$$\beta : C^{(n)} \longrightarrow \mathrm{Pic}^n(C) \qquad (C \text{ smooth curve of genus } g)$$
$$D \longmapsto \mathcal{O}_C(D)$$

Let $W_n := \beta(C^{(n)}) \subset \mathrm{Pic}^n(C)$ (L line bundles of deg n with $|L| \neq \emptyset$)

(R-R Thm) $\Rightarrow$ $W_n = \mathrm{Pic}^n(C)$ $\forall n \geq g$

For a $\underline{\text{gral}}$ $L \in \mathrm{Pic}^n(C)$. $1 \leq n \leq g$ $h^0(C, L) = 1$

$\Rightarrow$ β is birat. onto its image in this range.

$$W_n \underset{bir}{\simeq} C^{(n)}$$

Since β is proper. W_n is an $\underline{\text{irred.}}$ subvar.
In part.

$n = g-1$ $\quad W_{g-1} \subset \mathrm{Pic}^{g-1}(C)$ Canonical theta divisor

- There is a $\eta \in \mathrm{Pic}^{g-1}(C)$ s.t. $\alpha_\eta^* \Theta = W_{g-1}$

$$\alpha_\eta : \mathrm{Pic}^n(C) \longrightarrow \mathrm{Pic}^0(C) \simeq JC$$
$$L \longmapsto L \otimes \eta^{-1}$$

$\underline{\text{Thm}}$ (Riemann-Singularity thm)

For every $L \in \mathrm{Pic}^{g-1}(C)$ $\quad \mathrm{mult}_L W_{g-1} = h^0(C, L)$

§ Prym varieties.

§ $\underline{\text{Motivation}}$

$$\mathcal{M}_g = \{ C \text{ smooth. proj. curve of genus } g \} / \simeq$$
$$\mathcal{A}_g = \{ (A, \Theta) \mid A \text{ ppav.} \} / \simeq \; = \mathbb{H}_g / Sp_g(\mathbb{Q})$$

$\mathbb{H}_g := \{ \tau \in M_{g\times g}(\mathbb{C}) \mid \tau^t = \tau, \; \mathrm{Im}\,\tau > 0 \}$. Siegel upper. half space

$$Sp_g(\mathbb{Q}) = \left\{ M \in GL_{2g}(\mathbb{Q}) \;\middle|\; M \underbrace{\begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}}_{\left[\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)} M^t = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix} \right\}$$

Symplectic group. acts on $\mathbb{H}_g$:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_g(\mathbb{Q}) \qquad M \cdot \tau := (a + b\tau)(c + d\tau)^{-1}$$

$\tau \in \mathbb{H}_g \rightsquigarrow A_\tau = \mathbb{C}^g / \tau\mathbb{Z}^g \oplus \mathbb{Z}^g$

$A_\tau \simeq A_{\tau'} \iff \exists\, M \in Sp_{2g}(\mathbb{Q})$ s.t $\tau' = M \cdot \tau$

Obs $\dim \mathcal{A}_g = \dim \mathbb{H}_g = \dfrac{g(g+1)}{2}$

Def. The Torelli map $t : \mathcal{M}_g \longrightarrow \mathcal{A}_g$, $[C] \longmapsto [JC, \Theta]$

Thm The Torelli map is injective.

g	$\dim \mathcal{M}_g = 3g-3$ ($g\geq 2$)	$\frac{g(g+1)}{2} = \dim \mathcal{A}_g$
2	3	3
3	6	6
4	9	10
5	12	15
6	15	21
7	18	28

t, t, p_4, p_5, p_6

Filosophy: Study $\mathcal{A}_g$ through the geometry of curves

§ Prym varieties.

Consider a double cov. between smooth curves

$$\pi : \tilde{C} \longrightarrow C \qquad g(C) = g \qquad g(\tilde{C}) = \tilde{g}$$

Induces a homomorphism of groups the norm map

$$Nm_\pi : J\tilde{C} \longrightarrow JC$$
$$\Big[\sum_i n_i p_i\Big] \longmapsto \Big[\sum_i n_i \pi(p_i)\Big] \qquad \sum n_i = 0,\ n_i \in \mathbb{Z}.$$

Def. Define the Prym variety of π by

$P(\pi) = (\operatorname{Ker} Nm_\pi)^0 \subset J\tilde{C}$ which is a subvar. of dim.

