho$, where $\rho\colon P \longrightarrow \hat{P}$ is the polarization of P, and
(b) $\tilde{J} \cong J \times P/\{(\alpha, \psi\alpha) \mid \alpha \in J_2\}$.
If π is unramified, we get subgroups

$$\begin{array}{ccccccc} (0) & \subset & H_0 & \subset & H_1 & \subset & J_2 \\ & & \| & & \| & & \\ & & \{0, \mathfrak{A}\} & & \{\mathfrak{B} \mid e_2(\mathfrak{A}, \mathfrak{B}) = +1\} & & \\ & & \text{order } 2 & & \text{order } 2^{2g-1} & & \end{array}$$

and a symplectic isomorphism $\psi\colon H_1/H_0 \xrightarrow{\approx} P_2 = \ker \rho$ such that

$$\tilde{J} \cong J \times P/\{(\alpha, \psi\alpha) \mid \alpha \in H_1\}.$$

Corollary 2. If π is unramified or has only two branch points, then $\ker \rho = P_2$, hence $\rho = 2\lambda_\Xi$, where

$$\lambda_\Xi:\ P \xrightarrow{\approx} \hat{P}$$

is a principal polarization. Moreover, in these cases

$$\phi(J) = \{x \in \tilde{J} \mid \iota x = x\}.$$

II

4. RELATIONS BETWEEN THETA DIVISORS

The question arises: In the class of all positive divisors algebraically equivalent to $2\theta_X$, which ones arise as $\phi^{-1}(\theta_{Y,y})$, where $\theta_{Y,y} = T_y(\theta_Y)$ is a translate of θ_Y by y and ϕ^{-1} means its pullback as actual divisor, when defined? This class of divisors is the (disjoint) union of the linear systems $|\theta_X + \theta_{X,x}|$, $x \in X$. In particular, one can ask whether it ever happens that

$$\phi^{-1}\theta_{Y,y} = \theta_{X,x_1} + \theta_{X,x_2}$$

for some $y \in Y$, $x_1, x_2 \in X$. The situation seems to be that this does not occur in general, that it does occur for Jacobians, and that this special occurrence is the ultimate source of the "Schottky relations" satisfied by the theta nulls of Jacobians.

Let us see first what we can say about the situation in general. Since

$$\phi^{-1}(\theta_{Y,\phi(x)}) = \phi^{-1}(\theta_Y)_x$$

for all $x \in X$, we may as well restrict our attention to the divisors $\phi^{-1}(\theta_{Y,y})$ for $y \in P$. All these divisors are linearly equivalent, since

$$[\text{the div. class}\quad \phi^{-1}(\theta_{Y,y}) - \phi^{-1}(\theta_Y) \quad\text{in}\quad X] = \hat{\phi}(\lambda_{\theta_Y}(y))$$

and this is 0 if $y \in P$. Moreover, if we replace θ_X and θ_Y by suitable translates, we can then assume that θ_X and θ_Y are symmetric divisors (invariant under -1_X and -1_Y) and that†

$$\phi^{-1}(\theta_{Y,y}) \in |2\theta_X|, \qquad \text{all} \qquad y \in P.$$

Therefore we get a morphism (we change the sign of y to simplify the proposition that follows):

$$\delta:\ P - \{y \mid \phi(X) \subset \theta_{Y,-y}\} \longrightarrow |2\theta_X|$$
$$y \longmapsto \phi^{-1}(\theta_{Y,-y}).$$

† In fact, first take any symmetric θ_X and θ_Y. Then $\phi^{-1}(\theta_Y) = 2\theta_X + D$, where $2D \equiv 0$, hence $e_*^{\theta_Y}(\phi(x)) = e_2(D, x)$ for all $x \in X_2$. This is a homomorphism from $\phi(X_2)$ to $\{\pm 1\}$: Extend it to a homomorphism f: $Y_2 \to \{\pm 1\}$ and represent f by $f(x) = e_2(y, x)$, for some $y \in Y_2$. Then $\theta_{Y,y}$ is still symmetric and $e_2^{\theta_{Y,y}}(\phi(x)) = 1$, all $x \in X_2$, hence $\phi^{-1}(\theta_{Y,y})$ is totally symmetric, i.e., $\in |2\theta_X|$.

Moreover, because the polarization θ_Y on Y, pulled back to $X \times P$, splits into a product, it follows that we can write

$$\sigma^*(\mathcal{O}_Y(\theta_Y)) = p_1{}^*(\mathcal{O}_X(2\theta_X)) \otimes p_2{}^*(L_\rho),$$

where L_ρ is a symmetric invertible sheaf on P representing the polarization ρ. I claim the following:

Proposition. δ is essentially the morphism of P to projective space defined by the section of L_ρ. More precisely

$$\{y \mid \phi(X) \subset \theta_{Y,-y}\} = \{y \mid s(y) = 0 \quad \text{for all} \quad s \in \Gamma(L_\rho)\}$$

—call this set B_ρ (for base points)—and there is an isomorphism

$$i: \ \mathbb{P}(\Gamma(L_\rho)) \hookrightarrow |2\theta_X|$$

of $\mathbb{P}(\Gamma(L_\rho))$ with a linear subspace of $|2\theta_X|$ such that the diagram

$$\begin{array}{ccc} & \xrightarrow{\phi_\rho} & \mathbb{P}(\Gamma(L_\rho)) \\ P - B_\rho & & \downarrow i \\ & \xrightarrow{\delta} & |2\theta_X| \end{array}$$

commutes, where ϕ_ρ is the canonical morphism defined by sections of L_ρ.

Proof. We abbreviate $\mathcal{O}_X(\theta_X)$ to L_X and $\mathcal{O}_Y(\theta_Y)$ to L_Y. Now, according to the general theory of Mumford [9, Section 1] (see also Mumford [10, Section 23]), the isomorphism

$$\sigma^* L_Y \cong p_1{}^* L_X{}^2 \otimes p_2{}^* L_\rho$$

defines a lifting of the group H:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{G}(p_1{}^*L_X{}^2 \otimes p_2{}^*L_\rho) & \longrightarrow & X_2 \times \ker \rho & \longrightarrow & 0 \\ & & & & \cup & & \cup & & \\ & & & & H^* & \xrightarrow{\approx} & H & & \end{array}$$

and the pullback $\sigma^*(s_0)$ of the unique section $s_0 \in \Gamma(L_Y)$ (unique up to scalars) is the unique element of $\Gamma(L_X{}^2) \otimes \Gamma(L_\rho)$ fixed by H^*. But for any such H^*, it is easy to describe the element fixed by H^*: in fact

$$\mathcal{G}(p_1{}^*L_X{}^2 \otimes p_2{}^*L_\rho) \cong \mathcal{G}(L_X{}^2) \times \mathcal{G}(L_\rho)/\{(\lambda, \lambda^{-1}) \mid \lambda \in \mathbb{G}_m\}$$

and any such H^* contains a subgroup $H_0{}^*$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{G}(L_X{}^2) & \longrightarrow & X_2 & \longrightarrow & 0 \\ & & & & \cup & & \cup & & \\ & & & & H_0{}^* & \xrightarrow{\approx} & H_0 & & \end{array}$$

Then if $Z(H_0{}^*)$ is the centralizer of $H_0{}^*$, we get a Heisenberg group:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow Z(H_0{}^*)/H_0{}^* \longrightarrow H_1/H_0 \longrightarrow 0$$

and H^* itself is defined by an isomorphism ψ^*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & Z(H_0^*)/H_0^* & \longrightarrow & H_1/H_0 & \longrightarrow & 0 \\ & & \Big\downarrow \lambda^{-1} & & \Big\downarrow \psi^* & & \Big\downarrow \psi & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{G}(L_\rho) & \longrightarrow & \ker \rho & \longrightarrow & 0 \end{array}$$

by this connecting link

$$H^* = \{(x, \psi^* x) \mid x \in Z(H_0^*)/H_0^*\}.$$

Now the subspace $\Gamma(L_X{}^2)^{H_0^*}$ of H_0^* invariants is the unique irreducible representation of $Z(H_0^*)/H_0^*$ on which $\mathbb{G}_m$ acts identically, and the dual $\mathrm{Hom}(\Gamma(L_\rho), k)$ is the unique irreducible representation of $\mathcal{G}(L_\rho)$ on which $\mathbb{G}_m$ acts by $\lambda \longmapsto \lambda^{-1} \cdot$ (identity). Therefore ψ^* defines an isomorphism of these representations:

$$\chi\colon\ \Gamma(L_X{}^2)^{H_0^*} \xrightarrow{\approx} \mathrm{Hom}(\Gamma(L_\rho), k).$$

