1) Generalities on abelian varieties
2) Prym varieties and Prym map.
deg 27 3) Compute deg ($\mathcal{P}_6 : \mathcal{R}_6 \to \mathcal{A}_5$) (Donagi-Smith, Donagi)
deg 16. 4) Compute deg ($\mathcal{P}_4(3) : \mathcal{R}_4(3) \to \mathcal{B}_6$] (Faber).
5) Overview of finite Prym maps.

§ Abelian varieties. / $\mathbb{C}$.

Def. . A cplx. torus A is a quotient $A = V/\Lambda$ $\quad V \simeq \mathbb{C}^g$, Λ = full rank lattice

$\Lambda \simeq \mathbb{Z}^{2g}$

- A polarization on A is an ample $\overset{L}{\vee}$ line bundle on A.
- An abelian variety is a cplx torus admitting a polarization

(A, L) a polarized abelian var.

Rmk. A inherites the group structure "+" from V, A is an abelian group.

By def. of ampleness: $\exists\, k > 0$ s.t. $L^{\otimes k}$ is gen. by global sections.

$$\begin{aligned} \varphi_{L^{\otimes k}} : A &\hookrightarrow \mathbb{P}H^0(A, L^{\otimes k})^\vee \simeq \mathbb{P}^N \\ x &\longmapsto [s_0(x) : \dots : s_N(x)] . \end{aligned}$$

embedding ($k \geq 3$)

- An abelian variety is an abelian group + projective variety.

Denote $\mathrm{Pic}(A)$ = group of holomorphic line bundles on A.
$\cong H^1(A, \mathcal{O}_A^*)$

Consider the s.e.s.

$$0 \to \mathbb{Z} \to \mathcal{O}_A \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}_A^* \to 0$$

$$\begin{tikzcd} \rightsquigarrow \arrow[r] & H^1(A,\mathbb{Z}) \arrow[r] & H^1(A,\mathcal{O}_A) \arrow[r] & H^1(A,\mathcal{O}_A^*) \arrow[r, "c_1"] & H^2(A,\mathbb{Z}) \\ & & & L \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & c_1(L) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$c_1(L)$ = 1st Chern class.

We have $\boxed{H^2(A,\mathbb{Z}) \cong \mathrm{Alt}^2(\Lambda,\mathbb{Z})} = \{ E : \Lambda \times \Lambda \to \mathbb{Z}$ alternating bil forms $\}$

More generally. $\Lambda^n H^1(\Lambda, \mathbb{Z}) \xrightarrow{\sim} H^n(A, \mathbb{Z})$. $n \geq 1$

Def. $NS(A) := \mathrm{Im}\left(c_1 : H^1(A, \mathcal{O}_A^*) \longrightarrow H^2(A, \mathbb{Z}) \right)$

Néron-Severi group.

Prop. Let $E : V \times V \longrightarrow \mathbb{R}$. be an alternating bilinear form.
Then. TFAE:

(i) $\exists L$ line bundle on A st. $E = c_1(L)$.

(ii) $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ $\forall v, w \in V$.

Prop. There is 1-1 corresp.

$$\{ H \text{ hermitian form on } V. \} \longleftrightarrow \left\{ \begin{array}{l} E : V \times V \longrightarrow \mathbb{R}. \text{ alternating} \\ E(iv, iw) = E(v, w) \end{array} \right\}$$

$$H \longmapsto E = \mathrm{Im} H$$

$$H(v, w) = E(iv, w) + iE(v, w). \longmapsfrom E$$

- Appell-Humbert thm.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Lambda, \overset{\{z \in \mathbb{C} \mid |z| = 1\}}{\overset{\shortparallel}{\mathbb{C}_1}}) \arrow[r] \arrow[d, phantom, "\wr"] & \mathcal{P}(\Lambda) \arrow[r] \arrow[d, "\wr"'] & NS(A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(A) \arrow[r] & \mathrm{Pic}(A) \arrow[r] & NS(A) \arrow[r] & 0 \\
& (0, \chi) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & & H \arrow[u, phantom, "\rotatebox{90}{$\in$}"] &
\end{tikzcd}$$

$\mathrm{Pic}\, A \ni L \longleftrightarrow (H, \chi)$ χ Semi-character. for H

$\chi : \Lambda \longrightarrow \mathbb{C}_1$.

