Hyperkähler manifolds:

C^∞ manifolds: For a C^∞ manifold M, we denote T_M the tangent bundle of M and T_M^* the cotangent bundle.

For any (k, l), the sections of the bundle

$$T_M^{\otimes k} \otimes (T_M^*)^{l} \quad \text{are called } (k,l)\text{-tensors.}$$

Sections of $\Lambda^k T_M^*$ are differential k-forms.

Alternatively, vector fields (sections of T_M) can be defined as first order differential operators on C^∞ functions.

In a local coordinate chart $(x^1, \dots, x^n)$, the (local) vector fields $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ form a basis of the tangent bundle.

The local 1-forms $dx^1, \dots, dx^n$ form a basis of differential 1-forms.

local (k, l)-tensor:

$$T = \sum T^{i_1, \dots, i_k}_{j_1, \dots, j_l} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l}$$

<u>The Lie bracket</u>: Given two vector fields

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} \qquad w = \sum_{i=1}^{n} w^i \frac{\partial}{\partial x^i}$$

The Lie bracket $[v,w]$ can be locally expressed as

$$[v,w] = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

for C^∞ functions f on M:

$$[v,w](f) := v(w(f)) - w(v(f))$$

$$\left(v(f) = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i} \right)$$

<u>Connections:</u> For a C^∞ vector bundle E on M, a connection is a linear map

$$\nabla : C^\infty(E) \longrightarrow C^\infty(E \otimes T_M^*)$$

(equivalently $\nabla : C^\infty(E \otimes T_M) \longrightarrow C^\infty(E)$)

satisfying the Leibnitz rule:

$$\nabla(fe) = f\nabla(e) + e \otimes df$$

for all C^∞ sections e of E and C^∞ functions f on M.

For any vector field v, ∇ defines a linear map

$$\nabla_v : C^\infty(E) \longrightarrow C^\infty(E)$$

$$e \longmapsto \nabla_v(e) := (\nabla(e))(v)$$

or if we think of $\nabla : C^\infty(E \otimes T_M) \to C^\infty(E)$

$$\nabla_v(e) = \nabla(e \otimes v)$$

When $E = T_M$, the torsion of a connection

$$\nabla : C^\infty(T_M \otimes T_M) \longrightarrow C^\infty(T_M)$$

is defined as

$$T(v \otimes w) := \nabla_v(w) - \nabla_w(v) - [v, w]$$

We say ∇ is torsion free or symmetric

if $T = 0$.

Curvature: The curvature of a connection ∇ is

a linear map $R : C^\infty(E) \longrightarrow C^\infty(E \otimes \Lambda^2 T_M^*)$

or $R : C^\infty(E \otimes \Lambda^2 T_M) \longrightarrow C^\infty(E)$

or $R \in C^\infty(\mathrm{End}(E) \otimes \Lambda^2 T^*_M)$

$$\shortparallel$$

$$C^\infty(E^* \otimes E \otimes \Lambda^2 T^*_M)$$

defined via its action on sections e of E and vector fields v, w as:

$$R\big(e \otimes (v \wedge w)\big) = \nabla_v\big(\nabla_w(e)\big) - \nabla_w\big(\nabla_v(e)\big) - \nabla_{[v,w]}(e)$$

In local coordinates: $(x^1, \dots, x^n)$:

$$R\Big(e \otimes \Big(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}\Big)\Big) = \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}}(e) - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}}(e)$$

because $\Big[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big] = 0$

Def: We say ∇ is flat if $R=0$

Riemannian manifolds:

A C^∞ manifold is called Riemannian if it has a Riemannian metric, i.e., a $(2,0)$-tensor $g \in C^\infty((T_M^*)^{\otimes 2})$ which is symmetric, i.e.,

$$g \in C^\infty(\mathrm{Sym}^2 T_M^*)$$

and defines a positive definite quadratic form on the tangent space T_xM $\forall x \in M$.

$(g_x(v,v) > 0 \quad \forall v \in T_xM, v \neq 0)$.

Parallel transport:

Given a C^∞ vector bundle E on M with a connection $\nabla : C^\infty(E) \to C^\infty(E \otimes T_M^*)$,

and a smooth curve $\gamma : [0,1] \to M$.

The pull-back $\gamma^* E$ is a C^∞ vector bundle on $[0,1]$ with fiber $E_{\gamma(t)}$ at $t \in [0,1]$.

The connection ∇ defines a connection

$$\begin{tikzcd} \gamma^*\nabla : \gamma^* E \arrow[r] \arrow[rr, bend right, "\gamma^*\nabla"'] & \gamma^* E \otimes \gamma^* T_M^* \arrow[r, two heads] & \gamma^* E \otimes T^*_{[0,1]} \end{tikzcd}$$

In local coordinates:

$$\gamma(t) = (x^1(t), \dots, x^n(t))$$

$$\dot{\gamma}(t) := (\dot{x}^1(t), \dots, \dot{x}^n(t))$$

For all sections e of E

$$\nabla_{\dot{\gamma}(t)} e = \nabla_{\sum_{i=1}^{n} \dot{x}^i(t) \frac{\partial}{\partial x^i}} (e)$$

$$= \sum_{i=1}^{n} \dot{x}^i(t) \nabla_{\frac{\partial}{\partial x^i}} (e)$$

<u>Def & Prop.:</u> Put $x = \gamma(0)$, $y = \gamma(1)$.

Then, for all $e \in E_x = (\gamma^* E)_0$, there exists a unique smooth section s of $\gamma^* E$ s.t. $s(0) = e$ and $\gamma^* \nabla(s) = 0$, i.e., $\nabla_{\dot{\gamma}(t)}(s) = 0$.

The parallel transport of e along γ to y is $P_\gamma(e) := s(1) \in E_y = (\gamma^*E)_1$.

The map

$$P_\gamma : E_x \longrightarrow E_y$$

is a linear isomorphism.

<u>Holonomy:</u> <u>Def. & Prop.:</u> If γ is a loop ($\gamma(0) = \gamma(1) = x = y$), then $P_\gamma \in GL(E_x)$.

The holonomy $Hol_x(\nabla)$ at x is the image of P:

$$\mathrm{Hol}_x(\nabla) := \{ P_\gamma \mid \gamma \text{ loop based at } x \}.$$

it has the following properties:

(1) $\mathrm{Hol}_x(\nabla)$ is a Lie subgroup of $GL(E_x)$:

composition:
$$\gamma\delta\,(t) = \begin{cases} \delta(2t) & \text{if } t \in [0, \tfrac{1}{2}] \\ \gamma(2t-1) & \text{if } t \in [\tfrac{1}{2}, 1] \end{cases}$$

$$\gamma^{-1}(t) = \gamma(1-t)$$

$$P_{\gamma\delta} = P_\gamma \circ P_\delta \quad \text{and} \quad (P_\gamma)^{-1} = P_{\gamma^{-1}}$$

(2) If γ is a path from x to y, then
$$\mathrm{Hol}_y(\nabla) = P_\gamma \, \mathrm{Hol}_x(\nabla) \, P_\gamma^{-1}$$

Hence, up to conjugation, $Hol_x(\nabla)$ depends only on the connected component of M containing x (as a subgroup of $GL_m = GL(E_x)$ if $E_x \cong \mathbb{R}^m$)

(3) If M is simply connected, then $Hol_x(\nabla)$ is connected:
(any loop can be shrunk to a point, $\Rightarrow$ path in $Hol_x(\nabla)$ from P_{γ_0} to P_{γ_1})

(4) Relation with the curvature R of ∇:

$$\mathrm{Hol}_x(\nabla) \subset GL(E_x) \quad \text{Lie subgroup}$$

Lie algebra: $\quad \mathfrak{hol}_x(\nabla) \subset \mathrm{End}(E_x)$

recall $\quad R(\nabla) \in C^\infty(\mathrm{End}(E) \otimes \Lambda^2 T_M^*)$

at $x \in M$: $\quad R(\nabla)_x \in \mathrm{End}(E_x) \otimes \Lambda^2 T_x^* M$

<u>Claim:</u> $\quad R(\nabla)_x \quad \mathfrak{hol}_x(\nabla) \otimes \Lambda^2 T_x^* M.$

The connection ∇ induces connections on all $E^{\otimes k} \otimes (E^*)^{\otimes l}$ and all exterior and symmetric powers of E, E^* and their tensor products as well. We will denote these also by ∇.