If $\beta_1, \ldots, \beta_d$ is a basis of $\Gamma(L_\rho)$ and $\alpha_1, \ldots, \alpha_d$ is the basis of $\Gamma(L_X{}^2)^{H_0^*}$ such that $\chi(\alpha_i)(\beta_j) = \delta_{ij}$, then it is immediate that $\sum \alpha_i \otimes \beta_i \in \Gamma(L_X{}^2) \otimes \Gamma(L_\rho)$ is H^* invariant. Thus

$$\sigma^*(s_0) = \sum_{i=1}^{d} p_1{}^*\alpha_i \otimes p_2{}^*\beta_i;$$

hence for all $y \in P$

$$\phi^{-1}(\theta_{Y,-y}) = \text{zero set of } \ \mathrm{res}_{X \times \{y\}}(\sigma^* s_0) = \text{zero set of } \sum_{i=1}^{d} \beta_i(y) \cdot \alpha_i. \qquad (4.1)$$

Thus, first of all

$$\begin{aligned} \phi^{-1}(\theta_{Y,-y}) = X \Longleftrightarrow \textstyle\sum \beta_i(y) \cdot \alpha_i \equiv 0 &\Longleftrightarrow \beta_i(y) = 0, \quad \text{all } i \\ &\Longleftrightarrow y \quad \text{is a base point of} \quad \Gamma(L_\rho), \end{aligned}$$

and second if $l \in \mathrm{Hom}(\Gamma(L_\rho), k)$ is "homogeneous coordinates" for a point of $\mathbb{P}(\Gamma(L_\rho))$, then set $i(l) =$ the divisor $(\sum l(\beta_i) \cdot \alpha_i = 0)$. Then (4.1) implies that $\delta = i \cdot \phi_\rho$. Q.E.D.

Now starting from the other direction, $|2\theta_X|$ contains the reducible divisors $\theta_{X,x} + \theta_{X,-x}$, $x \in X$. Therefore we get a morphism:

$$\begin{aligned} \phi_X'\colon\ X &\longrightarrow |2\theta_X| \\ x &\longmapsto \theta_{X,x} + \theta_{X,-x}. \end{aligned}$$

I claim the following

Proposition (Wirtinger). There is a nondegenerate inner product $B\colon \Gamma(L_X{}^2) \otimes \Gamma(L_X{}^2) \longrightarrow k$ (which is symmetric or skew-symmetric depending on whether $\mathrm{mult}_0\, \theta_X$ is even or odd) such that if B induces the isomorphism B',

$$\mathbb{P}(\Gamma(L_X{}^2)) \xrightarrow{\approx} \mathbb{P}(\Gamma(L_X{}^2)^*) = |2\theta_X|,$$

then the diagram

$$\begin{array}{ccc} & & \mathbb{P}(\Gamma(L_X{}^2)) \\ & \overset{\phi_X}{\nearrow} & \\ X & & \wr\big\downarrow B' \\ & \underset{\phi_X{}'}{\searrow} & \\ & & |2\theta_X| \end{array}$$

commutes, where ϕ_X is the canonical morphism defined by sections of $L_X{}^2$.

Proof. In this case we use the morphism

$$\begin{aligned} \xi\colon X \times X &\longrightarrow X \times X \\ (x,y) &\longmapsto (x+y,\, x-y), \end{aligned}$$

and the isomorphism

$$\xi^*(p_1{}^*L_X \otimes p_2{}^*L_X) \cong p_1{}^*L_X{}^2 \otimes p_2{}^*L_X{}^2$$

(cf. Mumford 9, Section 2). Let $\{s_\alpha\}$ be a basis of $\Gamma(L_X{}^2)$. Then we can write

$$\xi^*(p_1{}^*\theta_X \otimes p_2{}^*\theta_X) = \sum_{\alpha,\beta} c_{\alpha\beta}\, p_1{}^*s_\alpha \times p_2{}^*s_\beta$$

for some matrix $c_{\alpha\beta} \in k$; or, more transparently,

$$\theta_X(u+v)\theta_X(u-v) = \sum c_{\alpha\beta}\, s_\alpha(u) \cdot s_\beta(v), \qquad \forall u, v \in X. \tag{4.2}$$

As a section of L_X, θ_X is even or odd depending on $\mathrm{mult}_0\, \theta_X$ and hence interchanging u and v in this formula, we find $c_{\alpha\beta}$ is symmetric or skew-symmetric in these two cases. Moreover, the element $\xi^*(p_1{}^*\theta_X \otimes p_2{}^*\theta_X)$ is invariant under the action of $\Delta(X_2) = \{(x,x) \mid x \in X_2\}$ on $p_1{}^*L_X{}^2 \otimes p_2{}^*L_X{}^2$ [via a suitable lifting of $\Delta(X_2)$ into $\mathcal{G}(p_1{}^*L_X{}^2 \otimes p_2{}^*L_X{}^2)$]. And since X_2 acts irreducibly on $\Gamma(L_X{}^2)$, this element cannot lie in any proper subspace $W_1 \otimes W_2$ of $\Gamma(L_X{}^2) \otimes \Gamma(L_X{}^2)$. This implies that $\det c_{\alpha\beta} \neq 0$, hence $c_{\alpha\beta}$ defines a form B. Finally, for each fixed v the formula (4.2) implies

$$u \in \mathrm{support}(\theta_{X,v} + \theta_{X,-v}) \Longleftrightarrow u \in \mathrm{zeros}\Big(\sum c_{\alpha\beta}\, s_\beta(v) s_\alpha\Big)$$

which gives us immediately

$$\phi_X{}'(v) = B'(\phi_X(v)). \quad \text{Q.E.D.}$$

Corollary 1. In the abstract situation (X, θ_X), (Y, θ_Y), ϕ, we get a diagram:

$$\begin{array}{ccccc} P - B_\rho & \xrightarrow{\phi_\rho} & \mathbb{P}(\Gamma(L_\rho)) & & \\ & & & \overset{i}{\searrow} & \\ & & & & |2\theta_X|. \\ & & & \underset{\approx}{\overset{B'}{\nearrow}} & \\ X & \xrightarrow{\phi_X} & \mathbb{P}(\Gamma(L_X{}^2)) & & \end{array}$$

Then for all $y \in P - B_\rho$, $x \in X$

$$\phi^{-1}(\theta_{Y,y}) = \theta_{X,x} + \theta_{X,-x} \Longleftrightarrow i(\phi_\rho(y)) = B'(\phi_X(x)).$$

The most important case here is when $\ker \rho = P_2$, so that there is a theta divisor θ_P on P with $\rho = 2\lambda_{\theta_P}$ and $L_\rho = L_P{}^2$, where $L_P = \mathcal{O}_P(\theta_P)$. Then Corollary 1 becomes the following.

Corollary 2. In the abstract situation (X, θ_X), (Y, θ_Y), ϕ, when $\rho = 2\lambda_{\theta_P}$, we get a diagram

$$\begin{array}{ccccc} P & \xrightarrow{\phi_P} & \mathbb{P}(\Gamma(L_P{}^2)) & & \\ & & & \overset{i}{\searrow} & \\ & & & & |2\theta_X| \\ & & & \underset{B'}{\overset{\approx}{\nearrow}} & \\ X & \xrightarrow{\phi_X} & \mathbb{P}(\Gamma(L_X{}^2)) & & \end{array}$$

and for all $y \in P$, $x \in X$

$$\phi^{-1}(\theta_{Y,y}) = \theta_{X,x} + \theta_{X,-x} \Longleftrightarrow i(\phi_P(y)) = B'(\phi_X(x)).$$

5. THE SPLITTING OF $\phi^{-1}(\theta_{Y,y})$ FOR JACOBIANS

Now return to the double covering $\pi\colon \tilde{C} \longrightarrow C$. Recall the geometric meaning of the theta divisors $\Theta \subset J$, $\tilde{\Theta} \subset \tilde{J}$:

(1) Let J_k be the variety of invertible sheaves on C of degree k, and $\tilde{J}_k$ be the variety of invertible sheaves on $\tilde{C}$ of degree k [so that if we choose a *base point* on J_k or $\tilde{J}_k$, $J_k \cong J$ and $\tilde{J}_k \cong \tilde{J}$, but without such a choice $J_k(\tilde{J}_k)$ is merely a principal homogeneous space over J $(\tilde{J})$]. Note that π^* induces: $\pi^*\colon J_k \longrightarrow \tilde{J}_{2k}$ because $\deg \pi^* L = 2 \cdot \deg L$. Moreover, note that there is a canonical group structure on the big schemes:

$$\coprod_{k \in \mathbb{Z}} J_k \quad \text{and} \quad \coprod_{k \in \mathbb{Z}} \tilde{J}_k.$$

(2) Then we can find Θ *canonically* in J_{g-1} by

$$\Theta = \{L \in J_{g-1} \mid \Gamma(L) \neq (0)\} \subset J_{g-1}$$

and similarly

$$\tilde{\Theta} = \{L \in \tilde{J}_{\tilde{g}-1} \mid \Gamma(L) \neq (0)\} \subset \tilde{J}_{\tilde{g}-1} = \tilde{J}_{2g+n-2}.$$

(3) The various translates of the theta divisors in J and $\tilde{J}$ are given by Θ_{-y}, and $\tilde{\Theta}_{-\tilde{y}}$ for $y \in J_{g-1}$ and $\tilde{y} \in \tilde{J}_{\tilde{g}-1}$. To ask whether $\phi^{-1}(\theta_{Y,y})$ splits in this case is therefore the same as asking for points $y \in \tilde{J}_n$ if

$$(\pi^*)^{-1}(\tilde{\Theta}_{-y}) = \Theta_{x_1} + \Theta_{x_2}$$

for some $x_1, x_2 \in J$.

