From yesterday:

$\mathcal{J}_5 \ni JX \rightsquigarrow \tilde{C} = \tilde{\Theta}_{sing}$ ← smooth genus 11 curve $\rightsquigarrow \mathrm{Prym}(\tilde{C}/C) = JX$

ι -1

$\downarrow$

$C = \tilde{\Theta}_{sing}/_{\pm 1}$ ← Plane quintic.

Corollary $\mathcal{P}_6|_{R_Q}$ is bijective

⚠ one still has to prove: that $\mathcal{P}_6$ is no ramified on R_Q.

Trigonal construction allow us to define

$$\tau : \mathcal{J}^1_{4,g-1} \longrightarrow \overline{R_g} \xrightarrow{\bar{P}_g} A_{g-1}$$
$$(X, g^1_4) \longmapsto [\tilde{C} \to C]$$

allowable.

Prop. (1) $P_g(\tau(X)) = JX$

(2) $\Psi : \tilde{C} \longrightarrow JX = P_g(\tilde{C} \to C)$ is the Abel-Prym map.
$(a,b) \longmapsto \varphi(a) + \varphi(b)$

$\tilde{C} \subset X^{(2)}$

$\Psi : \tilde{C} \longrightarrow J\tilde{C} \xrightarrow{1-\iota} P$

Abel-Prym.

$\varphi : X \longrightarrow JX$ Abel-Jacobi.

Proof. Universal property of Prym varieties

X a b c d $\rightsquigarrow$ $a - b$, $c - d$

$X \xrightarrow{4:1} \mathbb{P}^1$

$$\begin{tikzcd}
 & \tilde{C} \arrow[loop above, "\iota"] \arrow[r, "\psi"] \arrow[dl, "\alpha_{\tilde{c}}"'] \arrow[d, "\psi_{\tilde{c}}"] & JX \arrow[d, "t_{-\psi(\tilde{c})}"] \\
J\tilde{C} \arrow[r, "1-\iota"'] & P \arrow[r, "\tilde{\psi}"'] & JX
\end{tikzcd}$$

ψ symmetric.

$\psi \circ i = -\psi$

Obs. $\psi \circ \iota(a+b) = \psi(c+d) = \varphi(c) + \varphi(d) = -\varphi(a) - \varphi(b)$

since $\varphi(a) + \varphi(b) + \varphi(c) + \varphi(d) = 0$ on JX

It suffices to show $[\psi(C)] \in H_2(JX, \mathbb{Z})$ is

$\frac{2}{(g-2)!}\theta^{(g-2)}$ $\theta = [\Theta]$ class of the princ. pol. in JX.

(Masiewicki's criterion). Use a degeneration:

$X_t \rightsquigarrow X_0 \cup \mathbb{P}^1$

$f_t \downarrow 4:1$

$\mathbb{P}^1$

Assume the limit divisor class. $f_0|_{X_0} : X_0 \xrightarrow{3:1} \mathbb{P}^1$

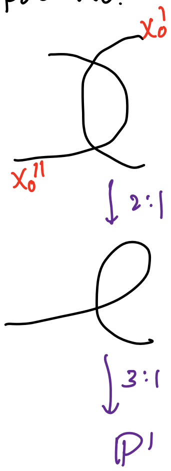
$D_0 = [T] + [p_0]$ for some $p_0 \in X_0$.

trigonal $\rightsquigarrow$ const.

$\tilde{C} = X_0' \cup X_0''$ / $p_1' \sim p_2''$, $p_2' \sim p_1''$

$\downarrow$

C $X_0 / p_1 \sim p_2$



Norm $(C) = X_0$

The class of $[\Psi_t(\tilde{C})]$ does not change in the degeneration.

In the limit $[\Psi(\tilde{C})] = [\mathcal{C}(X_0') \cup \mathcal{C}(X_0'')]$

$= 2[\mathcal{C}(X)] = \frac{2}{(g-2)!}\theta^{(g-2)}$ (Matsusaka's criterion)

$g(X) = g-1$

∎

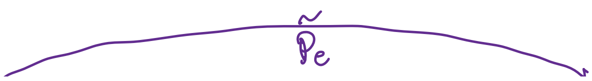
Let $\mathcal{R}_{T,g} = \{ [\tilde{C} \xrightarrow{2:1} C] \in \mathcal{R}_g \mid C \text{ trigonal} \}$

$\overline{\mathcal{R}_{T,g}} \subset \overline{\mathcal{R}_g}$ $\tau : \mathcal{Y}^1_{4,g-1} \to \overline{\mathcal{R}_g}$

By the inverse const. of the trigonal, $\mathrm{Im}\,\tau = \overline{\mathcal{R}_{g,T}}$.