Def: A tensor S is called (covariantly) constant if $\nabla(S) = 0$

Theorem: For a tensor S, $\nabla(S) = 0$ iff S is fixed by $Hol_x(\nabla)$, iff

$$P_\gamma(S(x)) = S(y) \quad \forall\, x, y \in M.$$

Levi-Civita connection: Suppose (M, g) is a Riemannian manifold. The fundamental theorem of Riemannian geometry:

Theorem: There exists a unique torsion free connection ∇ on T_M s.t. $\nabla g = 0$. This unique connection is called the Levi-Civita or Riemannian connection of (M, g).

Let $\nabla : T_M \longrightarrow T_M \otimes T_M^*$ be the L.C. connection.

$R(\nabla) : T_M \longrightarrow T_M \otimes \Lambda^2 T_M^*$ (1,3) tensor

$g : T_M \xrightarrow{\cong} T_M^*$ metric. (0,2) tensor

define the (0,4)- tensor $\tilde{R}$ as the composition:

$$\begin{tikzcd}
T_M \arrow[r, "R"] \arrow[rr, bend right, "\tilde{R}"'] & T_M \otimes \Lambda^2 T_M^* \arrow[r, "g \otimes Id"] & T_M^* \otimes \Lambda^2 T_M^*
\end{tikzcd}$$

One can show $\tilde{R} \in \mathrm{Sym}^2(\Lambda^2 T_M^*)$

$$\begin{array}{c} \mathrm{Sym}^2(\Lambda^2 T_M^*) \\ \cap \\ \Lambda^2 T_M^* \otimes \Lambda^2 T_M^* \\ \cap \\ T_M^* \otimes T_M^* \otimes \Lambda^2 T_M^* \end{array}$$

Bianchi identities (in notes) (more symmetries of $\tilde{R}$)

<u>Also</u>: $\tilde{R}_x \in \mathrm{Sym}^2 \mathrm{hol}_x(\nabla) \subset \mathrm{Sym}^2(\Lambda^2 T_x^* M)$

$$\begin{array}{c} \mathrm{Sym}^2 \mathrm{hol}_x(\nabla) \\ \cap \\ \mathrm{hol}_x(\nabla) \otimes \Lambda^2 T_x^* M \end{array}$$

<u>Notation</u>: Since ∇ is uniquely determined by g, we write $\mathrm{Hol}_x(g) := \mathrm{Hol}_x(\nabla)$, $\mathrm{hol}_x(g) = \mathrm{hol}_x(\nabla)$.

Symmetric and locally symmetric spaces:

<u>Def</u>: A Riemannian manifold is called (locally) reducible if every point has a neighborhood isometric to a product. It is called irreducible if it is not (locally) reducible.

<u>Prop.</u>: Suppose a neighborhood of $x \in M$ is isometric to the product $(M_1, g_1) \times (M_2, g_2)$.

Then $$\mathrm{Hol}_x(g_1 \times g_2) = \mathrm{Hol}_x(g_1) \times \mathrm{Hol}_x(g_2)$$

<u>Theorem</u>: If (M, g) is irreducible at x, then $\mathbb{R}^n = T_x M$ is an irreducible representation of $\mathrm{Hol}_x(g)$.

Def: A Riemannian manifold is called symmetric if, $\forall\, p \in M$, $\exists$ an isometry $s_p : M \longrightarrow M$ such that $s_p^2 = Id$ and p is an isolated fixed point of s_p.

Def: A Riemannian manifold is locally symmetric if every point has an open neighborhood isometric to an open subset of a symmetric space. It is called nonsymmetric if it is not locally symmetric.

Theorem: (M, g) is locally symmetric iff $\nabla R = 0$

Geodesics and completeness:

<u>Def</u>: A geodesic is a parametrized smooth curve $\gamma : (a,b) \longrightarrow M$ s.t., $\forall\ t \in (a,b)$, $\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0$.

Can prove are (locally) length minimizing.

<u>Theorem</u>: $\forall\ p \in M$, $v \in T_p M$, $\exists!$ geodesic $\gamma : (a,b) \longrightarrow M$ s.t. $\gamma(0) = p$, $\dot\gamma(0) = v$.

<u>Def</u>: A manifold (M,g) is complete if every geodesic can be defined on $\mathbb{R} \supset (a,b)$.

Theorem: Suppose (M,g) is a connected, simply connected symmetric space. Then (M,g) is complete. If we put

$$G := \{ s_p \circ s_q \mid p, q \in M \} \subset \mathrm{Isom}(M).$$

Then G is a connected Lie group. Choose $p \in M$ and let $H \subset G$ be the stabilizer of p. Then H is a closed Lie subgroup of G and the map

$$\begin{aligned} G/H &\longrightarrow M \\ g &\longmapsto g(p) \end{aligned}$$

is a diffeomorphism.

De Rham's theorem:

(M,g) Riemannian, complete, simply connected.
Then (M,g) is isometric to a product

$$M_0 \times M_1 \times \cdots \times M_k$$

where M_0 is a Euclidean space, $M_1, \dots, M_k$ are irreducible. The decomposition is unique up to reordering $M_1, \dots, M_k$. The holonomy group of (M,g) is the product of the holonomy groups of $M_1, \dots, M_k$.

Recall that when M is connected, then, up to conjugation, $\mathrm{Hol}_M(g) := \mathrm{Hol}_x(g)$ is a well-defined
(for any $x \in M$)
Lie subgroup of $GL_n(\mathbb{R})$.

<u>Def</u>: The restricted holonomy group $\mathrm{Hol}(g)^0$ is the connected component of the identity of
$$\mathrm{Hol}(g) := \mathrm{Hol}_M(g) \subset GL_n(\mathbb{R}).$$

<u>Berger's classification theorem</u>: Suppose (M, g) is complete, connected, non symmetric, irreducible. Then the restricted holonomy group $\mathrm{Hol}(g)^0$ is one of the following.

(1) $\mathrm{Hol}(g)^0 \cong SO(n)$ (automorphisms of $\mathbb{R}^n$, generic metric)

(2) $n = 2m \geq 4$, $\mathrm{Hol}(g)^0 = U(m) \subset SO(n)$ (automorphisms of $\mathbb{C}^m$, Kähler)

(3) $n = 2m \geq 4$, $\mathrm{Hol}(g)^0 = SU(m)$ (automorphisms of $\mathbb{C}^m$, Calabi-Yau, Kähler, Ricci-flat)

(4) $n = 4r \geq 4$, $\mathrm{Hol}(g)^0 = Sp(r) \subset SO(n)$ (automorphisms of $\mathbb{H}^r$, hyperKähler, Ricci-flat, Kähler)

(5) $n = 4r \geq 8$ $\mathrm{Hol}(g)^0 = Sp(r)Sp(1) \subset SO(n)$ (automorphisms of $\mathbb{H}^r$, quaternionic Kähler, not Kähler,

not Ricci-flat, Einstein)

(6) $n=7$, $\mathrm{Hol}(g)^\circ = G_2 \subset SO(7)$
(automorphisms of $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$, exceptional, Ricci-flat)

(7) $n=8$, $\mathrm{Hol}(g)^\circ = \mathrm{Spin}(7) \subset SO(8)$
(automorphisms of $\mathbb{O} \cong \mathbb{R}^8$, exceptional, Ricci-flat)

Kähler manifolds:

M a complex manifold, multiplication by i defines $I : T_M \longrightarrow T_M$ satisfying $I^2 = -Id$. the complex structure operator of M.