The double covering π gives us a unique divisor class $\mathfrak{A}$ such that:

$$2\mathfrak{A} \equiv \sum_{i=1}^{2n} P_1, \qquad P_i = \text{branch points},$$

$$\pi^{-1}\mathfrak{A} \equiv \sum_{i=1}^{2n} Q_i, \qquad Q_i = \pi^{-1}(P_i).$$

In fact, if $\mathbb{R}(\tilde{C}) = \mathbb{R}(C)(\sqrt{f})$, then $(f) = \sum_{i=1}^{2n} P_i - 2\mathfrak{A}$ on C and $(\sqrt{f}) = \sum_{i=1}^{2n} Q_i - \pi^{-1}(\mathfrak{A})$ on $\tilde{C}$ for some divisor $\mathfrak{A}$. Sheaf-theoretically, if $\tilde{C} = \mathbf{Spec}(\mathcal{O}_C \oplus L)$ as in Section 1, $L = \mathcal{O}_C(-\mathfrak{A})$. We then have the following.

Proposition. Let $x_1, \ldots, x_d$ be any d closed points on $\tilde{C}$ such that $\pi x_i \neq \pi x_j$, all $i \neq j$. Then for all invertible sheaves L of degree $g-1$ on C

$$\Gamma\left(\tilde{C}, \pi^* L\left(\sum_{i=1}^{d} x_i\right)\right) \neq (0) \Longleftrightarrow \Gamma(C, L) \neq (0) \quad \text{or} \quad \Gamma\left(C, L\left(\sum_{i=1}^{d} \pi x_i - \mathfrak{A}\right)\right) \neq (0).$$

Proof. Note that $\pi_*(\mathcal{O}_{\tilde{C}}) = \mathcal{O}_C \oplus \mathcal{O}_C(-\mathfrak{A})$, where $\mathcal{O}_C$ is the subsheaf of functions even under the involution ι, and $\mathcal{O}_C(-\mathfrak{A})$ is the subsheaf of odd functions. Therefore

$$\pi_*(\pi^* L(\sum x_i + \sum \iota x_i)) \cong L(\sum \pi x_i) \otimes \pi_* \mathcal{O}_{\tilde{C}} \cong L(\sum \pi x_i) \oplus L(\sum \pi x_i - \mathfrak{A}).$$

This sheaf has subsheaves as follows:

$$\begin{array}{ccccc} \pi_*(\pi^* L(\sum x_i + \sum \iota x_i)) & \cong & L(\sum \pi x_i) & \oplus & L(\sum \pi x_i - \mathfrak{A}) \\ \cup & & & & \\ \pi_*(\pi^* L(\sum x_i)) & & \cup & & \cup \\ \cup & & & & \\ \pi_*(\pi^* L) & \cong & L & \oplus & L(-\mathfrak{A}) \end{array}$$

but the middle sheaf does not break up into even and odd pieces, because $x_i \neq \iota x_j$ for any i, j. In fact at every point πx_i the middle sheaf is generated by $L \oplus L(-\mathfrak{A})$, plus a section (s_1, s_2) with nonzero images $\bar{s}_1 \in L(\pi x_i)/L$ and $\bar{s}_2 \in L(\pi x_i - \mathfrak{A})/L(-\mathfrak{A})$. It follows that the middle sheaf fits into an exact sequence:

$$0 \longrightarrow L \longrightarrow \pi_*(\pi^* L(\sum x_i)) \longrightarrow L(\sum \pi x_i - \mathfrak{A}) \longrightarrow 0.$$

This gives:

$$\begin{aligned} 0 \longrightarrow \Gamma(C, L) &\longrightarrow \Gamma(\tilde{C}, \pi^* L(\sum x_i)) \\ &\longrightarrow \Gamma(C, L(\sum \pi x_i - \mathfrak{A})) \xrightarrow{\ \delta\ } H^1(C, L) \longrightarrow \cdots \end{aligned}$$

which gives the implication "$\Longrightarrow$" of the lemma immediately. As for "$\Longleftarrow$," the only problem would be if

$$\begin{array}{l} \Gamma(C, L) = (0) \\ \delta\colon\ \Gamma(C, L(\sum \pi x_i - \mathfrak{A})) \longrightarrow H^1(C, L) \quad \text{injective} \\ \qquad\qquad \# \\ \qquad\qquad (0). \end{array}$$

But $\deg L = g - 1$, so $\chi(L) = 0$ and this is impossible. Q.E.D.

Corollary 1. Let $x_1, \ldots, x_n$ be any n closed points on $\tilde{C}$ such that $\pi x_i \neq \pi x_j$, all $i \neq j$. If†

$$y = \sum_{i=1}^{n} x_i \in \tilde{J}_n \quad \text{and} \quad x = \sum_{i=1}^{n} \pi x_i - \mathfrak{A} \in J_0,$$

† To simplify notation, we are identifying divisor classes of degree k with point of J_k, e.g., writing x for $t(x)$, etc.

then

$$(\pi^*)^{-1}(\tilde{\Theta}_{-y}) = \Theta + \Theta_{-x}.$$

Proof. Set-theoretically, this is just a translation of the proposition. Since the divisor $\pi^{-1}(\tilde{\Theta}_{-y})$ is algebraically equivalent to 2Θ, there can be no multiplicities and the equality holds between divisors, too.

Note what happens if $\pi x_i = \pi x_j$ but $x_i \neq x_j$. Then

$$L(\pi x_i) \subset \pi_*\left(\pi^* L\left(\sum_{i=1}^{n} x_i\right)\right)$$

and since $\deg L(\pi x_i) = g$, $\Gamma(L(\pi x_i))$ is always nontrivial. Therefore in this case

$$\pi^*(J_{g-1}) \subset \tilde{\Theta}_{-y}.$$

To rephrase this corollary in a form parallel to the general description of Section 4, we must choose suitable symmetric representatives of Θ and $\tilde{\Theta}$ in J and $\tilde{J}$ themselves (instead of in J_{g-1} and $\tilde{J}_{2g+n-2}$). In fact, choose:

(a) Theta characteristics ζ and $\tilde{\zeta}$ on C and $\tilde{C}$, i.e., divisor classes such that $2\zeta = K$ (the canonical class on C) and $2\tilde{\zeta} = \tilde{K}$ (the canonical class on $\tilde{C}$), and moreover such that

$$Nm\, \tilde{\zeta} = K + \mathfrak{A}.$$

[To see that this is possible, let $\mathfrak{B}$ be a divisor class on C such that $2\mathfrak{B} \equiv \mathfrak{A} - \sum_{i=1}^{n} P_i$ (half of the P_i only). Then set $\tilde{\zeta} = \pi^{-1}(\zeta + \mathfrak{B}) + \sum_{i=1}^{n} Q_i$.]

(b) ζ and $\tilde{\zeta}$ define theta divisors $\Theta_0 = \Theta_{-\zeta}$ and $\tilde{\Theta}_0 = \tilde{\Theta}_{-\tilde{\zeta}}$ in J and $\tilde{J}$ which are well known to be symmetric. Moreover, I claim that because of our careful choice of $\tilde{\zeta}$, $(\pi^*)^{-1}\tilde{\Theta}_0 \in |2\Theta_0|$. This follows, in fact, from the next Corollary soon to be stated.

(c) Now write $\tilde{\zeta} = \pi^{-1}\zeta + \delta$ and note that $2\delta = \sum_{i=1}^{2n} Q_i$, $\mathrm{Nm}\, \delta = \mathfrak{A}$, and $\deg \delta = n$.

We make the following important definition.