$\chi(\lambda + \mu) = \chi(\lambda) \cdot \chi(\mu) \exp(\pi i \, \mathrm{Im} H(\lambda, \mu))$

$\rightsquigarrow NS(A) \cong \mathrm{Pic}(A) / \mathrm{Pic}^0(A)$ is a torus $\forall \lambda, \mu \in \Lambda$

$\bar{\Omega} := \mathrm{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C})$. $\mathbb{C}$-anti-linear forms $l : V \to \mathbb{C}$

$\mathbb{R}$- linear form $\langle \, , \, \rangle : \bar{\Omega} \times V \longrightarrow \mathbb{R}$ non-deg.

$\langle l, v \rangle := \mathrm{Im}\, l(v)$

$\hat{\Lambda} := \{ l \in \bar{\Omega} \mid \langle l, \Lambda \rangle \subseteq \mathbb{Z} \}$ dual lattice of Λ

$\rightsquigarrow \hat{A} : \bar{\Omega}/\hat{\Lambda}$

Prop. $\bar{\Omega} \xrightarrow{\alpha} \mathrm{Hom}(\Lambda, \mathbb{C}_1)$ induces an isom. $\boxed{\hat{A} \xrightarrow{\sim} \mathrm{Pic}^0(A)}$ identified.

$\ell \longmapsto \exp(2\pi i \langle \ell, \cdot \rangle)$

Proof. $\langle , \rangle$ is non-deg. $\Rightarrow$ this map is surjectiv and $\mathrm{Ker}\,\alpha = \hat{\Lambda}$ ☒

For (A, L) polarized ab. var.

$$\begin{tikzcd} t_x^* L \arrow[d] & L \arrow[d] \\ A \arrow[r, "t_x"'] & A \end{tikzcd}$$

$t_x : A \to A$ translation, $a \mapsto a+x$

$$\varphi_L : A \longrightarrow \hat{A} = \mathrm{Pic}^0(A)$$
$$x \longmapsto t_x^* L \otimes L^{-1}$$

φ_L is a homomorphism of groups by the thm. of square.

In fact, φ_L is an isogeny, i.e. a surjective morphism (hom. of groups) between ab. varieties of the same dimension.

More generally: " $\hat{}$ " is a contravariant functor. (cplx. tori)

$$f : X_1 \longrightarrow X_2 \rightsquigarrow \hat{f} : \hat{X}_2 \longrightarrow \hat{X}_1$$

Def. $K(L) := \ker \varphi_L$ finite abelian subgroup of A.

$= \Lambda(L)/\Lambda \qquad \Lambda(L) = \{ v \in V \mid \mathrm{Im}\, H(v, \Lambda) \subseteq \mathbb{Z} \}$.

Rmk φ_L depends only on $c_1(L) = H$

= Summarizing =

Different encarnations of a pol. L on A

- A first Chern class $c_1(L) \in H^2(A, \mathbb{Z})$ L ampl l.b. on A
- A non-degenerated alternating form $E : V \times V \to \mathbb{R}$
  $E(\Lambda, \Lambda) \subset \mathbb{Z}$, $E(iv, iw) = E(v, w)$ $\forall v, w \in V$
- A non-deg. Hermitian form. $H : V \times V \to \mathbb{C}$ with $\mathrm{Im}\, H(\Lambda, \Lambda) \subseteq \mathbb{Z}$
- An isogeny $\Phi_L : A \longrightarrow \hat{A} = \mathrm{Pic}^0(A)$
- A Weil divisor $\Theta \subset A$ s.t. $\{ x \in A \mid t_x^* \Theta \sim \Theta \}$
  theta divisor   linear equiv.

is finite $\qquad \left(L = \mathcal{O}_A(\Theta) \right)$ $\qquad$ Ker Φ_L.

$\Bigg($ L, L' pol. define the same pol. when. $\varphi_L = \varphi_{L'}$

$\iff L \cong t_x L' \iff \Theta \underset{\text{alg.}}{\equiv} \Theta'$ $\Bigg)$

Let E be an alternating form repres. a polarization L on $A = V/_\Lambda$
There exists. a basis.

