$\tilde{C} \in \overline{\mathcal{M}}_{2g-1}$ $\qquad C \in \mathcal{M}_g$ $\qquad \pi : \underset{\circlearrowleft \mathbb{Z}_2}{\tilde{C}} \longrightarrow \underset{\tilde{C}/\mathbb{Z}_2}{\overset{\shortparallel}{C}}$ doubl.

$Nm : \mathrm{Pic}(\tilde{C}) \longrightarrow \mathrm{Pic}(C) \quad \rightsquigarrow \quad Nm_\pi : J\tilde{C} \longrightarrow JC$

generalized Jacobian

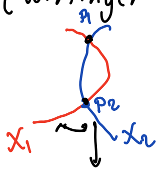
$n : N \longrightarrow C$ normalization

$\begin{matrix} L \\ \downarrow \\ C \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} n^*L \\ \downarrow \\ N \end{matrix} \quad +$ "descent data"

<u>Example</u> (Wirtinger cover). $\qquad X \in \mathcal{M}_{g-1} \qquad X_1 = X_2 = X$

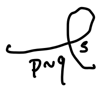
$p, q \in X.$

$\tilde{C} = X_1 \cup X_2 \Big/ \begin{matrix} p_1 \sim q_2 \\ p_2 \sim q_1 \end{matrix}$



$\iota : \tilde{C} \longrightarrow \tilde{C}$

$\iota(X_1) = X_2$



$C = X/_{p \sim q}$ $\qquad L \in \mathrm{Pic}(C).$

$\varphi_s : \tilde{L}_p \xrightarrow{\simeq} \tilde{L}_q$ $\qquad \tilde{L} = n^* L \in \mathrm{Pic}(N)$

$\varphi_s \in \mathbb{C}^*$ after choosing a trivialization around p and q. $\qquad n : X \longrightarrow C$

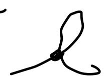
More generally:

$$0 \longrightarrow (\mathbb{C}^*)^b \longrightarrow JC \xrightarrow{n^*} JN \longrightarrow 0$$

b = 1. Betti number of the dual graph $\Gamma(C)$ of C.

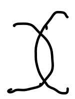
$b_1(\Gamma(C)) = |\text{Edges}| - |\text{Vertices}| + |\text{conn. comp.}|$

e.g. C

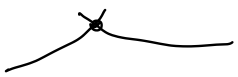


$b_1 = 1$

$\tilde{C}$



$b_1 = 2 - 2 + 1$



$b_1 = 1 - 2 + 1 = 0$

Lemma If $L \in \mathrm{Pic}(\tilde{C})$. s.t. $\mathrm{Nm}\, L \simeq \mathcal{O}_C$ then
$L \simeq M \otimes \iota^* M^{-1}$ for some $M \in \mathrm{Pic}(\tilde{C})$

M can be chosen of multidegree $(0, 0 \dots 0)$ or $(1, 0 \dots 0)$

$\rightsquigarrow$ $\ker \mathrm{Nm}_\pi$ has 2 comp.

Def. $P := \{ M \otimes \iota^* M^{-1} \mid \deg M = (0, \dots 0) \}$ connected alg. group.

Prym on the generalized Jacobian

Prop. P is an abelian var. of dim. $p_a(C) - 1$ (under the assumption $(*)$ from yesterday)

Proof.

$$\begin{tikzcd}
0 \arrow[r] & \tilde{T}_2 \arrow[r] \arrow[d] & P \times \mathbb{Z}/2 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{T} \arrow[r] \arrow[d, "\mathrm{Nm}"] & J\tilde{C} \arrow[r, "\tilde{n}^*"] \arrow[d, "\mathrm{Nm}_\pi"] & J\tilde{N} \arrow[r] \arrow[d, "\mathrm{Nm}_\pi"] & 0 \\
0 \arrow[r] & T \arrow[r] \arrow[d] & JC \arrow[r] \arrow[d] & JN \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

complete subvar. of $J\tilde{N}$

$T, \tilde{T}$ are the groups of mult. deg. $(0, \dots 0)$ with singular support.