Definition. If $x_1, \ldots, x_n$ are points of $\tilde{C}$, we wish to find elements

$$z = \tfrac{1}{2}\sum_{i=1}^{n}(x_i - \iota x_i) \in P, \qquad w = \tfrac{1}{2}\left(\sum_{i=1}^{n} \pi x_i - \mathfrak{A}\right) \in J.$$

We say that $z \in P$ and $w \in J$ are *compatible solutions* of the equations

$$2z = \sum_{i=1}^{n}(x_i - \iota x\ ), \qquad 2w = \sum_{i=1}^{n} \pi x_i - \mathfrak{A}$$

if $z + \pi^* w = \sum_{i=1}^{n} x_i - \delta$.

(Note that such a pair z, w always exists: In fact $J \times P \longrightarrow \tilde{J}$ is surjective, so $\sum_{i=1}^{n} x_i - \delta$ can be written $z + \pi^* w$, where $z \in P$ and $w \in J$. Taking Nm and $1 - \iota$, it follows that $2z$ and $2w$ have the required values.)

We can now state the following result.

Corollary 2. If $x_1, \ldots, x_n$ are any points of $\tilde{C}$ such that $\pi x_i \neq \pi x_j$, all $i \neq j$, and

$$z = \tfrac{1}{2}\sum_{1}^{n}(x_i - \iota x_i), \qquad w = \tfrac{1}{2}\left(\sum_{1}^{n} \pi x_i - \mathfrak{A}\right)$$

are compatible halves in P and J, then

$$(\pi^*)^{-1}(\tilde{\Theta})_{0,-z}) = \Theta_{0,w} + \Theta_{0,-w}.$$

Proof. Note that

$$\tilde{\Theta}_{0,-z} = (\tilde{\Theta}_{-\tilde{\zeta}})_{-\Sigma x_i+\delta+\pi^* w} = \tilde{\Theta}_{\pi^*(w-\zeta)-\Sigma x_i};$$

hence

$$\begin{aligned}(\pi^*)^{-1}(\tilde{\Theta}_{0,-z}) &= (\pi^*)^{-1}\tilde{\Theta}_{\pi^*(w-\zeta)-\Sigma x_i} = ((\pi^*)^{-1}\tilde{\Theta}_{-\Sigma x_i})_{w-\zeta} = (\Theta + \Theta_{\mathfrak{A}-\Sigma\pi x_i})_{w-\zeta} \\ &= \Theta_{0,w} + \Theta_{0,w+\mathfrak{A}-\Sigma\pi x_i} = \Theta_{0,w} + \Theta_{0,-w}. \quad \text{Q.E.D.}\end{aligned}$$

In case there are zero or two branch points, we can (a) work out more precisely what pairs (z, w) are compatible and (*b*) combine the result with Corollary 2 in Section 4 to obtain the following.

Corollary 3 (Schottky–Jung). If $\tilde{C}$ is *unramified* over C, so that $2\mathfrak{A} = 0$ as divisor class, then choose a divisor class $\mathfrak{B}$ on C such that $2\mathfrak{B} = \mathfrak{A}$. Choose any theta characteristic ζ on C and take $\tilde{\zeta} = \pi^{-1}(\zeta + \mathfrak{B})$ as theta characteristic on $\tilde{C}$. [Note that $2\tilde{\zeta} = \pi^{-1}(2\zeta + 2\mathfrak{B}) = \pi^{-1}(K + \mathfrak{A}) = \tilde{K}$ and $\mathrm{Nm}\,\tilde{\zeta} = 2\zeta + 2\mathfrak{B} = K + \mathfrak{A}$ as required.] These define $\Theta_0 \subset J$, $\tilde{\Theta}_0 \subset \tilde{J}$ and we have the following.

(i) $(\pi^*)^{-1}(\tilde{\Theta}_0) = \Theta_{0,\mathfrak{B}} + \Theta_{0,-\mathfrak{B}}$.

(ii) If $\Xi \subset P$ is a symmetric theta divisor on the Prym P, we get a canonical diagram

$$\begin{array}{ccccc} P & \xrightarrow{\phi_P} & \mathbb{P}(\Gamma(L_P{}^2)) & & \\ & & & \overset{i}{\searrow} & \\ & & & & |2\Theta_0| \\ & & & \underset{B'}{\overset{\approx}{\nearrow}} & \\ J & \xrightarrow{\phi_J} & \mathbb{P}(\Gamma(L_J{}^2)) & & \end{array}$$

where ϕ_P and ϕ_J are the Kummer maps defined by the linear systems $|2\Xi|$ and $|2\Theta_0|$, and i and B' are as in Section 4; and

$$i(\phi_P(0)) = B'(\phi_J(\mathfrak{B})).$$

Corollary 4 (Fay) (see *Note*, p. 350). If $\tilde{C}$ has two branch points over C, then choosing suitable theta characteristics on C and $\tilde{C}$, we get a symmetric theta divisor Ξ on P and a canonical diagram

$$\begin{array}{ccc} P & \xrightarrow{\phi_P} & \mathbb{P}(\Gamma(L_P{}^2)) \\ & & \wr\big\downarrow j \\ J & \xrightarrow{\phi_J} & \mathbb{P}(\Gamma(L_J{}^2)) \end{array}$$

where $j = (B')^{-1} \cdot i$ is now an isomorphism. Then for every $x \in \tilde{C}$ there are compatible halves:

$$z = \tfrac{1}{2}(x - \iota x) \in P, \qquad w = \tfrac{1}{2}(\pi x - \mathfrak{A}) \in J$$

such that

$$j(\phi_P(z)) = \phi_J(w).$$

As mentioned in the Introduction, one would hope that these last two corollaries can be used to find strong polynomial identities for the "theta-null werte" of Jacobians. Unfortunately, whereas for the projective embedding of any principally polarized Abelian variety (X, θ) defined by $|4\theta|$ one knows simple identities satisfied by the image of $0 \in X$ (namely Riemann's identities; cf. Mumford [9, Section 3]) for the morphism defined by $|2\theta|$ no analogous simple identities seem to be known. In classical terms, the problem is: Find identities for the set of 2^n functions of Z

$$f_a(Z) = \theta[{}^a_0](0, Z), \qquad a = [a_1, \ldots, a_n], \qquad a_i = 0 \quad \text{or} \quad 1.$$

($Z \in \mathfrak{H}_n$, Siegel's upper half-space). If $n = 3$, there appears to be a unique irreducible identity of order eight, which applied to $\phi_P(0)$ in Corollary 3 leads to the usual "Schottky relation" on the theta nulls of a curve C of genus 4.

To explain the strength of Corollary 4, for instance, it may be helpful to contrast it with the following result: If (X, θ_X) and (Y, θ_Y) are two principally polarized Abelian varieties and if $k \geqslant 4$, consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_X} & \mathbb{P}(\Gamma(L_X{}^k)) \\ & & \wr\big\downarrow j \\ Y & \xrightarrow{\phi_Y} & \mathbb{P}(\Gamma(L_Y{}^k)) \end{array}$$

where ϕ_X and ϕ_Y are the canonical maps defined by $|k\theta_X|$ and $|k\theta_Y|$ and j is an isomorphism under which the translations by X_k [which extend uniquely to projective transformations on $\mathbb{P}(\Gamma(L_X{}^k))$] correspond to translations by Y_k. (For any X and Y a finite number of such j's always exist.) Then

$$j(\phi_X(X)) \cap \phi(Y) \neq \varnothing$$

implies

$$j(\phi_X(X)) = \phi_Y(Y);$$

hence $X \cong Y$. In other words, distinct Abelian varieties, projectively embedded by somewhat more ample linear systems, *never meet*!

III

6. GEOMETRIC DESCRIPTION OF SING Ξ, UNRAMIFIED CASE

We now consider only an unramified $\pi\colon \tilde{C} \longrightarrow C$. Recall that in this case

(a) genus $C = \dim J = g$, genus $\tilde{C} = \dim \tilde{J} = 2g - 1$, $\dim P = g - 1$, and $\ker \rho = P_2 \cong \{0, \mathfrak{A}\}^\perp/\{0, \mathfrak{A}\}$, $\mathfrak{A} \in J_2$ defining π (hence P principally polarized).

(b) $\{x \in \tilde{J} \mid \iota x = x\} = \pi^* J$, and $\{x \in \tilde{J} \mid \text{Nm}\, x = 0\} \cong P \times \mathbb{Z}/2\mathbb{Z}$.