A metric is called Hermitian if

$$g(v,w) = g(Iv, Iw) \quad \forall\ v, w \text{ vector fields.}$$

The (1,1)-form associated to g and I is

$$\omega(v,w) := g(Iv, w) \quad \forall\ v, w \text{ vector fields.}$$

ω is a (1,1)-form means

$$\omega(Iv, Iw) = \omega(v,w) \quad \forall\ v, w.$$

One also checks that ω is anti-symmetric.

It is easy to check that any two of $\{I, g, \omega\}$ determines the third.

Definition and proposition: The metric g is Kähler with respect to I if one of the following equivalent conditions hold:

(1) $d\omega = 0$

(2) $\nabla \omega = 0$

(3) $\nabla I = 0$.

ω is then called the Kähler form of g.

Equivalently $Hol(g)$ preserves I and ω.

The subgroup of $SO(n)$ preserving I is $U(m)$ $(n = 2m)$.

So M is Kähler iff $\mathrm{Hol}(g) \subset U(n)$.

Recall the Ricci curvature $\mathrm{Ricci}: T_M \to T_M^*$.

We define the Ricci form

$$\rho(v,w) := \mathrm{Ricci}(Iv, w) \quad \forall\, v, w \text{ vector fields.}$$

equivalently
$$\rho: \begin{tikzcd} T_M \arrow[r, "I"] \arrow[rr, bend right] & T_M \arrow[r, "\mathrm{Ricci}"] & T_M^* \end{tikzcd}$$

and
$$\omega: \begin{tikzcd} T_M \arrow[r, "I"] \arrow[rr, bend right] & T_M \arrow[r, "g"] & T_M^* \end{tikzcd}$$

<u>Proposition:</u> ρ is a $(1,1)$-form. Its cohomology class in $H^2(M, \mathbb{R})$ is $[\rho] = 2\pi\, c_1(K_M)$
$$= 2\pi\, c_1(T_M^*).$$

Ricci-flatness: We say M is Ricci-flat if $\rho = 0$.

Can check: ρ is the curvature form of the connection induced by the Levi-Civita connection on $K_M := \Lambda^n T^*_M$.

So if $\rho = 0$, then K_M is a flat bundle.

In this case M is Calabi-Yau, $K_M \cong \mathcal{O}_M$ and $\mathrm{Hol}(g) \subset SU(n)$.

Recall: The Ricci curvature is the $(0,2)$-tensor defined as

$$\mathrm{Ricci}_x : T_x M \times T_x M \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto \mathrm{tr}\left(u \mapsto R_x(u, w)v\right)$$

if we write $R = \sum R^a{}_{bcd} \frac{\partial}{\partial x^a} \otimes dx^b \otimes (dx^c \wedge dx^d)$

then $\mathrm{Ricci} = \sum \mathrm{Ricci}_{ab}\, dx^a \otimes dx^b$

where $\mathrm{Ricci}_{ab} = \sum_c R^c{}_{acb}$

The hyperkähler case:

$$\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

$$i^2 = j^2 = k^2 = ijk = -1$$

The Lie group $Sp(n)$ is the group of $\mathbb{R}$-linear endomorphisms of $\mathbb{H}^n$ preserving a quaternionic Hermitian form q:

$$q(av, bw) = \bar{a}b\, q(v,w) \quad \forall\ a, b \in \mathbb{H},\ v, w \in \mathbb{H}^n$$

where if $a = \lambda + \mu i + \nu j + \rho k$, then $\bar{a} = \lambda - \mu i - \nu j - \rho k$.

Such a q can be represented by an $n \times n$ matrix A with entries in $\mathbb{H}$ s.t. $A \bar{A}^t = Id_{\mathbb{H}^n}$

Each time we choose $i \in \mathbb{H}$ s.t. $i^2 = -1$, we get an embedding $\boxed{Sp(n) \subset SU(2n)}$: complete i to a quaternionic triple (i, j, k), represent multiplication by i, j, k on $\mathbb{H}^n$ by matrices I, J, K. We can write

$$A = H + \Omega J$$

($\mathbb{C}$) Hermitian w.r.t. i

anti-symmetric with entries only involving i

Then: $n \times n$ matrix $\in Sp(n) \iff$ commutes with A

$\iff$ commutes with H and Ω.

(can get $Sp(n) \subset U(2n)$ by thinking of $U(2n)$ as the group of transformations of $\mathbb{H}^n$ commuting with $\mathbb{H}$)

Given a Riemannian manifold M with $Hol(g) \subset Sp(n)$, we can identify T_pM with $\mathbb{H}^n$ to obtain a sphere of complex structures:

$$\lambda = aI + bJ + cK$$

with $a^2 + b^2 + c^2 = 1$ $\quad \left((a,b,c) \in S^2 \subset \mathbb{R}^3 \right)$

I, J, K are obtained from $T_pM \cong \mathbb{H}^n$

Can check $\nabla\lambda = 0 \quad \forall \lambda = aI + bJ + cK$.

$\Rightarrow$ g is Kähler w.r.t. all these complex structures.

So we have a sphere of Kähler structures on M.

Comparison: Calabi-Yau case: $\mathrm{Hol}(g) = SU(n)$

$\exists!$ complex Kähler structure

Hyperkähler case: $\mathrm{Hol}(g) \subseteq Sp(n)$

at least a sphere of Kähler complex structures.

Def: M is irreducible Hyperkähler if $\mathrm{Hol}(g) = Sp(n)$.

In such a case there is exactly one sphere of Kähler complex structures.

Extreme case: If $\mathrm{Hol}(g)=0$ we can have more Kähler complex structures.

example: $M=$ complex torus, then $\mathrm{Hol}(g)=0$

The decomposition theorem:

Let (M,I,g) be compact Kähler, complete, Ricci-flat manifold. Then:

(1) The universal cover of M is isomorphic to

$$\mathbb{C}^k \times \prod_i V_i \times \prod_j X_j \quad \text{where:}$$

$\mathbb{C}^k$ has the standard Kähler metric, and, for all i

V_i is compact, simply connected (irreducible) with holonomy $SU(m_i)$

and $\forall j$, X_j is compact simply connected (irreducible) with holonomy $Sp(r_j)$

(2) there exists a finite étale cover M' of M isomorphic to $T \times \prod_i Y_i \times \prod_j X_j$ where T is a complex torus.

Hyperkähler manifolds are "holomorphic symplectic"

<u>Prop.</u>: Suppose (M, I, g) is compact Kähler, simply connected, Ricci-flat of dim $2r$ ($/\mathbb{C}$) with holonomy group $Sp(r)$. Then

(1) $\exists$ a holomorphic 2-form φ on M which is non-degenerate everywhere as a map $T_M \to T_M^*$.

i.e., $\varphi : T_M \xrightarrow{\cong} \Omega^1_M$.

(unique up to multiplication by a scalar)

(2) $\forall \; 0 \le p \le n$

$$H^0(M, \Omega_M^{2p+1}) = 0 \qquad H^0(M, \Omega^{2p}_M) = \mathbb{C}\varphi^p .$$

<u>Def</u>: A complex manifold M is called holomorphic symplectic if $\exists \; \varphi : T_M \xrightarrow{\cong} \Omega^1_M$ holomorphic 2-form. M is called irreducible if φ is unique up to multiplication by a scalar.