$\pi^* : T \to \tilde{T}$ induces an isom.

and since $\mathrm{Nm} \circ \pi^* = $ mult. by 2

$\mathrm{Nm}_{|\tilde{T}}$ is surjective

$\tilde{T}_2 = \ker(\mathrm{Nm}_{|\tilde{T}}) = \{$points of order 2 in $\tilde{T}\}$ $\simeq (\mathbb{C}^*)^b$

$\Rightarrow P$ is complete reduced and connected ☒

Rmk $g : P \to R$ is an isogeny. with
$|\ker g| = 2^{t-1}$ $\quad t = \dim T$

Def.
$\Theta_L = \{ M \in JC \mid h^0(L \otimes M) \geq 1 \}$ $\quad L \in \mathrm{Pic}^{g-1}(C)$
$(L = \pi^* L_0)$

Thm. Θ_L induces twice a principal pol.

$n : N \to C$ $\qquad \Theta' \subset JN$ $\qquad (n^*)^{-1}(\Theta') \equiv \Theta_L$

$n^* : JC \to JN$

$\cup$

$\odot^1$ $\qquad$ for a good choice of L

e.g. $\quad 2 \deg L = \deg \omega_C$

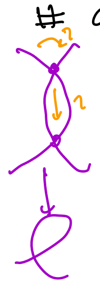
Def. $(\tilde{C}, \iota)$ $\quad (\tilde{C} \in \overline{\mathcal{M}}_{2g-1}, \ \iota^2 = \mathrm{id})$ is allowable.

(admissible for Beauville)

if P is an abelian variety.
$\quad \overset{\shortparallel}{\mathrm{Prym}(\tilde{C}, \iota)}$

This def. is equiv. to.

$(\tilde{C}, \iota)$ is allowable if the only fixed pts of ι are nodes where the 2 branches are not exchanged and
$\#$ of nodes exchaged under ι $=$ $\#$ irred. comp. exabanged under ι.



is allowable

$$\overline{R_g} := \{ [\tilde{C} \to C] \text{ allowable cover} \} \subset \overline{\overline{R}} \leftarrow \text{comp. by admissible}$$

Thm The Prym map $\mathcal{P}_g$ extends to a proper map
$$\overline{\mathcal{P}_g} : \overline{R_g} \longrightarrow A_{g-1}$$

$\boxed{\text{Case } g=6}$

$$JX \in \underset{\substack{\uparrow \\ \text{Jacobian locus}}}{\overset{\text{dim } 12}{\mathcal{J}_5}} \subset A_5 \qquad \text{generic Jacobian}$$

$(X \in \mathcal{M}_5 \text{ generic})$

What $[\tilde{C} \to C] \in \overline{R_6}$ give $\mathcal{P}_6(\tilde{C} \to C) = JX$?

(1) $C \in \mathcal{M}_6$ plane quintic and $\pi : \tilde{C} \xrightarrow{\ \eta \in JC[2]\ } C$. even double cover.

deg 1 $\left(\begin{aligned} h^0(L \otimes \eta) &\equiv 0 \mod 2 \ \text{even cover} \\ &\equiv 1 \mod 2 \ \text{odd cover} \end{aligned} \right)$ $\qquad C \overset{\varphi}{\hookrightarrow} \mathbb{P}^2$

$\underset{\substack{\uparrow \\ \deg 5.}}{L} = \varphi^* \mathcal{O}_{\mathbb{P}^2}(1)$

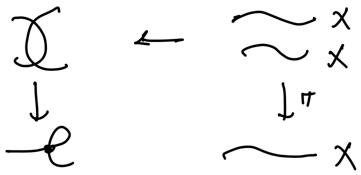
moduli count : $12 = \dim \mathcal{J}_5$

(2) C trigonal curve. $C \xrightarrow[3:1]{h} \mathbb{P}^1$ $\deg R_h = 16$

moduli count $16 - 3 = 13 > 12$

10 $\pi: \tilde{C} \longrightarrow C$ double unramified

(3) $\begin{array}{c} \tilde{C} \\ \pi \downarrow \\ C \end{array}$



$$\begin{array}{c} JX = \mathrm{Prym} \\ \downarrow \\ JX \times JX \\ \downarrow \\ JX \end{array}$$

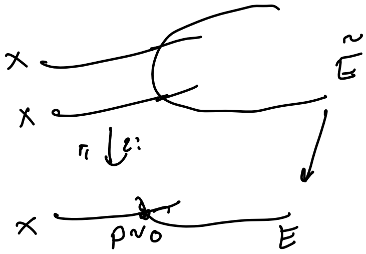
16

moduli count. $\dim M_5 + 2 = 14$.