[Use the fact that in the notation of (i)–(viii) of Section 2, $a = g$, $b = g - 1$, and $c = g - 1$.] In fact, in a previous paper [11] we have shown by a different argument that if we look at the principal homogeneous space $\tilde{J}_{2g-2}$ instead of $\tilde{J} = \tilde{J}_0$, and at

$$\text{Nm}\colon \tilde{J}_{2g-2} \longrightarrow J_{2g-2},$$

then $\mathrm{Nm}^{-1}(K)$ ($K \in J_{2g-2}$ the canonical divisor class) breaks into two components P^+, P^- such that:

$\forall$ invertible sheaves L_α on $\tilde{C}$, corresponding to $\alpha \in \tilde{J}_{2g-2}$,
if $\mathrm{Nm}\, L \cong \Omega_C^1$, then

$$\dim \Gamma(L_\alpha) \text{ even} \Longleftrightarrow \alpha \in P^+, \qquad \dim \Gamma(L_\alpha) \text{ odd} \Longleftrightarrow \alpha \in P^-; \tag{6.1}$$

moreover, for some α,

$$\dim \Gamma(L_\alpha) = 0 \quad \text{and} \quad \dim \Gamma(L_\alpha) = 1.$$

Translating these back to $\tilde{J}_0$ by any $\alpha \in \mathrm{Nm}^{-1}(K)$, P^+ and P^- correspond to P and its nontrivial coset in ker Nm. Now the theta divisors of C and $\tilde{C}$ live canonically in J_{g-1} and $\tilde{J}_{2g-2}$ and Riemann's theorem (see Kempf [7], and Szpiro [16]) asserts

$$\forall \text{ invertible sheaves } L_\alpha \text{ on } C \text{ (resp. } \tilde{C}) \text{ corresponding to } \alpha \in J_{g-1} \text{ (resp. } \tilde{J}_{2-g2}), \tag{6.2}$$
$$\dim \Gamma(L_\alpha) = \text{mult. of } \alpha \text{ on } \Theta \text{ (resp. } \Theta).$$

Combining (6.1) and (6.2), we find the following result.

Proposition. (a) $\tilde{\Theta} \supset P^-$; (b) $\tilde{\Theta} \cdot P^+ = 2\Xi$, where $\Xi \subset P^+$ is a canonical representative of the theta divisor on P.

Proof. In fact

$$\alpha \in P^- \Longrightarrow \dim \Gamma(L_\alpha) \text{ odd} \Longrightarrow \dim \Gamma(L_\alpha) \geqslant 1 \Longrightarrow \alpha \in \tilde{\Theta}$$

and

$$\alpha \in \tilde{\Theta} \cap P^+ \Longrightarrow \dim \Gamma(L_\alpha) \text{ even and positive} \Longrightarrow \dim \Gamma(L_\alpha) \geqslant 2$$
$$\Longrightarrow \alpha \text{ singular on } \tilde{\Theta};$$

hence $\tilde{\Theta} \cdot P^+$ consists entirely in multiple components. But the principal polarization on $\tilde{J}$ restricts to twice that on P, so $\Theta \cdot P^+$ is in the algebraic equivalence class 2Ξ. It is easy to check that such a divisor can never have a component of multiplicity $\geqslant 3$ (or else the morphism it defines would not collapse an involution $x \longrightarrow x_0 - x$). Thus $\Theta \cdot P^+ = 2D$, D algebraically equivalent to Ξ, hence equal to it after a suitable translation. Q.E.D.

Corollary.

$$\mathrm{Sing}\, \Xi = \{x \in P^+ \mid \text{mult. at } x \text{ of } \tilde{\Theta} \geqslant 4\} \cup \left\{ x \in P^+ \,\middle|\, \begin{array}{l} \text{mult. at } x \text{ of } \tilde{\Theta} = 2, \text{ but} \\ T_{x,P^+} \subset (\text{tangent cone to } \Theta \text{ at } x) \end{array} \right\}.$$

In order to apply this corollary, we must know how to compute the tangent cone to $\tilde{\Theta}$. In general, suppose J is any Jacobian and $\Theta \subset J_{g-1}$. If L_α on C corresponds to the point $\alpha \in J_{g-1}$, then not only is

$$\dim \Gamma(L_\alpha) = \text{mult. at } \alpha \text{ of } \Theta,$$

but if $k = \dim \Gamma(L_\alpha)$, $s_1, \ldots, s_k$ is the basis of $\Gamma(L_\alpha)$, $t_1, \ldots, t_k$ is the basis of $\Gamma(\Omega \otimes L_\alpha^{-1})$, and $s_i \otimes t_j \in \Gamma(\Omega)$ defines the differential ω_{ij} at $\alpha \in J_{g-1}$, then identifying $\Gamma(\Omega)$ to the cotangent space m_α/m_α^2 of J_{g-1} at α, Kempf [7] proves that $\det(\omega_{ij})=0$ is the tangent cone to Θ at α.

Now if L_α is a sheaf on $\tilde{C}$ such that $\text{Nm}\, L_\alpha \cong \Omega_C$, then

$$L_\alpha \otimes \iota^* L_\alpha = \pi^* \text{Nm}\, L_\alpha \cong \pi^*\Omega_C \cong \Omega_{\tilde{C}};$$

hence choosing such an isomorphism ϕ, we may use the pairing

$$\begin{aligned} \langle\, ,\rangle\colon\ \Gamma(L_\alpha)\otimes\Gamma(L_\alpha) &\longrightarrow \Gamma(\Omega_{\tilde{C}}) \\ (s,t) &\longmapsto \phi(s\otimes\iota^* t)=\langle s,t\rangle \end{aligned}$$

instead of

$$\Gamma(L_\alpha)\otimes\Gamma(\Omega_{\tilde{C}}\otimes L_\alpha^{-1}) \xrightarrow{\ \otimes\ } \Gamma(\Omega_{\tilde{C}}).$$

Now ι induces $\iota^*\colon \Gamma(\Omega_{\tilde{C}}) \longrightarrow \Gamma(\Omega_{\tilde{C}})$, too: In fact, this is just the automorphism found by decomposing

$$\begin{aligned} \Gamma(\Omega_{\tilde{C}}) \cong \Gamma(\pi^*\Omega_C) &\cong \Gamma(\pi_*\pi^*\Omega_C) \\ &\subset \Gamma(\Omega_C)+\Gamma(\Omega_C(\mathfrak{A})) \\ &\qquad\qquad \| \\ &\qquad \text{the "Prym differentials"} \end{aligned}$$

and letting $\iota^* = +1$ on $\Gamma(\Omega_C)$, $\iota^* = -1$ on $\Gamma(\Omega_C(\mathfrak{A}))$. It is easy to check that

$$\iota^*(\langle s,t\rangle)=\langle t,s\rangle;$$

hence the above pairing splits into two pairings:

$$\text{Symm}^2\Gamma(L_\alpha)\longrightarrow\Gamma(\Omega_C), \qquad \Lambda^2\Gamma(L_\alpha)\longrightarrow\Gamma(\Omega_C(\mathfrak{A})).$$

Moreover, in the identification $\Gamma(\Omega_C)+\Gamma(\Omega_C(\mathfrak{A}))\cong\Gamma(\Omega_{\tilde{C}})\cong$ cotangent space $T^*_{\alpha,\tilde{J}_{2g-2}}$, clearly the even and odd subspaces under ι^* go over as follows: $\Gamma(\Omega_C)\cong$ cotangent space at α to the coset $\alpha+\pi^*(J_{2g-2})$, and $\Gamma(\Omega_C(\mathfrak{A}))\cong$ cotangent space at α to $P^{\pm}$.

Taking a basis $s_1,\dots,s_k$ of $\Gamma(L_\alpha)$, let $\omega_{ij}=\langle s_i,s_j\rangle$. Then $\iota^*\omega_{ij}=\omega_{ji}$, hence decomposing ω_{ij}.

$$\omega_{ij}=\omega_{ij}^+ + \omega_{ij}^-, \qquad \omega_{ij}^+\in\Gamma(\Omega_C), \qquad \omega_{ij}^-\in\Gamma(\Omega_C(\mathfrak{A})),$$

It follows that ω_{ij}^+ is symmetric and ω_{ij}^- is skew-symmetric. Therefore if $\alpha\in P^+$, $\det(\omega_{ij})=0$ is the tangent cone to $\tilde{\Theta}$ at α, and $\det(\omega_{ij}^-)=0$ is the tangent cone to $\tilde{\Theta}\cdot P^+$ at α. But $\det(\omega_{ij}^-)=Pf(\omega_{ij}^-)^2$ ($Pf=$ Pfaffian), so that $Pf(\omega_{ij}^-)=0$ is the tangent cone to Ξ at α (unless it vanishes identically).

We use this to establish the following result.