Examples: Surfaces: $Sp(1) = SU(2)$

hyperkähler ↗ ↖ Calabi-Yau

$\Rightarrow$ hyperkähler = Calabi-Yau.

K3 surfaces, Complex tori

Def: A K3 surface is a compact complex manifold of dim. 2 s.t. $\Omega^2_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

One can prove: they are simply connected

integral cohomology is torsion free

Kähler (hard) unique Kähler metric

e.g.: (1) Double covers of $\mathbb{P}^2$ branched along smooth sextics.

(2) Smooth quartics in $\mathbb{P}^3$

(3) $(2,3)$ complete intersections in $\mathbb{P}^4$

(4) $(2,2,2)$ " " " $\mathbb{P}^5$.

Aside: $Sp(n) \subset$ automorphisms of $\mathbb{H}^n$

$Sp(1) = SU(2) = S^3 \subset \mathbb{H}$

group of unit length quaternions.

$Sp(1)$ acts on $\mathbb{H}^n$ via scalar matrix multiplication

$Sp(n)\, Sp(1) =$ subgroup of $\mathrm{Aut}(\mathbb{H}^n)$ generated by the actions of $Sp(n)$ and $Sp(1)$.

Higher dimensional compact hyperkähler manifolds.

Hilbert schemes of points:

S = compact complex manifold of dim. 2

$S^n :=$ the n-th Cartesian power of S

$S^{(n)} := \quad S^n / \mathfrak{S}_n \qquad$ the n-th symmetric power of S.

$\pi : S^n \twoheadrightarrow S^{(n)}$

$S^{(n)} =$ effective zero cycles

$\quad p \in S^n \qquad p = (x_1, \dots, x_n) \qquad x_i \in S$

$\quad \bar{p} = \pi(p) \qquad \bar{p} = x_1 + \dots + x_n$

$\Delta_{ij} :=$ the diagonal of S^n where $x_i = x_j$

The action of $\mathfrak{S}_n$ is not free on the diagonals.

The stabilizer of a generic point of $\Delta_{ij} = \{Id, (ij)\}$

$\pi : S^n \twoheadrightarrow S^{(n)}$ is étale away from $\bigcup_{ij} \Delta_{ij}$

$\operatorname{codim}(\Delta_{ij} \subset S^n) = 2$

π is ramified exactly along $\cup \Delta_{ij}$

$\Rightarrow$ $S^{(n)}$ cannot be smooth

$S^{(n)}$ has a natural desingularization:

the Hilbert scheme $S^{[n]}$ of length n

Artinian (analytic) subschemes of S.

$$S^{[n]} \longrightarrow S^{(n)}$$

$$Z \longmapsto \text{underlying zero cycle.}$$

$S^n_* \subset S^n$ → open set where at most 2 coordinates are equal.

$S^{(n)}_* \subset S^{(n)}$ image.

$S^{[n]}_* \subset S^{[n]}$ inverse image in $S^{[n]}$

Given a cycle $2x_1 + \cdots + x_{n-1} \in S^{(n)}_*$ the datum of an Artinian subscheme of length n supported on $2x_1 + \cdots + x_{n-1}$ is equivalent to the datum of a tangent direction to S at x_1.

So the set of Artinian subschemes supported on $2x_1 + \cdots + x_n$ can be identified with $\mathbb{P}T_{x_1} S$.

Denote $D \subset S^{(n)}$ be the diagonal, $D_* = D \cap S^{(n)}_*$

Theorem: (1) The complex analytic pair $\left(S^{(n)}_*, D_*\right)$ is locally isom. to $\left(B \times C, B \times \{0\}\right)$ where B is a complex ball, C is a cone with vertex 0 over a smooth conic in $\mathbb{P}^2$.

(2) $$S^{[n]}_* = Bl_{D_*} S^{(n)}_*$$

(3) We have the Cartesian diagram

$$\begin{tikzcd}
Bl_{\Delta_*}(S^n_*) \arrow[r, "\eta"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\square"] & S^n_* \arrow[d, "\pi"', "\text{quotient by } \mathfrak{S}_n"] \\
Bl_{D_*} S^{(n)} = S^{[n]}_* \arrow[r, "\varepsilon"] & S^{(n)}_*
\end{tikzcd}$$

$$\Rightarrow \quad S^{[n]}_* = Bl_{\Delta_*}(S^n_*) / \mathfrak{S}_n$$

(the action of $\mathfrak{S}_n$ lifts to $Bl_{\Delta_*}(S^n_*)$.)

<u>Proposition:</u> If $K_S = \Omega^2_S$ is trivial, then $S^{[n]}$ admits a holomorphic symplectic form.

<u>Idea of proof:</u> Choose a generator ω of $H^0(S, K_S)$.

$\psi := pr_1^*\omega + \dots + pr_n^*\omega \qquad pr_i : S^n \twoheadrightarrow S$ i-th projection.

invariant under $\mathfrak{S}_n$.

pull-back to $Bl_\Delta(S^n_*)$:

$\eta^*(\psi|_{S^n_*})$ invariant under the action of $\mathfrak{S}_n$

$\Longrightarrow \quad \exists$ hol. φ on $S^{[n]}_*$ s.t. $\eta^*\psi|_{S^n_*} = \rho^*\varphi$

Because $S^{[n]} \setminus S^{[n]}_*$ has codim. 2, φ extends to all of $S^{[n]}$.

Need to show φ is everywhere nondegenerate:

show $\Lambda^n \varphi$ does not vanish anywhere.

$\Lambda^n \varphi$ is a section of $K_{S^{[n]}} = \Omega^{2n}_{S^{[n]}}$

$\Lambda^n \varphi$ does not vanish anywhere $\Leftrightarrow$ $\mathrm{Div}(\Lambda^n \varphi) = 0$.

In fact $S^{[n]}$ is <u>irreducible</u> holomorphic symplectic (if S is $K3$).

We compute the fundamental group and cohomology of $S^{(n)}$: $S^n \longrightarrow S^{(n)}$ and $Bl_\Delta(S^n_*) \longrightarrow S^{[n]}_*$ are Galois with Galois group $\mathfrak{S}_n$:

$$\begin{tikzcd}
 & & & \pi_1(S^{[n]}) \arrow[d, equal] & \\
1 \arrow[r] & \mathfrak{S}_n \arrow[r] \arrow[dd, equal] & \pi_1(\mathrm{Bl}_\Delta(S^n_*)) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \pi_1(S^{(n)}_*) \arrow[r] \arrow[dd] & 0 \\
 & & \pi_1(S^*_n) \arrow[d, equal] & & \\
0 \arrow[r] & \mathfrak{S}_n \arrow[r] & \pi_1(S^n) \arrow[r] & \pi_1(S^{(n)}) \arrow[r] & 0
\end{tikzcd}$$

$$\Rightarrow \quad \pi_1(S^{[n]}) \xrightarrow{\cong} \pi_1(S^{(n)})$$

<u>Lemma:</u> (1) $H^i(S^{(n)}, \mathbb{Q}) = H^i(S^n, \mathbb{Q})^{\mathfrak{S}_n}$

(2) $H^2(S^{[n]}, \mathbb{Q}) = H^2(S^{(n)}, \mathbb{Q}) \oplus \mathbb{Q}[E]$

(3) $H^2(S^{(n)}, \mathbb{Q}) = H^2(S, \mathbb{Q}) \oplus \Lambda^2 H^1(S, \mathbb{Q})$

<u>Idea of proof</u>: (1) Standard

(2) blow up formula (replace $S^{[2]}$ with $S^{[n]}_*$)

(3) use (1) with Künneth $\checkmark$ $\square$

Fact from algebraic topology:

$$\pi_1(S^{(n)}) = H_1(S,\mathbb{Z}) \Rightarrow \pi_1(S^{[n]}) = H_1(S,\mathbb{Z}).$$

<u>Corollary</u>: If S is a K3 surface, then $\pi_1(S^{[n]}) = 1$.

and $H^2(S^{[n]},\mathbb{Q}) = H^2(S,\mathbb{Q}) \oplus \mathbb{Q}[E]$.