Rmk $X \in \mathcal{M}_5$ always has a g^1_4

$\boxed{g=6}$



$$\begin{tikzcd}
\widetilde{\mathcal{R}}_T \subset Bl_{\overline{\mathcal{R}}_T}\overline{\mathcal{R}}_6 = \widetilde{\mathcal{R}}_6 \arrow[d, "\mathbb{P}^1\text{-bdl}"'] \arrow[r, "\widetilde{\mathcal{P}}_6"] & \widetilde{\mathcal{A}}_5 = Bl_{\mathcal{J}_5}\mathcal{A}_5 \supset \widetilde{\mathcal{J}}_5 \arrow[d, "\mathbb{P}^2\text{-bdl}"] \\
\overline{\mathcal{R}}_T \underset{\text{codim } 2}{\subset} \overline{\mathcal{R}}_6 \arrow[r, "\mathcal{P}_6"] & \mathcal{A}_5 \underset{\text{codim } 3}{\supset} \mathcal{J}_5
\end{tikzcd}$$

Recall: $d\mathcal{P}_6^* : Sym^2 H^0(C, \omega_C \otimes \eta) \longrightarrow H^0(C, \omega_C^2)$

Prop. $\mathrm{Ker}\, d\mathcal{P}_6^* = \{$ Quadrics in $\mathbb{P}^4$ containing $\Psi(C)\}$

$\Psi : C \longrightarrow \mathbb{P}H^0(C, \omega_C \otimes \eta)^\vee \simeq \mathbb{P}^4$ Prym-canonical.

$\Phi : X \hookrightarrow \mathbb{P}H^0(C, \omega_X)^\vee \simeq \mathbb{P}^4$ canonical embedding

$X \in \mathcal{M}_5$ $\{X$ non-hyp., non-trigonal, no vanishing theta null$\}$

$(h^0(X, \varkappa) > 0$ $\quad \varkappa$ even theta-char $\quad \varkappa^{\otimes 2} \simeq \omega_X)$

$\alpha(X) = \{L \in Pic^4(X) : h^0(L) \geq 2\}$

$$\alpha(X) = \overset{(-1)}{\circlearrowright} \widetilde{C} \quad \widetilde{W}_{sing} \quad \text{smooth curve genus } 11$$

$$2:1 \downarrow \pi \text{ étale}$$

plane quintic $\longrightarrow F = \widetilde{W}_{sing}/(\pm 1)$ genus 6

K-g^1_4 $\leftrightarrow$ g^1_4 on X $\leftrightarrow$ family of planes on a singular quadric S containing $\Phi(X) \subset \mathbb{P}^4$

$\rightsquigarrow$ S contains 2 families of planes cutting a g^1_4 on X.

$\Rightarrow$ $\overline{\mathcal{P}}_6 : \overline{\mathcal{R}}_{T,6} \longrightarrow \mathcal{J}_5$ has 1-dim'l fiber.

(because generic $X \in \mathcal{M}_5$ has 1-dim'l family of singular quadrics $\supset \Phi(X)$)

Thm The local deg of $\overline{\mathcal{P}}_6$ at $\overline{\mathcal{R}}_T$ is 10

· $f_{*,z}$ is injective on the normal space at z $\quad \mathbb{P}(f_{*,z})$

$\Leftrightarrow$ surjective on conormal. $\quad \widetilde{\mathcal{P}}_{e,z}$

$$d\mathcal{P}_6^*|_{\text{conor.}} : \underset{\underset{T^*\mathcal{J}_5 \oplus N^*(\quad)}{\cap}}{\underset{\text{rk. } 3}{N^*(\mathcal{J}_5 \setminus \mathcal{A}_5)}} \longrightarrow \underset{\underset{T^*\overline{\mathcal{R}}_T \oplus N^*(\quad)}{\cap}}{\underset{\text{rk } 2}{N^*(\overline{\mathcal{R}}_T \setminus \overline{\mathcal{R}}_6)}}$$

$\uparrow$

$$\begin{array}{ccc} \cong & & \cong \\ T^*\mathcal{A}_5 & \xrightarrow[dp_6^*]{} & T^*\overline{\mathcal{R}}_6 \end{array}$$

this is surjective because $\dim\left(\operatorname{Ker} dp_6^*|_{\text{conor}}\right) \leq 1$ ∎