Proposition. If $\text{Nm}\, L_\alpha=\Omega_C$ and $\dim\Gamma(L_\alpha)=2$, then

$T_{\alpha,P^+}\subset$ tangent cone to $\tilde{\Theta}$ at $\alpha \Longleftrightarrow L_\alpha\cong\pi^*(\mathfrak{A})(\sum x_i)$ for some points $x_i\in\tilde{C}$ and a sheaf M on C such that $\dim\Gamma(M)=2$.

Proof. Let s, t be a basis of $\Gamma(L_\alpha)$. In the proceding notation

$$(\omega_{ij}^-)=\begin{pmatrix} 0 & \langle s,t\rangle-\langle t,s\rangle \\ \langle t,s\rangle-\langle s,t\rangle & 0\end{pmatrix}.$$

So the linear form $\langle s, t\rangle - \langle t, s\rangle$ is the tangent cone to Ξ unless $\alpha \in \operatorname{Sing}\Xi$. Thus

$$\begin{aligned} T_{\alpha, P^+} \subset \text{tangent cone to } \tilde{\Theta} \text{ at } \alpha &\Longleftrightarrow \langle s, t\rangle = \langle t, s\rangle \\ &\Longleftrightarrow s \otimes \iota^* t = t \otimes \iota^* s \Longleftrightarrow \iota^*(s/t) = s/t \\ &\Longleftrightarrow s/t \in \mathbb{R}(C). \end{aligned}$$

In classical language, $s/t \in \mathbb{R}(C)$ says "the pencil defined by L_α is pulled back from a pencil on C." In modern language, let $\sum x_i$ be the base points of $\Gamma(L_\alpha)$, let $\mathfrak{B}$ be the poles of s/t on C, and let $M = \mathcal{O}_C(\mathfrak{B})$. Then $L_\alpha \cong \pi^* M(\sum x_i)$ and 1, $s/t \in \Gamma(M)$; hence $\dim \Gamma(M) \geqslant 2$. Clearly $\dim \Gamma(M) = 2$ since $\dim \Gamma(L_\alpha) = 2$. Q.E.D.

7. DIM SING Ξ

Notations are as in Section 6. Recall that if C is a curve, a theta characteristic of C is a sheaf such that $L^2 \cong \Omega_C$; L is even or odd if $\dim \Gamma(L)$ is even or odd. We wish to prove the following theorem.

Theorem.
(a) C hyperelliptic $\Longrightarrow (P, \Xi)$ is a hyperelliptic Jacobian (hence $\dim \operatorname{Sing}\Xi = g - 4$) or a product of two such (hence $\dim \operatorname{Sing}\Xi = g - 3$).
(b) $g = 3$, C not hyperelliptic $\Longrightarrow (P, \Xi)$ is a two-dimensional Jacobian.
(c) $g = 4$, C not hyperelliptic $\Longrightarrow (P, \Xi)$ is a three-dimensional Jacobian, and Ξ is singular iff P is a hyperelliptic Jacobian iff $\exists$ is an even theta characteristic L with $\Gamma(L) \neq (0)$ and $L(\mathfrak{A})$ even.
(d) Assuming C not hyperelliptic and $g \geqslant 5$, then $\dim \operatorname{Sing}\Xi \leqslant g - 5$ and

$$\dim \operatorname{Sing}\Xi = g - 5 \Longrightarrow \begin{cases} C \text{ trigonal,} \\ \text{or } C \text{ double cover of an elliptic curve,} \\ \text{or } g = 5 \text{ and } \exists \text{ even theta characteristic } L \text{ with} \\ \Gamma(L) \neq (0) \text{ and } L(\mathfrak{A}) \text{ even,} \\ \text{or } g = 6 \text{ and } \exists \text{ odd theta characteristic } L \text{ with} \\ \dim \Gamma(L) \geqslant 3, \text{ and } L(\mathfrak{A}) \text{ even.} \end{cases}$$

In fact, in part (d), "$\Longleftarrow$" apparently also holds, but we will omit the proof of this. We first want to point out the following corollary.

Corollary. If $g \geqslant 5$ and C is neither trigonal, a double cover of an elliptic curve, nor of the preceding two special types of genus 5 or 6, then the polarized Abelian variety (P, Ξ) is *not* a Jacobian or a product of Jacobians.

This follows from the theorem and the fact that $\dim \operatorname{Sing}\Theta \geqslant \dim J - 4$ for polarized Jacobians (J, Θ) [1]. It would be quite interesting to find out in the special cases exactly which (P, Ξ) is a Jacobian.

Proof of Theorem. As shown in the previous section, the singularities of Ξ canonically embedded in P^+ arise from two sources.

Case 1: sheaves L_α such that $\operatorname{Nm} L_\alpha = \Omega_C$, $\dim \Gamma(L_\alpha) \geqslant 2$ and even, and $L_\alpha = \pi^* M(\sum x_i)$, where $\dim \Gamma(M) \geqslant 2$.

Case 2: sheaves L_α such that $\operatorname{Nm} L_\alpha = \Omega_C$, $\dim \Gamma(L_\alpha) \geqslant 4$ and even.

Note that in case 1

$$\Omega_C = \operatorname{Nm} L_\alpha = \operatorname{Nm}(\pi^* M(\textstyle\sum x_i)(= M^2(\sum \pi x_i),$$

so M satisfies the two conditions: (a) $\dim \Gamma(M) \geqslant 2$ and (b) $\dim \Gamma(\Omega_C \otimes M^{-2}) \geqslant 1$. Also, if there are no x_i, i.e., $L_\alpha = \pi^* M$, then $\dim \Gamma(L_\alpha)$ even implies (c) If $\Omega_C \cong M^2$, $\dim \Gamma(M) + \dim \Gamma(M \otimes \mathfrak{A})$ even.

Conversely, if M satisfies (a)–(c), choose an effective divisor $\sum \pi x_i$ in the linear system $\Gamma(\Omega_C \otimes M^{-2})$ and set $L_\alpha = \pi^* M(\sum x_i)$. This falls in case 1 unless there is at least one x_i and $\dim \Gamma(L_\alpha)$ odd. But as shown in a previous work [11, p. 187], we can then replace *one* x_i by $\iota(x_i)$ to make $\dim \Gamma(L_\alpha)$ even. So all M satisfying (a)–(c) define L_α in case 1.

It is not so easy to construct all the L_α in case 2 directly from sheaves on C. However, I claim the following.

Lemma. If $\dim \operatorname{Sing} \Xi \geqslant g - 5$, then almost all $\alpha \in \operatorname{Sing} \Xi$ correspond to sheaves L_α in case 1.

Proof. Suppose $Z \subset \operatorname{Sing} \Xi$ were a component of dimension $\geqslant (g-5)$ such that

$$\dim \Gamma(L_\alpha) \geqslant 4, \qquad \text{all} \quad \alpha \in Z.$$

According to previous results [11, pp. 186–188], $\dim \Gamma(L_\alpha) = 4$ for almost all $\alpha \in Z$. Let $Z_0 \subset Z$ be the open subset where $\dim \Gamma(L_\alpha) = 4$. We wish to apply the following quite general result.

Proposition. Let C be any curve, $Z \subset J_d$ a subvariety, $Z_0 \subset Z$ an open set, and assume that for some k

$$\dim \Gamma(L_\alpha) = k, \qquad \text{all} \quad \alpha \in Z_0.$$

Then identifying T_{α, J_α} to $H^1(\mathcal{O}_C)$, hence to the dual of $\Gamma(\Omega_C)$, I claim

$$\operatorname{Im}[\Gamma(L_\alpha) \otimes \Gamma(\Omega_C \otimes L_\alpha^{-1}) \longrightarrow \Gamma(\Omega_C)]^\perp \supset T_{\alpha, Z}$$

for all $\alpha \in Z_0$.

This is proved for $k = 2$ in Lemma 2.5 of Saint-Donat [14] but the proof extends verbatim to all k. Applying this to our case, let

$$W_\alpha = \operatorname{Im}[\Lambda^2 \Gamma(L_\alpha) \longrightarrow \Gamma(\Omega_C \otimes \mathfrak{A})].$$

Identifying $\Gamma(\Omega_C \otimes \mathfrak{A})$ with T^*_{α, P^+}, we find $T_{\alpha, Z} \subset W_\alpha{}^\perp$. Since the codimension of Z in P^+ is $\leqslant 4$, it follows that $\dim W_\alpha \leqslant 4$. But $\dim \Lambda^2 \Gamma(L_\alpha) = 6$, so the kernel of $\Lambda^2 \Gamma(L_\alpha) \longrightarrow \Gamma(\Omega_C \otimes \mathfrak{A})$ has dimension at least two. Now the set of decomposable 2-forms $s \wedge t$ in $\Lambda^2 \Gamma(L_\alpha)$ forms a cone in $\Lambda^2 \Gamma(L_\alpha)$ of codimension one, so at least one such $s \wedge t$ lies in the kernel. But for any s, t we find

$$\begin{aligned} s \wedge t = 0 &\Longleftrightarrow \langle s, t \rangle = \langle t, s \rangle \\ &\Longleftrightarrow s/t \in \mathbb{R}(C) \qquad \text{(as before).} \end{aligned}$$

Therefore, exactly as in the last section, $L_\alpha \cong \pi^* M(\sum x_i)$ where $\dim \Gamma(M) \geqslant 2$. Q.E.D.