In particular, $H^0(\Omega^2_{S^{[n]}}) = H^{2,0}(S^{[n]}) = H^{2,0}(S)$

$\cong H^0(\Omega^2_S)$ has dim. 1

$\Rightarrow$ $S^{[n]}$ is irreducible holomorphic symplectic.

$b_2(S^{[n]}) = 23$

<u>Case</u>: $S = A$ is a complex torus of dim. 2.

$A^{[n+1]}$ is holomorphic symplectic.

$$\pi_1(A^{[n+1]}) = \underset{= \pi_1(A, \mathbb{Z})}{H_1(A, \mathbb{Z})} \cong \mathbb{Z}^4 \neq 0$$

$$H^2(A^{[n+1]}, \mathbb{Q}) = H^2(A, \mathbb{Q}) \oplus \Lambda^2 H^1(A, \mathbb{Q}) \oplus \mathbb{Q}[E].$$

nontrivial $(2,0)$-forms.

Consider the addition map $s : A^{(n+1)} \longrightarrow A$

<u>compose</u>:

$$\begin{tikzcd} A^{[n+1]} \arrow[r, "\rho"] \arrow[rr, bend right, "\sigma"'] & A^{(n+1)} \arrow[r, "s"] & A \end{tikzcd}$$

Put $K_n := \sigma^{-1}(0)$

<u>Definition</u>: K_n is the $(n+1)$st generalized Kummer variety.

The complex torus A acts on itself by translation

$a \in A \qquad t_a : A \longrightarrow A$
$\qquad\qquad\qquad\qquad x \longmapsto x + a$

t_a acts on $A^{[n+1]}$ by pull-back: $Z \longmapsto t_a^* Z$.

We have the Cartesian diagram:

$$\begin{array}{ccc}
A \times A^{[n+1]} & \xrightarrow{(a,Z) \longmapsto t_a^* Z} & A^{[n+1]} \\
\big\downarrow & & \big\downarrow \sigma \\
A \times A & \xrightarrow{(a,x) \longmapsto t_{(n+1)a}^* x} & A
\end{array}$$

which induces the Cartesian diagram:

$$\begin{tikzcd}
A \times K_n \arrow[rr, "{(a,Z) \mapsto t_a^* Z}"] \arrow[d] & & A^{[n+1]} \arrow[d, "\sigma"] \\
A \arrow[rr, "(n+1)\mathrm{Id}_A"] & & A
\end{tikzcd}$$

$\Rightarrow$ σ is a smooth map and all its fibers are isomorphic to K_n.

<u>Proposition</u>: K_n is irreducible holomorphic symplectic (and simply connected), and

$$H^2(K_n, \mathbb{Q}) \cong H^2(A, \mathbb{Q}) \oplus \mathbb{Q}[E \cap A^{[n+1]}]$$

$$b_2(K_2) = 7$$

Note: $\dim S^{[n]} = 2n = \dim K_n$

They are not deformation equivalent: different b_2.

Two more known examples: not deformation equivalent to these: 10-diml. and 6-diml.

Big open problem: Are there other irreducible holomorphic symplectic manifolds which are not deformation equivalent to one of the above?

Moduli of hyperkählers, the Beauville-Bogomolov form, the period domain, the period map and Torelli

Generalities: Given a differentiable manifold X, there can be many complex structures on X.

We define the Teichmüller space of X as:

$$\mathrm{Teich}(X) := \{\text{complex structures on } X\} \big/ \sim^{\circ}$$

where two complex structures I, J are equivalent:

$I \sim^{\circ} J$ if $\exists$ a diffeomorphism $\varphi : X \to X$

s.t. $\varphi^* I = J$ and φ is isotopic (homotopic) to Id_X.

The moduli space of complex structures on X is

$$\mathrm{Compl}(X) := \{\text{complex structures on } X\} \big/ \sim$$

where $I \sim J$ if $\exists$ diffeo. $\varphi : X \to X$ s.t. $\varphi^* I = J$.

If we denote $\mathrm{Diff}(X)$ the group of diffeo. of X and $\mathrm{Diff}^0(X)$ its connected comp. of Id_X, then

$$G := \mathrm{Diff}(X) \big/ \mathrm{Diff}^0(X)$$

is the (discrete) group of components of $\mathrm{Diff}(X)$. and

$$\mathrm{Compl}(X) = \mathrm{Teich}(X) \big/ G$$

A priori, we are interested in $\mathrm{Compl}(X)$, but it does not have so many "nice" properties. $\mathrm{Teich}(X)$ is much "nicer". In practice, we work with small open sets of $\mathrm{Teich}(X)$: they describe small deformations of complex structures.

Deformations:

<u>Def</u>: A family of complex manifolds is a smooth proper morphism of complex spaces:

$$\mathcal{X} \longrightarrow S$$

<u>Definition</u>: A deformation of (X, I) is a family $\pi: \mathcal{X} \longrightarrow S$ of compact complex manifolds with a point $s_0 \in S$ and an isomorphism $X_0 := \pi^{-1}(s_0) \cong X$ as complex manifolds

A deformation is universal if, for any deformation $\pi': \mathcal{X}' \longrightarrow S'$, $\exists!$ morphism $\varphi: S' \longrightarrow S$ s.t. $\varphi(s_0') = s_0$ and $\mathcal{X}' \to S'$ is the pull-back of $\mathcal{X} \to S$ via φ. In other words, we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ S' \arrow[r, "\varphi"] & S \end{tikzcd}$$

The universal deformation is unique up to unique isomorphism and we denote it $\mathcal{X} \longrightarrow Def(X)$.

Kuranishi's theorem: Suppose (X, I) is a compact complex manifold with $H^0(X, T_X) = 0$ (T_X the holomorphic tangent bundle, $H^0(X, T_X) = 0$ means $\nexists$ global holomorphic vector fields on X), then a universal deformation of (X, I) exists and it is universal for all of its fibers.

Unobstructedness for K-trivial Kähler manifolds;

(X, I) compact complex, $H^0(X, T_X) = 0$, $\mathcal{X} \to Def(X)$ the Kuranishi family of X (the universal deformation).

We assume that $Def(X)$ is very small, i.e.,

$\pi : \mathcal{X} \to Def(X)$ is a germ of a deformation.

For $t \in Def(X)$ close to $s_0 \in Def(X)$ s.t. $\pi^{-1}(s_0) \cong (X, I)$

we have $$T_t\, Def(X) = H^1(X_t, T_{X_t})$$

$$\begin{array}{ccc} X_t & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ t & \in & Def(X) \end{array}$$

in particular $T_{s_0} Def(X) = H^1(X, T_X)$

The obstructions to deformations (to various orders) provide local analytic equations for $Def(X)$ in a neighborhood of $0 \in H^1(X, T_X)$.

We say the deformations are unobstructed if the obstructions are 0, i.e., a small open set of $\mathrm{Def}(X)$ can be identified with a neighborhood of $0 \in H^1(X, T_X)$.