Basis of V as $\mathbb{R}$-VS. $\rightsquigarrow$ $\lambda_1, \lambda_2 \ldots \lambda_g, \mu_1, \ldots \mu_g$ of Λ wrt which.

E is given by the matrix

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}_{2g \times 2g}$$

$D = \operatorname{diag}(d_1 \ldots d_g) \qquad d_i \mid d_{i+1} \quad i = 1, 2 \ldots g-1 \qquad d_i > 0$ (non-deg E)

Def. The vector $(d_1 \ldots d_g)$ is called the type of the polarization
A polarization is principal if it is of type

$$\underbrace{(1 \ldots 1)}_{g = \dim A}$$

Theorem (Riemann-Roch) $\quad L$ non-deg. (positive definite $c_1(L)$)

$$\chi(L) = \operatorname{Pf}(E) = d_1 d_2 \cdots d_g$$

$$=: \sum_{\nu=0}^{g} (-1)^\nu h^\nu(A, L)$$

$$= h^0(A, L)$$

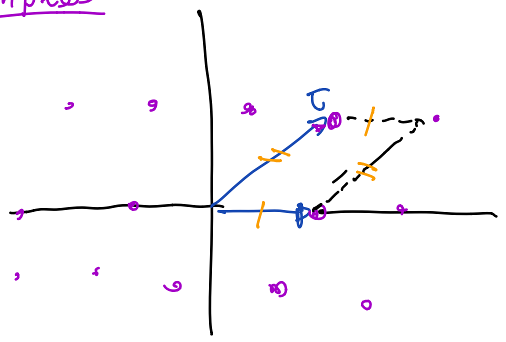
$\left[\det E = \operatorname{Pf}(E)^2 \right]$

$\boxed{\text{Rmk. } L \text{ is principal. } h^0(C, L) = 1}$

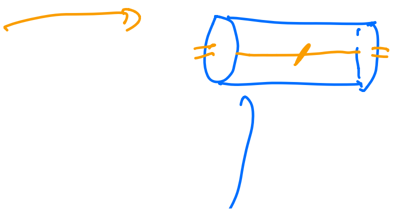
Moreover. $\deg \Phi_L = \det E$ so $\chi(L)^2 = \deg \Phi_L$

Examples $\qquad \mathbb{C} \qquad \operatorname{Im}\tau > 0$

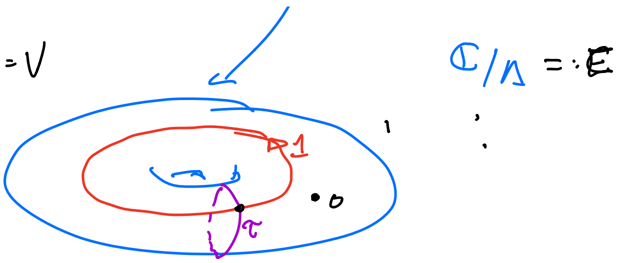
$g = 1$



$\Lambda = \tau\mathbb{Z} \oplus 1 \cdot \mathbb{Z} \qquad \mathbb{C}/\Lambda$



In the basis $\{1, \tau\}$ of $\mathbb{C} = V$



$E : V \times V \longrightarrow \mathbb{R}$ given by $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

intersection form on $H_1(E, \mathbb{Z}) \simeq \mathbb{Z}^2$

$\rightsquigarrow$ principal pol. ab. variety of dim 1 $\Leftrightarrow$ elliptic curve. $(g=1)$

$L = \mathcal{O}_E(0)$ of $\mathrm{Div}^1(E)$ $\rightsquigarrow$ $h^0(L) = 1$

$L^{\otimes 3} = \mathcal{O}_E(3 \cdot 0)$ $\leftarrow$ deg. 3 $\quad h^0(E, L^{\otimes 3}) = 3$

$E \xhookrightarrow[\deg 3]{} \mathbb{P}H^0(E, L^{\otimes 3}) \simeq \mathbb{P}^2$ $\rightsquigarrow$ E plane cubic.