$\dim P = \dim J\tilde{C} - JC = \tilde{g} - g$

It carries a natural polarization

$$\tilde{\Theta}|_{P(\pi)}$$

Thm (Mumford) $\tilde{\Theta}|_{P(\pi)} \underset{\text{alg. equiv.}}{\equiv} 2\Xi$, with Ξ principal pol on $P(\pi)$

$\Leftarrow$ π is étale or π is ramified in exactly 2 pts.

Rmk. 1) The def. also holds for $\deg(\pi) > 2$

2) For $d>2$ there are only 2 other cases when $\tilde{\Theta}|_{P(\pi)}$ induces a principal pol:

- π triple (non-cyclic) covering, $g(C) = 2$ $(\tilde{g} = 4)$
- π any finite map $\underset{(d)}{}$. $\tilde{C} \to C$ $g(\tilde{C}) = 2$, $g(C) = 1$

Alternatively.

$$\begin{array}{c} \tilde{C} \circlearrowleft \iota \text{ involution} \\ \pi \downarrow \\ C = \tilde{C}/\iota \end{array}$$

$\iota \in \mathrm{Aut}(\tilde{C}) \rightsquigarrow \mathrm{End}(J\tilde{C})$

$P(\pi) := \mathrm{Im}(1-\iota) \subset J\tilde{C}$

So $P(\pi)$ is the "ι-anti-invariant" part of $J\tilde{C}$

$\pi^* : JC \longrightarrow J\tilde{C}$

$L \longmapsto \pi^* L$.

When π is étale: $\boxed{\operatorname{Ker} \pi^* = \langle \eta \rangle}$

$\eta^{\otimes 2} \simeq \mathcal{O}_C$

$\iota \circlearrowleft \pi^* JC$ is the "ι-invariant part"

$\pi^* JC = \mathrm{Im}(1+\iota)$

We have an isogeny (surj. + same dim) $\pi^* JC \times P(\pi) \xrightarrow{\text{sum.}} J\tilde{C}$

Assume π is étale (unramified)

$\operatorname{Ker} Nm_\pi = P \cup P_1$ 2 connected comp.

$0 \in P$

$P(\pi) = P = (1-\iota)\, \underbrace{\mathrm{Pic}^0(\tilde{C})}_{\simeq J\tilde{C}}$

$P_1 = (1-\iota)\, \mathrm{Pic}^1(\tilde{C}) \subset J\tilde{C}$

$(1-\iota)\cdot L = \underbrace{L \otimes (\iota^* L)^{-1}}_{\deg\ 0}$

$JC[2] := \{ \eta \in \mathrm{Pic}^0(C) \mid \eta^{\otimes 2} \simeq \mathcal{O}_C \}$ $\quad$ ($\otimes n$)

$= \{ \eta \in JC \mid 2\eta = 0 \}$

2-torsion points (n-torsion points)

$$|JC[2]| = 2^{2g}$$

(C,η) $\rightsquigarrow$ étale double cov. $\qquad \eta^{\otimes 2} \simeq \mathcal{O}_C$ endows

$\eta \in JC[2] \setminus \{0\}$ $\qquad \mathcal{O}_C \hookrightarrow \mathcal{O}_C \oplus \eta$ with a ring structur

$$\rightsquigarrow \tilde{C} := \operatorname{Spec}(\mathcal{O}_C \oplus \eta)$$
$$\downarrow \pi$$
$$C := \operatorname{Spec} \mathcal{O}_C$$

Def. $\mathcal{R}_g := \{ [C,\eta] \mid [C] \in \mathcal{M}_g,\ \eta \in JC[2] \setminus \{0\} \}$.

moduli space parametrizing étale double cov.

forgetful (finite) $\ 2^{2g}-1$

$$\mathcal{R}_g \xrightarrow{2^{2g}-1} \mathcal{M}_g \ni [C]$$

Rmk. By const. $\quad T_0 J\tilde{C} = H^0(\tilde{C}, \omega_{\tilde{C}})^\vee \circlearrowleft i$

$$\Rightarrow T_0 J\tilde{C} = H^0(\tilde{C}, \omega_{\tilde{C}})^+ \oplus H^0(\tilde{C}, \omega_{\tilde{C}})^-$$
$$\qquad\qquad \quad \cong H^0(C,\omega_C)^\vee \qquad\qquad \cong T_0 P(\pi)$$