Theorem of Tian–Todorov & Bogomolov: Deformations of K-trivial X are unobstructed.

In particular, if (X, I) is hyperkähler, then its deformations are unobstructed.

Other nice results: If X is Kähler, so is small deformation of X.

If X is Kähler and K-trivial, then small deformations of X are also Kähler and K-trivial.

If X holomorphic symplectic (Kähler), then small deformations of X are also holomorphic symplectic.

If X is irreducible holomorphic symplectic, then all fibers of any deformation of X are irreducible holomorphic symplectic.

The key to understanding the deformations is the period domain. This has to do with the second cohomology $H^2(X,\mathbb{Z})$. We need a polarization to define the period domain: the Beauville-Bogomolov form.

Suppose X is irreducible holomorphic symplectic

(= irreducible hyperkähler).

of complex dim. $2n$, and choose $\sigma \in H^0(X, \Omega^2_X)$ s.t.

$$\int_X (\sigma\bar{\sigma})^n = 1.$$

$$H^0(X, \Omega^2_X) = H^{2,0}$$

For $\alpha \in H^2(X, \mathbb{C})$,

$$q_X(\alpha) := \frac{n}{2}\int_X \alpha^2 (\sigma\bar{\sigma})^{n-1} + (1-n)\int_X \sigma^{n-1}\bar{\sigma}^n \alpha \int_X \sigma^n \bar{\sigma}^{n-1}\bar{\alpha}.$$

If $\alpha = \lambda\sigma + \beta + \mu\bar{\sigma}$ with $\beta \in H^{1,1}(X)$, then

$$q_X(\alpha) = \lambda\mu + \frac{n}{2}\int_X \beta^2 (\sigma\bar{\sigma})^{n-1}!$$

Beauville showed that $\exists\ d_X \in \mathbb{N}$ s.t.

$$\int_X \alpha^{2n} = d_X \left(q_X(\alpha)\right)^n \qquad \forall \alpha \in H^2(X,\mathbb{C}).$$

Therefore, if r_X is the real positive n-th root of d_X, then $\tilde{q}_X := r_X q_X$ is an n-th root of the n-th power cup product on $H^2(X,\mathbb{C})$.

<u>Note</u>: If $n=1$, then $\int_X \alpha^2 = d_X q_X(\alpha) = \tilde{q}_X(\alpha)$

$\tilde{q}_X$ is the intersection form..

Interestingly, this was discovered via the example of the Fano variety of lines of cubic fourfold:

If T is a cubic fourfold, then $F :=$ the
$\cap \mathbb{P}^5$

Fano variety of lines in T is irreducible holomorphic symplectic (projective). And

$$H^4(T, \mathbb{Z}) \xrightarrow{AJ} H^2(F, \mathbb{Z})$$

Abel-Jacobi

has an intersection form $\rightsquigarrow$ Beauville-Bogomolov form.

Properties of the quadratic form $\tilde{q}_X$:

$\tilde{q}_X$ is indivisible, non-degenerate, of signature $(3, b_2 - 3)$ on $H^2(X, \mathbb{R})$;

integer-valued on $H^2(X,\mathbb{Z})$.

<u>Furthermore</u>: $\tilde{q}_X(\sigma)=0$, $\tilde{q}_X(\sigma+\bar{\sigma})>0$

and for t close to 0 in $Def(X)$:

$$\underline{q_X(\sigma_t)=0 \quad \text{and} \quad q_X(\sigma_t+\bar{\sigma}_t)>0.}$$

<u>Definition</u>: The local period domain:

$$Q_X := \{\alpha \mid q_X(\alpha)=0,\ q_X(\alpha+\bar{\alpha})>0\}$$

$$\subset \bar{Q}_X := \{\alpha \mid q_X(\alpha)=0\} \subset \mathbb{P}H^2(X,\mathbb{C}).$$

The local period map $P_X : Def(X) \longrightarrow Q_X$

$$t \longmapsto [\sigma_t]$$

Properties: P_X is holomorphic because σ_t varies holomorphically with t.

We have the local Torelli theorem:

P_X is a local isomorphism

$\Longleftrightarrow$ dP_X is an isomorphism at 0.

The global period domain.

Definition: A lattice is the data of a free $\mathbb{Z}$-module Γ of finite rank with an integral nondegenerate quadratic form q_Γ.

Definition: Given a lattice (Γ, q_Γ), the period domain Q_Γ is:

$$Q_\Gamma := \{\alpha \mid q_\Gamma(\alpha)=0,\ q_\Gamma(\alpha+\bar{\alpha})>0\}$$

$$\subset \overline{Q}_\Gamma := \{\alpha \mid q_\Gamma(\alpha)=0\} \subset \mathbb{P}(\Gamma \otimes_{\mathbb{Z}} \mathbb{C}).$$

Moduli space of marked holomorphic symplectic manifolds:

Def: (1) A (Γ, q_Γ)-marking of X is a lattice isomorphism $\varphi : (H^2(X,\mathbb{Z}), \tilde{q}_X) \cong (\Gamma, q_\Gamma)$

(2) The pair (X,φ) is called a marked manifold.

(3) Two marked manifolds (X, φ), (X', φ') are isomorphic if $\exists$ $f : X \longrightarrow X'$ bihol. s.t. $\varphi' = \varphi \circ f^*$. We write $(X, \varphi) \cong (X', \varphi')$.

$\left(f^* : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})\right)$

(4) The moduli space of marked irreducible holomorphic symplectic manifolds is

$$\mathcal{M}_T := \left\{ (X, \varphi) \right\} \Big/ \cong .$$

Def: The global period map is:

$$P : \mathcal{M}_T \longrightarrow Q_\Gamma \subset \overline{Q}_\Gamma \subset \mathbb{P}(T \otimes_{\mathbb{Z}} \mathbb{C})$$
$$(X, \varphi) \longmapsto [\varphi(\sigma)]$$

<u>Theorem (Global Torelli (Verbitsky)):</u>

The map P is generically injective on each connected component of $\mathcal{M}_T$.

This is weaker than a usual global Torelli theorem.

<u>Examples:</u> (1) Two complex tori are isomorphic if and only if their first cohomologies are isomorphic as Hodge structures.

(2) Two Riemann surfaces are isomorphic if and only if their first cohomologies are isomorphic as Hodge structures and, under the given isomorphism, the intersection forms of the two curves coincide. (for short we will say the first cohomologies are Hodge isometric).

(3) Two K3 surfaces are isomorphic if and only if their second cohomologies are Hodge isometric.

Note that σ generates $H^{2,0}(X) \subset H^2(X,\mathbb{C})$

and $\tilde{q}_X(\sigma) = 0$

<u>also</u>: $\sigma^\perp$ (perp for $\tilde{q}_X$) $= H^{2,0} \oplus H^{1,1}$

$\bar{\sigma}$ generates $H^{0,2}$

$\Rightarrow$ $[\sigma]$ determines the Hodge structure on $H^2(X,\mathbb{C})$

For irreducible holomorphic symplectic manifolds, global Torelli fails:

<u>Examples</u>: (1) <u>Debarre</u> (1984):

$\exists$ nonisomorphic (but bimeromorphic) compact hyperkähler manifolds with Hodge isometric second cohomologies (not algebraic)

(2) Namikawa (2002): $K_2(T)$ $K_2(T^*)$ nonbirational projective 4-dim'l hyperkähler manifolds with Hodge isometric second cohomologies.

<u>Question</u>: Is there a "good" characterization of

irreducible holomorphic symplectic manifolds that are Hodge isometric but not isomorphic.

A few words about $\mathfrak{M}_\Gamma$ and the period map:
We can use the local period maps to show that $\mathfrak{M}_\Gamma$ is a smooth (non-Hausdorff) complex analytic space.