We are now ready to prove the theorem—or rather reduce it to a strengthened form of a theorem of Martens which is given in the appendix. First of all, say C is hyperelliptic: Let $p\colon C \longrightarrow \mathbb{P}^1$ be the double covering and let $\{z_1, \ldots, z_{2g+2}\}$ be the branch points. It is well known that all unramified double coverings $\pi\colon \tilde{C} \longrightarrow C$ arise as follows.

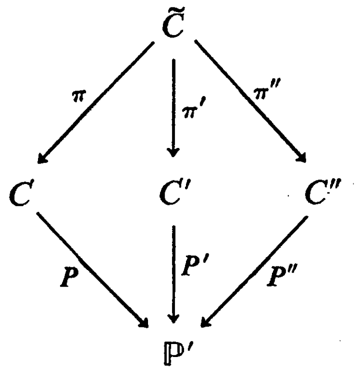
(a) Separate the z_i into two groups of even cardinality:

$$\{1, 2, \ldots, 2g+2\} = I' \cup I'', \qquad I' = 2h+2, \qquad I'' = 2k+2.$$

$I' \cap I'' = \phi$; hence $h + k + 1 = g$.

(b) Let $p'\colon C' \longrightarrow \mathbb{P}^1$ and $p''\colon C'' \longrightarrow \mathbb{P}^1$ be the hyperelliptic curves with branch points $\{z_i\}_{I'}$ and $\{z_i\}_{I''}$, respectively.

(c) Let $\tilde{C}$ be the normalization of $C \times_{\mathbb{P}^1} C'$. The $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts on $\tilde{C}$ and we get a tower of curves:



by dividing by its three subgroups of order two. Note that $\tilde{C} =$ norm. of $C \times_{\mathbb{P}^1} C'' = C' \times_{\mathbb{P}^1} C''$. I claim that in this situation

$$\operatorname{Prym}(\tilde{C}/C) \cong J' \times J''$$
$$\Xi \longleftrightarrow J' \times \Theta'' + \Theta' \times J'',$$

where J' and J'' are the Jacobians of C' and C''. (Note that if h or k is zero, one of the factors here disappears.)

Idea of Proof: Now we have $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acting on $\tilde{J}$ and up to 2-isogenies, $\tilde{J}$ splits into four "eigensubvarieties"; the part invariant under the whole group will be empty, and the other three pieces will be π^*J, J', and J''. One checks that $J' \times J''$ injects into $\tilde{J}$ by the natural map $(\pi')^* \times (\pi'')^*$, and that the image is P. Finally, one checks that the $\tilde{\Theta}$ polarization on $\tilde{J}$ splits into the sum of 2Θ, $2\Theta'$, and $2\Theta''$ on the three pieces; hence Ξ splits into the sum of Θ' and Θ''. The details are left to the reader.

Now suppose C is not hyperelliptic and that $\dim \operatorname{Sing} \Xi = v \geqslant g - 5$. Almost all of these singular points must define sheaves L_α in case 1: It follows that for some d there is a v-dimensional family of pairs $\{M, \sum_{i=1}^{e} y_i\}$ where (i) M is an invertible sheaf on C; (ii) $\sum y_i$ is an effective divisor of degree e; (iii) $\deg M = d$ and $2d + e = 2g - 2$; (iv) $\dim \Gamma(M) \geqslant 2$; (v) $M^2(\sum_{i=1}^{e} y_i) \cong \Omega_C$; and (vi) if $e = 0$, then $\dim \Gamma(M) + \dim \Gamma(M \otimes \mathfrak{A})$ even.

Now for each M the set of all divisors $\sum y_i$ of this type is a projective space whose dimension equals $\dim \Gamma(\Omega_C \otimes M^{-2}) - 1$. By Clifford's theorem, we can bound this by

$$\dim \Gamma(\Omega_C \otimes M^{-2}) - 1 < \tfrac{1}{2} \deg \Omega_C \otimes M^{-2} = g - 1 - d.$$

If $d < g - 1$, by Marten's theorem (see appendix), the dimension of the set of Ms of degree d with $\dim \Gamma(M) \geqslant 2$ is bounded by $d - 3$, and if C is not trigonal, a double cover of an elliptic curve, or a nonsingular quintic, then it is bounded by $d - 4$. Therefore

$$\nu = (\text{dim. of possible } M\text{s}) + (\text{dim. of possible } \textstyle\sum y_i) < g - 4$$

and $\nu < g - 5$, except in the aforementioned special cases. Also, if $d = g - 1$, then $M^2 \cong \Omega_C$, i.e., M is one of the finite set of theta characteristics: if $g \leqslant 5$, these can give us a $(\geqslant g - 5)$-dimensional singular locus on Ξ.

Finally, let us look at the low-genus cases: If $g = 3$, the only singularities on Ξ arise from theta characteristics M. But if C is not hyperelliptic, $\dim \Gamma(M) = 0$ or 1 for all M, so Ξ is nonsingular. Thus (P, Ξ) is a principally polarized two-dimensional Abelian variety with Ξ nonsingular: Hence it is a Jacobian. If $g = 4$ and C is not hyperelliptic, again singularities on Ξ can arise only from theta characteristics. In fact, in $\mathbb{P}^3$ the canonical model of C equals $F.G$, with F a quadric, C a cubic. And if F is nonsingular, again $\dim \Gamma(M) = 0$ or 1 for all theta characteristics M. But if F is a cone, there is one even M with $\dim \Gamma(M) = 2$—namely the M defined by the divisors C.(line on F). If also $\dim \Gamma(M \otimes \mathfrak{A})$ equals zero rather than one, then Sing Ξ has a single point. Thus (P, Ξ) is a principally polarized three dimensional Abelian variety with zero or one singularity on Ξ. Now either from the fact that the moduli space over $\mathbb{Z}$ of such varieties is irreducible six dimensional, hence Jacobians are dense in it, hence by Hoyt [6] every such variety is a Jacobian or product of Jacobians; or from Harris' thesis [5], it follows that (P, Ξ) is a Jacobian. Since a three dimensional Jacobian (J, Θ) has a singular Θ if and only if J comes from a hyperelliptic curve, this proves (c). As for (d), we have proved this already modulo noting that nonsingular quintics are precisely the nonhyperelliptic curves of genus six with sheaves N such that

$$N^2 \cong \Omega_C, \qquad \dim \Gamma(M) = 3.$$

[i.e., $N = \mathcal{O}_C(1)$]. This N defines sheaves M by $M = N(-z)$, $z \in C$, hence potential singularities of Ξ by

$$L_\alpha = \pi^*(N(-z))(x_1 + x_2).$$

Then x_1 and x_2 must satisfy

$$\Omega_C \cong N^2(-2z)(\pi x_1 + \pi x_2);$$

hence $\pi x_1 = \pi x_2 = z$. Therefore

$$L_\alpha = \pi^* N(x - ix) \qquad \text{or} \qquad L_\alpha = \pi^* N.$$

But one of these will be in P^+, the other in P^-, hence Ξ will either have a whole curve of singularities parametrized by x, or exactly one singularity, and in fact

$$\dim \text{Sing } \Xi = 1 \Longleftrightarrow \pi^* N(x - \iota x) \in P^+ \Longleftrightarrow \pi^* N \in P^-$$
$$\Longleftrightarrow \dim \Gamma(N) + \dim \Gamma(N \otimes \mathfrak{A}) \text{ odd } \Longleftrightarrow \dim \Gamma(N \otimes \mathfrak{A}) \text{ even.} \quad \text{Q.E.D.}$$

Precisely this final special case has turned out recently to be surprisingly interesting. The reason is that the Pryms (P, Ξ) arising from quintics $C \subset \mathbb{P}^2$ and double coverings

$\tilde{C} = \mathbf{Spec}(\mathcal{O}_C + \mathcal{O}_C(\mathfrak{A}))$ for which $\dim \Gamma(\mathcal{O}_C(1)(\mathfrak{A}))$ is *odd* include the intermediate Jacobians of cubic hypersurfaces in $\mathbb{P}^4$: By the corollary, these are *not* Jacobians and their Ξ has one singular point at which the tangent cone is in fact exactly the cubic hypersurface! Clemens and Griffiths [2] have given another proof that this intermediate Jacobian is not a Jacobian and have deduced from this that the cubic hypersurface is not rational. On the other hand, Clemens conjectures that when $\dim \Gamma(\mathcal{O}_C(1)(\mathfrak{A}))$ is *even*, then (P, Ξ) is a Jacobian.