Higher dimension: Jacobian of a curve

Let C be a smooth proj. alg. curve. $g = \mathrm{genus}(C)$

$H_1(C, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$

‖ group of closed paths in C.

We have a map:

holomorphic differentials on C.

$$\begin{array}{ccc} \mathbb{Z}^{2g} \simeq H_1(C, \mathbb{Z}) & \hookrightarrow & H^0(C, \omega_C)^\vee \simeq \mathbb{C}^g \\ \rotatebox{90}{$\in$} & & \\ \gamma & \longmapsto & \{\omega \longmapsto \int_\gamma \omega\} \end{array}$$

Def. The Jacobian of C is

$$JC := H^0(C, \omega_C)^\vee / H_1(C, \mathbb{Z}) \simeq \mathbb{C}^g / \mathbb{Z}^{2g}$$

$\dim JC = g(C) = g$. The intersection product on $H_1(C, \mathbb{Z}$

$$H_1(C, \mathbb{Z}) \times H_1(C, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(\gamma_1, \gamma_2) \longmapsto \gamma_1 \cdot \gamma_2$$

$\rightsquigarrow \quad E = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}$ principal polarization. denoted by $\Theta \subset JC$ (divisor).

$c_1(L) = E \qquad L = \mathcal{O}_{JC}(\Theta) \qquad h^0(JC, \Theta) = 1$

Rmk. The Jacobian of an elliptic curve is isomorphic to itself.

§ Abel-Jacobi map.

$\mathrm{Pic}^0(C)$ = group of line bundles of deg 0 on C.

$= \mathrm{Div}^0(C) / _{\text{Princ. div.}}$

Def. Abel-Jacobi map $\quad \alpha : \mathrm{Div}^0(C) \longrightarrow JC$

$$D = \sum_\nu p_\nu - q_\nu \longmapsto \left\{ \omega \longmapsto \sum_\nu \int_{p_\nu}^{q_\nu} \omega \right\} \quad \text{mod } H_1(C, \mathbb{Z})$$

Thm. This map induces an isom

$$\boxed{\mathrm{Pic}^0(C) \xrightarrow{\sim} JC}$$

Alternatively:

$$\alpha_{D_n} : C^{(n)} \longrightarrow JC \qquad C^{(n)} = C^n / S_n \text{ unordered } n\text{-tuple.}$$

$$\sum n_\nu p_\nu \longmapsto \left\{ \omega \longmapsto \sum_\nu \int_c^{p_\nu} \omega \right\} \quad \text{mod } H_1(C, \mathbb{Z})$$

$D_n = n \cdot c \qquad c \in C$ fix

$$\beta : C^{(n)} \longrightarrow \mathrm{Pic}^n(C) = \{\text{l.b. of deg. } n \text{ on } C\}$$
$$D \longmapsto \mathcal{O}_C(D)$$

$\beta^{-1}(L) = |L|$ linear system of L (effective div. $\sim_{\text{lin}} L$)

Prop. The "projectivized differential" of Abel-Jacobi map

$$\alpha_c : C \hookrightarrow JC \qquad c \in C \text{ fix}$$
$$p \longmapsto \mathcal{O}_C(p - c)$$

$$\begin{tikzcd}
\mathbb{P}(d\alpha_\alpha): \mathbb{P}(\mathbb{T}C) \arrow[r] \arrow[d, phantom, "\wr"] & \mathbb{P}(\mathbb{T}JC) \arrow[d, phantom, "\parallel"] & T_0 JC \simeq H^0(C,\omega_C)^\vee \\
C & \mathbb{P}(\underbrace{JC} \times \mathbb{P}H^0(C,\omega_C)^\vee)\,. \arrow[r, green] & \mathbb{P}H^0(C,\omega_C)^\vee.
\end{tikzcd}$$

$$p \longmapsfrom [p, \underbrace{[\omega_1(p) : \dots : \omega_g(p)]}]$$

is the canonical map.

$$\varphi_{\omega_C} : C \longrightarrow \mathbb{P}H^0(C,\omega_C)^\vee \simeq \mathbb{P}^{g-1}.$$

$$p \longmapsto [\omega_1(p) : \dots : \omega_g(p)]\,.$$