(étale) $\quad \pi^*\omega_C \cong \omega_{\tilde{C}}$

$$\begin{aligned}
H^0(\tilde{C},\omega_{\tilde{C}}) &\cong H^0(C, \pi_*(\omega_{\tilde{C}} \otimes \mathcal{O}_{\tilde{C}})) \\
&= H^0(C, \pi_*(\pi^*\omega_C \otimes \mathcal{O}_{\tilde{C}})) \\
&\simeq H^0(C, \omega_C \otimes \pi_*\mathcal{O}_{\tilde{C}}) = H^0(C, \omega_C \otimes (\mathcal{O}_C \oplus \eta)) \\
&\simeq H^0(C,\omega_C) \oplus \underbrace{H^0(C, \omega_C \otimes \eta)}_{\simeq (T_0 P)^\vee}
\end{aligned}$$

Prym-differential

Def. The Prym map.

$$\mathcal{P}_g : \mathcal{R}_g \longrightarrow \mathcal{A}_{g-1}$$
$$[C,\eta] \longmapsto (P, \Xi)$$

Obs. $\dim P = \tilde{g} - g$
$= (2g-1) - g$
$= g-1$

Thm (a) The Prym map is dominant if $g \leq 6$ (Wirtinger)

(b) $\mathcal{P}_g$ is generically injective if $g \geq 7$ (Friedman-Smith, Kanev, Welters, Debarre)

(c) The Prym map is never injectiv (Donagi's tetragonal const.)

Open question: Describe completely the non-injectivity locus of P_g.

To show P_g is generically finite, one has to show dP_g is inj. or equivalently

$$dP_g^* : T^*\mathcal{A}_{g-1} \longrightarrow T^*\mathcal{R}_g \quad \text{is surjectiv at the generic point.}$$

At $P = P_g(C,\eta) \in \mathcal{A}_{g-1}$ we have the identifications.

- $T^*_{[C,\eta]}\mathcal{R}_g \simeq T^*_{[C]}\mathcal{M}_g = H^0(C, \omega_C^2)$
- $T_P^*\mathcal{A}_{g-1} \simeq \mathrm{Sym}^2(T_0^*P) = \mathrm{Sym}^2(H^0(C, \omega_C \otimes \eta))$

$\rightsquigarrow$ $dP_g^* : \mathrm{Sym}^2(H^0(C, \omega_C \otimes \eta)) \longrightarrow H^0(C, \omega_C^2)$

is given by the mult. of sections

$H^0(C, \omega_C^2 \otimes \eta^2)$ $\simeq$ $\eta^2 \simeq \mathcal{O}_C$

Lemma dP_g^* is surj. for $g \geq 6$

$\S$ Case $P_6 : \mathcal{R}_6 \longrightarrow \mathcal{A}_5$

Thm (Donagi-Smith) $P_6 : \mathcal{R}_6 \to \mathcal{A}_5$ is generically finite of degree $\underline{\underline{27}}$

Plan:
- P_g extends to a proper.
- Extend the Prym. to boundary ("allowable" double cov.)
- Study the generalized theta div
- Compute the local degree along different loci on a particular fiber.

Def. A curve C is stable. is smooth or. it is connected., the only sing. are ordinary double pts and $\# \mathrm{Aut}(C) < \infty$

$\rightsquigarrow$ $\overline{\mathcal{M}_g}$ = moduli space of stable curves of genus g.

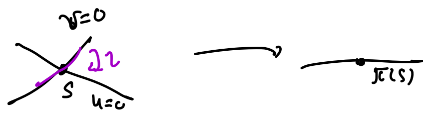
Let $\tilde{C} \in \overline{\mathcal{M}_{2g-1}}$ with $\iota : \tilde{C} \to \tilde{C}$ involution. Assume.