APPENDIX: A THEOREM OF MARTENS

The purpose of this appendix is to somewhat strengthen Marten's theorem [8, Theorem 1] (see also Saint-Donat [14, Theorem 2.4]) as follows.

Theorem. If C is a nonsingular curve of genus g, and $W_d \subset J_d$, $1 \leqslant d \leqslant g-1$, is the locus of invertible sheaves of degree d *with sections*, then

$$\exists d, \quad 2 \leqslant d \leqslant g-2, \quad \text{such that} \quad \dim \operatorname{Sing} W_d \geqslant g-3$$
$$\Longleftrightarrow C \text{ is (a) hyperelliptic, or (b) trigonal, or}$$
$$\text{(c) double cover of an elliptic curve, or}$$
$$\text{(d) nonsingular plane quintic.}$$

Proof. Recall that by Kempf's results [7,16]

$$\operatorname{Sing} W_d = (\text{locus of inv. sheaves } L, \quad \dim \Gamma(L) \geqslant 2);$$

hence, in Marten's notations, $\operatorname{Sing} W_d = G_d^1$. Thus he shows that

$$\exists d, \quad 2 \leqslant d \leqslant g-2, \quad \dim \operatorname{Sing} W_d \geqslant d-2 \Longleftrightarrow C \quad \text{hyperelliptic.}$$

Excluding this case, we assume $\dim \operatorname{Sing} W_d = d-3$ for some d. If $d=3$,

$$\operatorname{Sing} W_3 \neq \phi \Longleftrightarrow \exists L \quad \text{of degree three,} \quad \dim \Gamma(L) \geqslant 2$$
$$\Longleftrightarrow C \quad \text{trigonal.}$$

Excluding this case, we may assume $d \geqslant 4$ (hence $g \geqslant 6$) and C not trigonal. Consider a general L of degree d with $\dim \Gamma(L) = 2$ and look at the pairing

$$\underbrace{\Gamma(L)}_{\dim 2} \otimes \underbrace{\Gamma(\Omega \otimes L^{-1})}_{\dim (g-d+1)} \xrightarrow{\ \phi\ } \Gamma(\Omega).$$

If d is the smallest d for which $\dim \operatorname{Sing} W_d = d-3$, we can assume that $\Gamma(L)$ is base-point free. Let $\alpha, \beta \in \Gamma(L)$ be a basis. Now according to Kempf's results, the pairing ϕ allows us to compute the Zariski tangent space to $\operatorname{Sing} W_d$ at any point $L \in \operatorname{Sing} W_d$ such that $\dim \Gamma(L) = 2$: namely, identify

$$T_{L, \operatorname{Sing} W_d} \subset T_{L, J_d} \cong T_{0, J} \cong H^1(\mathcal{O}_C) \cong \text{dual of} \quad \Gamma(\Omega).$$

Then he shows that

$$\operatorname{Im} \phi = (T_{L, \operatorname{Sing} W_d})^{\perp}.$$

Therefore

$$\dim(\operatorname{Im}\phi) \leqslant g - d + 3.$$

But since α and β have no common zeros, we get an exact sequence:

$$0 \longrightarrow \alpha \otimes \beta \otimes \Gamma(\Omega \otimes L^{-2}) \longrightarrow \alpha \otimes \Gamma(\Omega \otimes L^{-1}) + \beta \otimes \Gamma(\Omega \otimes L^{-1}) \longrightarrow \operatorname{Im}\phi \longrightarrow 0; \quad \text{(A.1)}$$

hence

$$\dim \operatorname{Im}\phi = 2(g - d + 1) - \dim \Gamma(\Omega \otimes L^{-2}) = g + 3 - \dim \Gamma(L^2).$$

Therefore $\dim \Gamma(L^2) \geqslant d$. In other words, the L^2s define a $(d-3)$-dimensional subset of W_{2d} of points corresponding to Ms with $\dim \Gamma(M) \geqslant d$: In Martens's notation,

$$\dim G_{2d}^{d-1} \geqslant d - 3.$$

Applying his Theorem 1 again, the only cases where this might happen are: (i) $d = 4$, $\dim \operatorname{Sing} W_4 = 1$, or (ii) $d = 5$, $g = 7$, $\dim \operatorname{Sing} W_5 = 2$.

If (i) happens, fix one L_0 of degree four, $\dim \Gamma(L_0) = 2$, $\Gamma(L_0)$ base-point free, and let L be any other. Note that $\dim \Gamma(L_0 \otimes L) = 4$ in all cases where $L \not\approx L_0$ [e.g., by computing $\Gamma(L_0 \otimes L)$ by an exact sequence like (A.1)]. Therefore by Riemann–Roch,

$$\dim \Gamma(\Omega \otimes L_0^{-1}) = \dim \Gamma(L_0) + 2g - 6 - g + 1 = g - 3$$
$$\dim \Gamma(\Omega \otimes L_0^{-1} \otimes L^{-1}) = \dim \Gamma(L_0 \otimes L) + 2g - 10 - g + 1 = g - 5.$$

Let $P_1, \ldots, P_{g-6}$ be any $g - 6$ points on C in general position. Then

$$\Gamma\left(\Omega \otimes L_0^{-1} \otimes L^{-1}\left(-\sum_{i=1}^{g-6} P_i\right)\right) \neq (0),$$

hence if s_L is a section here, and $M = \Omega \otimes L_0^{-1}(-\sum_{i=1}^{g-6} P_i)$, we find

$$s_L \otimes \Gamma(L) \subseteq \Gamma(M)$$

for all L. Note that

$$\dim \Gamma(M) = \dim \Gamma(\Omega \otimes L_0^{-1}) - (g - 6) = 3.$$

Therefore $\Gamma(M)$ defines a rational map $\pi\colon C \longrightarrow \mathbb{P}^2$ such that every base-point-free pencil $\Gamma(L)$ of degree four defines a map $C \longrightarrow \mathbb{P}^1$ which is the composition of π and a projection of $\pi(C)$ to $\mathbb{P}^1$. But if $d = \operatorname{degree}(\pi(C))$, then projecting $\pi(C)$ from a point of $\mathbb{P}^2 - \pi(C)$, or from a simple point of $\pi(C)$, gives a map of degree d, or $d - 1$, from $\pi(C)$ to $\mathbb{P}^1$. Since $\pi(C)$ has only finitely many multiple points and there are supposed to be an infinite number of Ls, we conclude that either π birational and $d \leqslant 5$ or π of degree two, $d \leqslant 3$. Since $g \geqslant 6$, C is either a nonsingular plane quintic or a double covering of an elliptic curve. Both of these do have an infinite $\operatorname{Sing} W_4$, i.e., take the line bundles $[\mathcal{O}_C \otimes \mathcal{O}_{\mathbb{P}^2}(1)](-P)$, any $P \in C$, in the first case, or π^*L, any L of degree two on the elliptic curve, in the second case.

Finally, we want to exclude (ii). Assume we have a two-dimensional family of Ls such that

$$\deg L = 5, \qquad \dim \Gamma(L) = 2, \qquad \dim \Gamma(L^2) = 5.$$

By Riemann–Roch, $\Gamma(\Omega \otimes L^{-2}) \neq (0)$; hence $L^2 \cong \Omega(-P-Q)$ for some P, Q. Since the set of all L of degree five such that $L^2 \cong \Omega(-P-Q)$ for some P and Q is irreducible and two dimensional, it follows that dim $\Gamma(L) \geqslant 2$ for any such L. Especially if $M^2 \cong \Omega$, then dim $\Gamma(M(-P)) \geqslant 2$ for every $P \in C$. Therefore dim $\Gamma(M) \geqslant 3$. But for any principally polarized Abelian variety X and symmetric theta divisor $\Theta \subset X$, Θ cannot contain all points of order two on X (see Mumford [9, p. 346]). For Jacobians this means by Riemann's theorem that there is always an M with $M^2 \cong \Omega$, $\Gamma(M) = (0)$. This is a contradiction and (ii) never occurs. Q.E.D.

Note added in proof. Corollary 4 is Proposition 5.7 of Fay [4], in which there is a misprint: The lower limit of the integral in Fay should be D. To get our version, use the remarks at the top of p. 100.

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