Compute the deg on $\overline{\mathcal{R}}_{T,6}$

$X \in \mathcal{M}_5$ generic. $\quad \mathbb{P}^2_{JX} = \mathbb{P}\left(N_{JX}(\mathcal{Y}_5 \setminus \mathcal{A}_5)\right)$

$\mathcal{R} = \tilde{P}_e^{-1}(\mathbb{P}^2_{JX}) \subset \tilde{\mathcal{R}}_T$

Describe $\tilde{P}_e : \mathcal{R} \longrightarrow \mathbb{P}^2_{JX} = \Pi^\vee$

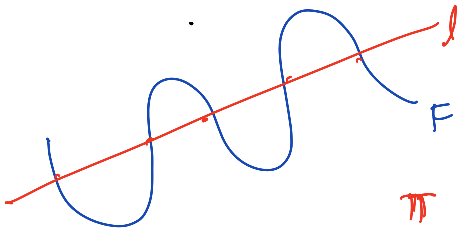
Π = Plane containing $F = \Theta_{sing}/\pm$. (genus 6)

$= \langle Q_0, Q_1, Q_2 \rangle$

$\alpha : \mathcal{M}_5 \longrightarrow \overline{\mathcal{R}}'_Q$

$X \longmapsto (\Theta_{sing}, (-1))$

↑ param. g^1_4's on X



point in $\mathbb{P}^2_{JX}$ $\longleftrightarrow$ line in Π $\longleftrightarrow$ pencil of quadrics in $\mathbb{P}^4$ $\supset \Phi(X)$

Viceversa

line in $\mathbb{P}^2_{JX}$ $\longleftrightarrow$ point in Π $\longleftrightarrow$ sing. quadric $Q \supset \Phi(X)$

$[\tilde{C}, C] \in \tilde{\mathcal{R}}_T$ st. $\operatorname{Prym}(\tilde{C}/C) = JX$.

$\mathbb{P}^1_{[\tilde{C},C]} \subset R \subset \tilde{\mathcal{R}}_T \qquad \tilde{P}_e| : \mathbb{P}^1_{[\tilde{C},C]} \longrightarrow \mathbb{P}^2_{JX}$ injective.

$= f_{t,z}$

its image is a line in $\mathbb{P}^2_{JX}$ $\longleftrightarrow$ sing. quadric $Q \supset \Phi(X)$

s.t. $Q \in \ker\left(dP^*|_{\text{con.}}\right)$

$\deg \tilde{P}_e = \# P_e^{-1}(p) \qquad (p \in \mathbb{P}^2_{JX} \text{ generic})$

$= \# \{ C \in \alpha(X) \mid \operatorname{Prym}(\tilde{C}/C) = JX \text{ and } p \in \tilde{P}_e(\mathbb{P}^1_C) \subset \mathbb{P}^2 \}$

$= \# \{ C \in \alpha(X) \mid \pi(C) \in \ell_p \subset \Pi^\vee \}$

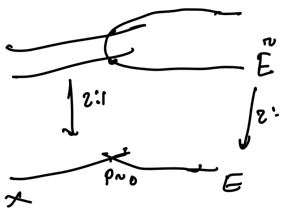
$$\begin{array}{c} \alpha(X) \\ {\scriptstyle 2:1}\downarrow \pi \\ F \hookrightarrow \Pi \end{array} \qquad \begin{array}{c} g^1_4 \text{ on } X \\ \updownarrow \\ (\tilde{C}, C) \end{array} \qquad \Rightarrow \deg \tilde{P}_e = (\deg \pi) \cdot (\deg F) = 2 \cdot 5 = 10$$ ∎

§ Boundary components

$\mathcal{R}_S = \{ [\tilde{C} \xrightarrow{2:1} C] \in \overline{\mathcal{R}_6} \mid [C] \in \overline{\mathcal{M}_6} \}$

Singular $\quad \mathcal{X} \longrightarrow \ell_x$

$\mathcal{R}_E = \left\{ \begin{array}{c} \tilde{C} \in \overline{\mathcal{R}_6} \\ \downarrow \\ C \end{array} \right.$



elliptic $\Big\}$

ellipti tails

Obs. $\overline{P_6}|_{R_S} : R_S \longrightarrow \mathcal{J}_5 \subset \mathcal{A}_5$ proper and surjective

dim 44 $\qquad$ dim 12

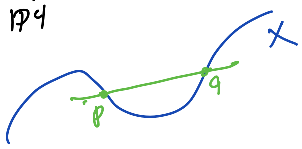
$X \in \mathcal{M}_5$ generic (has no autom.)