(∗) The fixed points of $\tilde{C}$ are exactly the singular pts. and a sing pt. the two branches are not exchanged under ι

<u>Lemma</u> $\tilde{C}/\iota$ has only ordinary double points. $C = \tilde{C}/\iota \in \overline{\mathcal{M}_g}$

<u>Proof.</u> $s \in \tilde{C}_{sing}$ fixed by ι

$$\hat{\mathcal{O}}_s \cong \mathbb{C}[[u,v]]/(u\cdot v)$$



- if ι exchanges the branches : one can choose u, v s.t.

$$\iota^* u = v \qquad \iota^* v = u$$

$\rightsquigarrow$ invariants under ι^* of $\hat{\mathcal{O}}_s$ is $\mathbb{C}[[u+v]]$ which is regular.

- if ι does not exchange the branches of s

$$\iota^* u = -u \qquad \iota^* v = -v$$

$\rightsquigarrow$ invariants under ι^* of $\hat{\mathcal{O}}_s$ is

$$\mathbb{C}[[u^2, v^2]] \simeq \mathbb{C}[[x,y]]/(xy)$$

Assume (∗) $\Rightarrow$ $C = \tilde{C}/\iota$ has only nodes

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{f}} & \tilde{C} \\ {\scriptstyle \pi'}\downarrow {\scriptstyle 2:1} & & \downarrow {\scriptstyle \pi} \\ N & \xrightarrow{f} & C \end{array}$$

<u>Lemma</u> $\pi^* \omega_C \simeq \omega_{\tilde{C}}$

$$p_a(\tilde{C}) = 2p_a(C) - 1$$

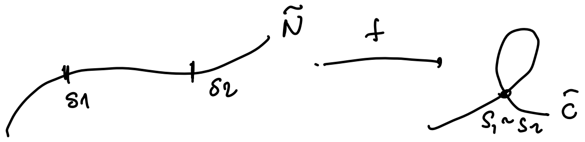
<u>Cartier divisors</u> on $\tilde{C}$

$\tilde{\mathcal{K}}$: ring of funct. on $\tilde{C}$

$\mathcal{K}$: " " on C.

Group of Cartier divisors

$$\operatorname{Div} \tilde{C} = \bigoplus_{x \in \tilde{C}_{reg}} \mathbb{Z} \cdot x + \bigoplus_{s \in \tilde{C}_{sing}} \tilde{\mathcal{K}}_s / \mathcal{O}_{\tilde{C},s}$$



ν_1, ν_2 the valuations of s_1, s_2 resp.

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \tilde{K}_s^* / \mathcal{O}_s^* \xrightarrow{(\nu_1, \nu_2)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

$$f \longmapsto (\nu_1(f), \nu_2(f))$$

$f \in \operatorname{Ker}(\nu_1, \nu_2) \Longrightarrow \nu_i(f) = 0 \quad i = 1, 2$ | no zeros, no poles at s_1, s_2

$$\Longrightarrow \frac{f(s_1)}{f(s_2)} \in \mathbb{C}^*$$

Choosing t_1, t_2 local par. around $s_1, s_2 \in \tilde{N}$

$$\tilde{K}_s / \mathcal{O}_{\hat{C}, s} \simeq \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z}$$

$a = u t_1^m$
$a = v t_2^n$

$$\begin{array}{ccc} \psi & & \\ a & \longmapsto & \left(\frac{u}{v}, m, n \right) \end{array}$$

$$\iota^* (z, m, n)_s = ((-1)^{m+n} z, m, n)$$

$$\begin{array}{ccccccc} \tilde{K}^* & \longrightarrow & \operatorname{Div}(\hat{C}) & \longrightarrow & \operatorname{Pic}(\tilde{C}) & \longrightarrow & 0 \\ {\scriptstyle Nm} \downarrow & & \downarrow {\scriptstyle \pi_*} & & \downarrow {\scriptstyle Nm} & & \\ K^* & \longrightarrow & \operatorname{Div}(C) & \longrightarrow & \operatorname{Pic}(C) & \longrightarrow & 0 \end{array}$$

$$\pi_* (\sum x_i) = \sum_i \pi(x_i)$$

$$\pi_* ((z, m, n)_s) = ((-1)^{m+n} z^2, m, n)_{\pi(s)}$$