Given an irreducible holomorphic symplectic manifold, choose a marking $\varphi : H^2(X,\mathbb{Z}) \to \Gamma$.
The marking gives an isomorphism

$$Q_X \xrightarrow{\simeq} Q_\Gamma$$

The Kuranishi family $\mathcal{X} \longrightarrow \mathrm{Def}(X)$ is locally isomorphic to the period domain Ω_Γ via the local Torelli theorem:

$$\begin{tikzcd}
\substack{\text{open ball} \\ \text{in } \mathrm{Def}(X)} \arrow[r, hook] & \Omega_X \arrow[r, "\cong"] \arrow[d, hook] & \Omega_\Gamma \arrow[d, hook] \\
& \mathbb{P}H^2(X, \mathbb{C}) \arrow[r, "\cong"] & \mathbb{P}(\Gamma \otimes_{\mathbb{Z}} \mathbb{C})
\end{tikzcd}$$

The open balls above cover $\mathfrak{M}_\Gamma$ and the analytic structures on the intersections of different open balls coincide because the Kuranishi family is universal for all its fibers.

$$\mathrm{Teich}(X) = \{\text{complex structures on } X\}/\sim_0$$

$$\downarrow \text{ can be infinite}$$

$$\mathfrak{M}_\Gamma = \{\text{marked complex structures}\}/\simeq$$

$$\downarrow$$

$$\mathrm{Compl}(X) = \{\text{complex structures on } X\}/\sim$$

$$= \mathrm{Teich}(X)/G$$

$$\mathrm{Teich}(X) \xrightarrow[\text{local isom.}]{} \mathfrak{M}_\Gamma \xrightarrow{P_\Gamma} Q_\Gamma \subset \overline{Q}_\Gamma \subset \mathbb{P}(\Gamma \otimes_{\mathbb{Z}} \mathbb{C})$$

$\mathfrak{M}_\Gamma$ is a non Hausdorff complex manifold.

Q_Γ is a Hausdorff, simply connected.

$$\begin{tikzcd} \mathcal{M}_\Gamma \arrow[r] \arrow[rr, bend right, "P"'] & \mathcal{M}_\Gamma^s \arrow[r, "P^s"] & Q_\Gamma \end{tikzcd}$$

$\mathcal{M}_\Gamma^s$ " $=$ " $\mathcal{M}_\Gamma \big/$ identify the infinitely near points of $\mathcal{M}_\Gamma$

$p \overset{i.n.}{\sim} q$ if every neighborhood of p contains q
and " q " p.

$\mathcal{M}_\Gamma^s =$ Hausdorff complex manifold.

<u>Saw</u>: P^s is a local isomorphism.

Theorem: P^s is surjective from any connected component of $\mathcal{M}_\Gamma^s$ to Q_Γ.

Corollary: P^s induces isomorphisms between components of $\mathcal{M}_\Gamma^s$ and Q_Γ.

Question: What is the difference between $\mathcal{M}_\Gamma^s$ and $\mathcal{M}_\Gamma$?

Proposition (Huybrechts): If (X,φ) and (X',φ') are two infinitely near points of $\mathcal{M}_\Gamma$, then X and X' are bimeromorphic and $P(X,\varphi)=P(X',\varphi')$ is contained in the hyperplane $Q_\Gamma \cap \alpha^\perp$ for some $\alpha \in \Gamma$.

Proposition (Huybrechts): Let $X \overset{f}{\underset{\cong}{\dashrightarrow}} X'$ be compact hyperkähler, bimeromorphic. Then there exists families of compact hyperkähler manifolds

$$\mathcal{X} \longrightarrow D \quad \text{and} \quad \mathcal{X}' \longrightarrow D$$ over a complex

disc D s.t.

(1) $X_0 \cong X$, $X'_0 \cong X'$

(2) $\exists$ bimeromorphic $F: \mathcal{X}' \longrightarrow \mathcal{X}$

$$\begin{tikzcd} \mathcal{X}' \arrow[rr] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & D & \end{tikzcd}$$

which is an isomorphism over $D \setminus \{0\}$

and induces f on $X_0 \cong X \dashrightarrow X'_0 \cong X$

$$F|_{X_0} = f.$$

Proposition (Huybrechts): The set hyperkähler complex structures on X with a fixed Hodge structure on $H^2(X,\mathbb{Z})$ is non-empty (period map is surjective) and consists of a finite number of bimeromorphic equivalence classes.

Surjectivity of the period map (Verbitsky):

Uses twistor lines = twistor conics.

Lattice (Γ, q_Γ) $\qquad Q_\Gamma \subset \overline{Q}_\Gamma \subset \mathbb{P}(\Gamma \underset{\mathbb{Z}}{\otimes} \mathbb{C})$

signature of $q_\Gamma \otimes \mathbb{R}$ is $(3, b_2 - 3)$ $\quad b_2 = \text{rank of } \Gamma$

Each choice of a 3-dimensional real plane $F_{\mathbb{R}}$ $\subset \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ positive for q_Γ gives a twistor conic:

$$F_{\mathbb{R}} \otimes \mathbb{C} \subset \Gamma \otimes \mathbb{C}$$

$$\mathbb{P}^2 \cong P := \mathbb{P}(\Gamma_{\mathbb{R}} \otimes \mathbb{C}) \subset \mathbb{P}(\Gamma \otimes \mathbb{C})$$

$$P \cap Q_\Gamma =: \text{a twistor conic.}$$

<u>Proposition</u>: Q_Γ is twistor path connected, i.e., any two points of Q_Γ can be joined by a connected sequence of twistor conics.

Twistor spaces:

Suppose (X, g) is hyperkähler.

We saw $\exists\ I, J, K$ Kähler (w.r.t. g) complex structures on X. $\forall\ (a,b,c) \in S^2 \subset \mathbb{R}^3$

$$\lambda := aI + bJ + cK \text{ is also a Kähler}$$

complex structure on (X, g). The Kähler form associated to λ is $\omega_\lambda(\cdot,\cdot) := g(\lambda\cdot,\cdot)$.

So we have a family $\{(X,\lambda) \mid \lambda \in S^2\}$ of compact Kähler manifolds.

Definition: The twistor space $\mathcal{X} \longrightarrow \mathbb{P}^1$ of (X, g) is the product $X \times \mathbb{P}^1$ (as a C^∞ manifold) with the <u>almost</u> complex structure

$$I_{X\times\mathbb{P}^1} : T_x X \oplus T_\lambda \mathbb{P}^1 \longrightarrow T_x X \oplus T_\lambda \mathbb{P}^1$$

$$(v, w) \longmapsto (\lambda(v), I_{\mathbb{P}^1}(w))$$

this is integrable by Hitchin, Karlhede, Lindström, Roček.

$$\begin{tikzcd} \mathcal{X} \supset \pi^{-1}(\lambda) \arrow[d, "\pi"'] \\ S^2 = \mathbb{P}^1 \ni \lambda \end{tikzcd}$$

$\pi^{-1}(\lambda)$ has complex structure I_λ.

We can choose consistent markings on all the fibers

to get $\quad P : \mathbb{P}^1 \longrightarrow Q_\Gamma$

$$\lambda \longmapsto [\sigma_{(X,\lambda)}]$$

where $\sigma_{(X,\lambda)}$ is a generator of $H^{2,0}(X,\lambda)) = H^0(\Omega^2_{(X,\lambda)})$.

<u>Lemma:</u> The image $P(\mathbb{P}^1)$ is a twistor conic.

<u>Corollary</u> (of this and the twistor path connectivity)

$$P : \mathcal{M}_\Gamma \longrightarrow Q_\Gamma$$

is surjective on any connected component of $\mathcal{M}_\Gamma$.