$\overline{\mathcal{P}_6}|_{\mathcal{R}_S}^{-1}(JX) = S^2(X)$ $\quad$ (choice of $p,q \in X$ to glue)

Prop. For $C = X/_{p\sim q}$ $\quad$ $\ker dP^*$ is 2 dim'l.

Recall. $X = Q_0 \cap Q_1 \cap Q_2$ $\quad$ the secant $\overline{p\cdot q}$ imposes 1 linear conditions on the quadrics.

$\cap$
$\mathbb{P}^4$



$f := \tilde{P}_e| : S^2X \longrightarrow \mathbb{P}^2_{JX} \simeq \mathbb{P}(N_{JX}(J^5 \setminus A_5))$

$\quad\quad (\mathbb{P}^2)^\vee = \Pi \leftarrow$ quadrics through $\Phi(X) \subset \mathbb{P}^4$

$f(p,q)$ = pencil of quadrics containing $\overline{\Phi(p)\cdot\Phi(q)}$

$p,q \in X$

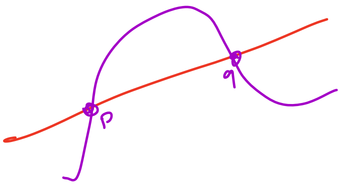
deg f = # chords of $\Phi(X)$ contained in the intersection of 2 quadrics in general position (pencil)

Lemma (A) The intersection of 2 quadrics (3-dim'l) in gral position in $\mathbb{P}^4$ contains 16 lines. $\leftarrow$ (del Pezzo surface of deg 4 $\quad$ $\mathbb{P}^2$ blow up in 5 point)

Lemma (B) $\Phi(X)$ meets each of the

easy. $\quad$ 16 lines twice.

$$\Rightarrow \quad \underline{\underline{\deg f = 16}}.$$



$L \subset Q_1 \cap Q_2$

$\Phi(X) \subset \mathbb{P}^4$

<u>Rmk</u> $\mathcal{R}_E$ doesn't contribute to the degree.

$$\begin{tikzcd}
\mathcal{R}_E \subset \overline{\mathcal{R}_6} \arrow[rr, "\overline{\mathcal{P}_6}"] \arrow[dr] \arrow[d] & & \mathcal{A}_5 \\
\text{blow-down} & \mathcal{R}' \arrow[ur] &
\end{tikzcd}$$

$\overline{\mathcal{R}_g}$

$\mathcal{R}'$: new compactification of $\mathcal{R}_6$
(" " of $\mathcal{M}_6$)

§ Another special fiber.

$$\mathcal{R}_Q' = \underset{\text{(even)}}{\mathcal{R}_Q} \cup \underset{\text{(odd)}}{\mathcal{R}_C}.$$

$\mathcal{C}$: moduli space of non-sing cubic 3-folds.

$X \subset \mathbb{P}^4$ $\quad$ $\dim X = 3$ cubic.

moduli count 10

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathcal{A}_5 \quad \text{(Torelli)} \\ X & \longmapsto & JX = H^{1,2}(X,\mathbb{C}) / H^3(X,\mathbb{Z}) \end{array}$$

Intermediate Jacobian

$$\begin{tikzcd}
\widetilde{\mathcal{R}}_C \subset \widetilde{\mathcal{P}}_6 : \widetilde{\mathcal{R}}_6 \arrow[r, "\widetilde{\mathcal{P}}_6"] \arrow[d] & \widetilde{\mathcal{A}}_5 \supset \widetilde{\mathcal{J}}_5 \arrow[d] \\
(\tilde{C},\eta) \in \overline{\mathcal{R}}_C \subset \mathcal{R}_6 \arrow[r, "\mathcal{P}_6"] & \mathcal{A}_5 \supset \mathcal{J}_5
\end{tikzcd}$$

$\mathcal{P}_6(\mathcal{R}_C) = JX.$

(conic bundle structure)

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\psi"] \arrow[d] & JX \arrow[d] \\
P \arrow[r, "\tilde{\psi}"] & JX
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "?\varphi"] \arrow[d] & P \arrow[d] \\
JX \arrow[r, "\tilde{\varphi}"] & P
\end{tikzcd}$$