(4) Elliptic tails

0

$\begin{array}{c} \tilde{C} \\ \downarrow \\ C \end{array}$



moduli count: $\dim M_5 + \dim A_1 + 1 = 14$ (divisor on $\bar{R}_6$)

How to compute a local degree

$f: X \longrightarrow Y$ proper dominant between n-dim'l var. X, Y.

$W \subset Y$ irred. subvar. of codim k

$f^{-1}(W)$ = finitely many components Z_i of codim l_i in X

d_i: local deg. of f along Z_i $\quad \deg f = \sum d_i$

Let $Z \subset W$ be one of these comp.

$$\begin{array}{ccccc} \tilde{Z} \subset \tilde{X} = Bl_Z X & \xrightarrow{\tilde{f}} & Bl_W Y = \tilde{Y} \supset \tilde{W} \\ \downarrow \qquad \downarrow & & \downarrow \qquad \swarrow \\ Z \subset X & \xrightarrow{f} & Y \supset W \end{array}$$

exc. div

$\rightsquigarrow f_*: \tilde{Z} \longrightarrow \tilde{W}$ $\qquad f_* = \tilde{f}|_{\tilde{Z}}$

$\tilde{Z} = \mathbb{P}(N(Z \setminus X))$ $\qquad TX = TZ \oplus T(Z \setminus X)$

$z \in Z$ $\quad w = f(z) \in W$

$$\begin{array}{ccc} df_z: T_z X & \longrightarrow & T_w Y. \\ \cup & & \cup \\ T_z Z & \longrightarrow & T_w W \end{array}$$

$\rightsquigarrow$ df_z induces $\quad f_{*,z} : N_{Z \setminus X, z} \longrightarrow N_{W \setminus Y, w}$

<u>Lemm</u> • $\tilde{f}$ is regular at generic $z \in \tilde{Z}$ $\Leftarrow\Rightarrow$ $f_{*,z}$ is not identically zero at generic $z \in Z$

• $\tilde{f}$ is regular $\forall \tilde{z}$ in the fiber over $z \in Z$

$\Leftarrow\Rightarrow$ $f_{*,z}$ is injective on the normal space to Z at z

In this $\tilde{f}|_{\text{fiber over } z}$ is the projectivized of the linear map $f_{*,z}$

<u>Lemma</u> Assume $f_{*,z}$ is injective on $N_{Z \setminus X, z}$ at <u>each</u> $z \in Z$

$\Rightarrow$ the local deg of f along Z equals the deg. of
$$f_* : \tilde{Z} \longrightarrow \tilde{W}$$

⚠ If. $f_{*,z}$ is not inj. at z $\Rightarrow$ $\tilde{f}$ is not reg. on neighborhood of $\tilde{z}$ $\rightsquigarrow$ blow up. again.

Subvar. mapping to $\tilde{W}$ $\Rightarrow$ $\tilde{Z}$ is only one of several comp

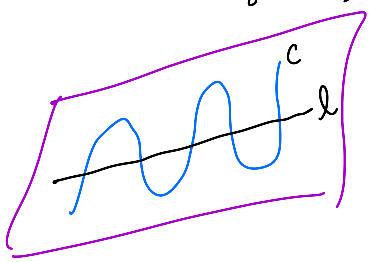
$\rightsquigarrow$ possibly $\deg f_* < \deg \tilde{f}|_{f^{-1}(\text{neigh. of } z)} = \text{local deg.}$

§ Plane quintics.