Calabi's conjecture, Yau's theorem:

Let (M, I) be a compact complex manifold and g a Kähler metric with Kähler form ω and Ricci form ρ. Let ρ' be a real $(1,1)$ form on (M, I) with cohomology class $[\rho'] = [\rho] \left(= 2\pi\, c_1(K_M)\right)$. Then there exists a unique Kähler metric g' on (M, I) whose Kähler form ω' satisfies $[\omega'] = [\omega]$ and such that ρ' is the Ricci form for g'.

Corollary: Suppose (M, I, g) is compact Kähler with $c_1(K_M) = 0$. Then, in each Kähler class on M, $\exists!$ Ricci-flat Kähler metric.

(Note: Def: A Kähler class is the cohomology class of a (1,1) which is Kähler w.r.t some metric on M).

Furthermore, the Ricci-flat Kähler metrics on M form a smooth family of dim $h^{1,1}(M)$, isomorphic to the Kähler cone of M.

Apply this to (X, λ) where $\lambda = aI + bJ + cK$.
$\in S^2 \cong \mathbb{P}^1$

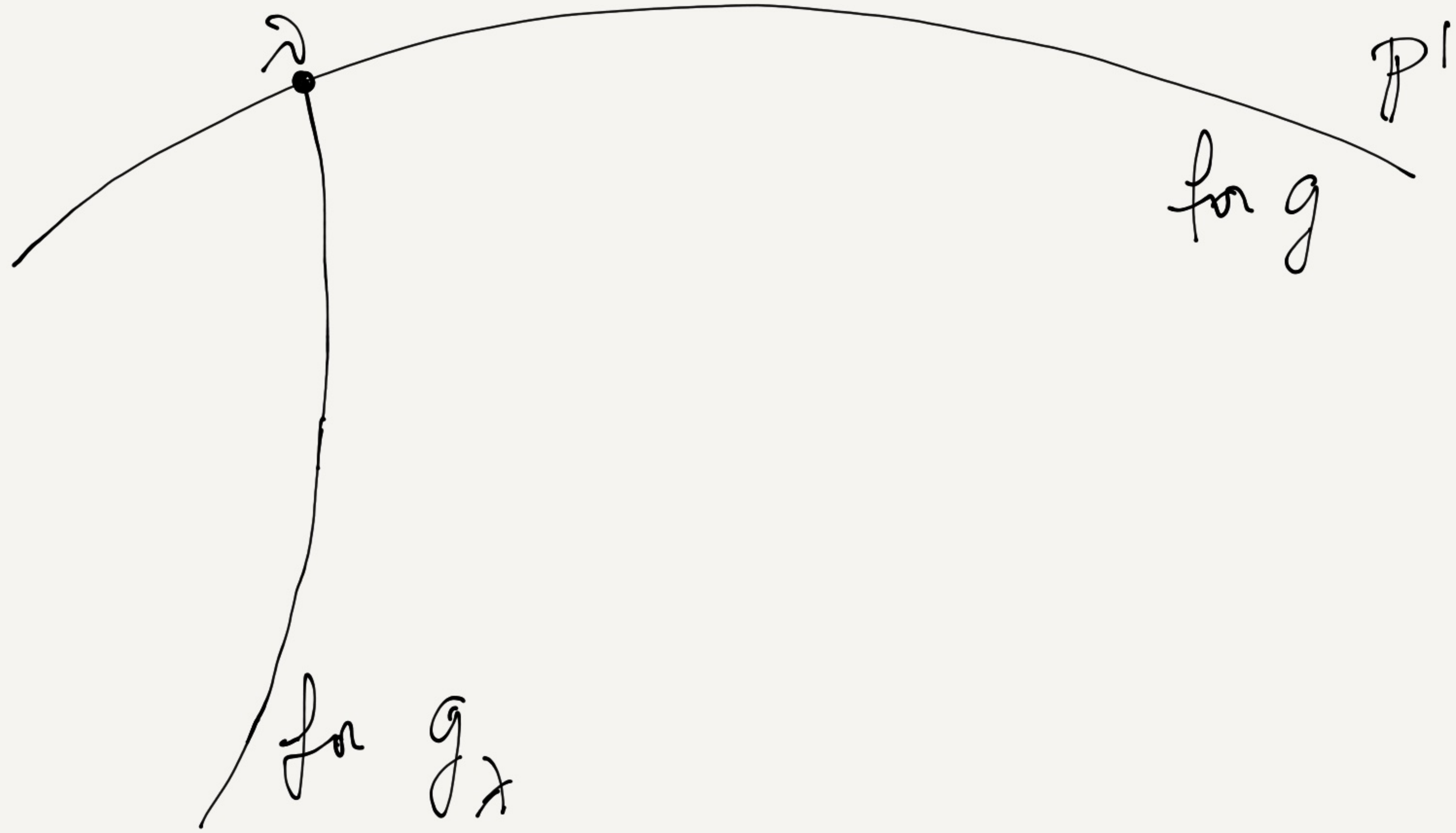
$\forall$ Kähler class $\alpha \in H^{1,1}(M)$

$\exists!$ hyperkähler metric g_λ, Kähler for λ.

s.t. $[\omega_{g_\lambda}] = \alpha$

$\rightsquigarrow$ construct the twistor family for g_λ.

Picture:



A few words about hyperholomorphic bundles:

<u>Def</u>: Given a Hermitian vector bundle B on X. with Hermitian connection θ, we say (B,θ) is hyperholomorphic if it is compatible with all the complex structures $\lambda \in S^2 = \mathbb{P}^1$.

B C^∞ complex vector bundle is Hermitian if it has a Hermitian metric $\langle\ ,\ \rangle$.

$\theta : B \longrightarrow B \otimes T_X^*$ Hermitian connection

$\Theta :=$ curvature of θ $\in \mathrm{End}(B) \otimes \Lambda^2 T_M^*$

If we are given a complex structure I on B, we say θ and I are compatible if

Θ is a $(1,1)$-form with respect to I.

Intuitive thinking: on the twistor space

$$\begin{tikzcd} X \times \mathbb{P}^1 = \mathcal{X} \arrow[d] & B \times \mathbb{P}^1 \text{ as } C^\infty \arrow[l] \\ \mathbb{P}^1 & \end{tikzcd}$$

<u>Verbitsky</u>: Given a vector bundle B on (X, I)

If $c_1(B)$ and $c_2(B)$ are of type $(1,1)$ and $(2,2)$ respectively with respect to all complex structures $\lambda \in S^2$, then B is hyperholomorphic.

To construct moduli spaces, we need stability:

Fix a Kähler form ω on X $\qquad n = \dim_{\mathbb{C}} X$

F coherent sheaf.

$$\deg(F) := \frac{1}{\mathrm{vol}(X)} \int_X c_1(F) \wedge \omega^{n-1}$$

$$\mathrm{vol}(X) := \int_X \omega^n$$

$$\mathrm{Slope}(F) := \frac{\deg(F)}{\mathrm{rank}(F)}$$

<u>Def</u>: F is called stable if $\forall$ subsheaves $F' \subset F$ s.t. $\mathrm{rank}\, F' < \mathrm{rank}\, F$, we have $\mathrm{Slope}(F') < \mathrm{Slope}(F)$.

semi-stable: slope$(F') \leq$ slope(F).

Theorem (Uhlenbeck–Yau): B indecomposable bundle on M compact Kähler, Then

B is stable $\Longleftrightarrow$ B has a Yang-Mills metric.

Def: a Hermitian metric is Yang-Mills if its curvature form is a multiple of Id.