$C \in \mathcal{M}_6$ plane quintic $\quad C \overset{\varphi}{\underset{g^2_5}{\hookrightarrow}} \mathbb{P}^2$

$\rightsquigarrow$ there is a natural theta characteristic $L = \varphi^*_d \mathcal{O}_{\mathbb{P}^2}(1)$

$L^{\otimes 2} \simeq \omega_C \quad (\deg L = 5)$

$C \hookrightarrow$ 

$\mathcal{R}'_Q = \{ [C, \eta] \in \mathcal{R}_6 \mid C \text{ plane quintic} \}$

$(\eta \leftrightarrow \tilde{C} \to C)$

$\{ [C, \eta \otimes L] \mid C \text{ plane quintic} \}$

$h^0(\eta \otimes L) \equiv 1 \mod 2$ η is odd

$\equiv 0$ η is even

$\rightsquigarrow \mathcal{R}'_Q = \mathcal{R}_Q \sqcup \mathcal{R}_C$

$\mathcal{R}_Q$ (even) $\xrightarrow{P_6} \mathcal{J}_5$

$\mathcal{R}_C$ (odd) $\xrightarrow{P_6} \mathcal{C}$ cubic 3-folds

Prop. (Mumford) $(C, \eta) \in \mathcal{R}'_Q$

$\mathrm{Prym}(C, \eta) \in N_1 \iff \eta$ is even. (i.e. $(C,\eta) \in \mathcal{R}_Q$)

Recall: A ppav. $(A, \Theta) \in N_k$ (Andreotti-Mayer loci). if $\dim(\mathrm{Sing}\,\Theta) \geq k$.

If $JX = \mathrm{Prym}(C, \eta) \in \mathcal{J}_5$ X generic (non-hyp., nor trigonal)

$\tilde{C} = \Theta_{sing} = \{ L \in \mathrm{Pic}^4(X) \mid h^0(L) \geq 2 \}$ $\iota : L \mapsto K - L$

$X \xrightarrow[|\omega_X|]{} \mathbb{P}^4$ can. embedding $X = Q_0 \cap Q_1 \cap Q_2$.

Any g^1_4 is cut out by 1-param. family of planes sweeping out a quadric (of rank 3 or 4) in $\mathbb{P}^4$ containing X.

$\Pi = \langle Q_0, Q_1, Q_2 \rangle \simeq \mathbb{P}^2$. $= \{ \underbrace{\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2}_{\lambda} \mid (\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2 \}$

disc locus $\Pi = \{ Q_\lambda \mid Q_\lambda$ is singular$\} = C$

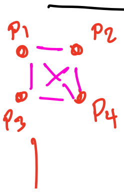
↳ quintic.

For each $\lambda \in C$ there 2-systems of 2-planes of Q_λ cutting the g^1_4

$\rightsquigarrow \tilde{C} \xrightarrow{2:1} C$

§ Trigonal curves

Recillas construction: (arbitrary g) $g(X) = g - 1$ X tetragonal



$\tilde{C} := \{ p_1 + p_2 \in X^{(2)} \mid p_1 + p_2 + p_3 + p_4 \in g^1_4 \}$ $\circlearrowleft \sigma$

$\tilde{C} \xrightarrow{2:1} C = \tilde{C}/\sigma$

σ free of fixed points

$X \xrightarrow[4:1]{f} \mathbb{P}^1$

$$\begin{tikzcd} \arrow[d, "g^1_4"'] & \arrow[d, "h", "3:1"'] \\ \mathbb{P}^1 & \mathbb{P}^1 \end{tikzcd}$$

Then:

- $C = \tilde{C}/\sigma$ trigonal
- $\pi : \tilde{C} \to C$ étale.
- h has same branch locus as f. $(2g+4)$
- $g(C) = g$

Inverse const.

$[\tilde{C} \xrightarrow{\pi} C] \in \mathcal{R}_T = \{ [\tilde{C} \to C] \in \mathcal{R}_g \mid C \text{ trigonal} \}$

$$\begin{tikzcd} C \arrow[d, "3:1"', "h"] \\ \mathbb{P}^1 \end{tikzcd}$$

$$\begin{tikzcd} & \tilde{X} \supseteq \sigma \arrow[dl, "2:1"'] \arrow[r, hook] \arrow[d, "\pi_1^{(3)}"] & \tilde{C}^{(3)} \supseteq \sigma \arrow[d, "\pi^{(3)}", "2^3 = 8 : 1"'] \\ \tilde{X}/\sigma = X \arrow[r, "4:1"] & \mathbb{P}^1 \simeq g^1_3 \arrow[r, hook] & C^{(3)} \ni p+q+r \end{tikzcd}$$