Rational curves on K3 surfaces and Euler characteristics of Moduli spaces

The goal is to go through a result of Beauville [2] (following work of Yau and Zaslow [6]) which uses hyperkähler geometry to count the number of rational curves in a very general K3 surface of degree 2d.

Problem 1. Assume that a K3 surface X admits an *elliptic pencil* – that is a map

$$\pi\colon X \to \mathbb{P}^1$$

so that the general fibers are smooth genus 1 curves. Assume that all the fibers that do not have geometric genus 1 are irreducible rational curves with a single node. Count the number of rational fibers. (Hint: If $R = \sqcup_{i=1}^n R_i$ is the union of rational curves, compute the topological euler characteristic using the formula:

$$e(X) = e(R) + e(X \setminus R)$$

and compute $e(R_i)$.)

Examples of Hyperkählers from K3 surfaces. Let X be a very general K3 surface of degree 2d with primitive line bundle L (with $L^2 = 2d$) and let $\Pi = \mathbb{P}(H^0(X, L)) \cong \mathbb{P}^{d-1}$. Moduli spaces of sheaves on X are frequently hyperkähler manifolds. Here are two examples:
(1) **Hilbert schemes of n points** on X – denoted $X^{[n]}$.
(2) **Compactified Jacobians** – denoted $\overline{\mathcal{J}}^d(X)$ – parametrizing coherent sheaves supported on curves $C \in \Pi$, which when thought of as sheaves on C are line bundles (or torsion-free sheaves of rank 1 when C is singular) of degree d.

Problem 3. Show that if X is a K3 surface, then Π contains only finitely many rational curves (curves with geometric genus 0).

Problem 4. Compute the dimension of $X^{[n]}$ and $\overline{\mathcal{J}}^d(X)$.

Problem 5. Show that the hyperkählers $X^{[g]}$ and $\overline{\mathcal{J}}^g(X)$ are birational.

There is a natural map

$$\pi\colon \overline{\mathcal{J}}^g(X) \to \Pi$$

which sends a coherent sheaf $\mathcal{F}$ to the curve in Π that it is supported on.

Problem 6. Show that the general fiber of π is an Abelian variety. Describe the fibers over a general point $C \in \Pi$.

Problem 7. (this is [2, Prop. 2.2]) Let C be an integral curve such that the normalization $\widehat{C}$ has genus ≥ 1. We show that $e(\overline{\mathcal{J}}^d(C)) = 0$ as follows.
(1) Find a line bundle $\mathcal{M}$ on C such that $\mathcal{M}$ is torsion of degree m (for any $m > 0$). (This uses the comparison between the Jacobian of C and of $\widehat{C}$.)
(2) Show that tensoring by $\mathcal{M}$ is a free action of $\mathbb{Z}/m\mathbb{Z}$ on $\overline{\mathcal{J}}^d(\widehat{C})$.
(3) Conclude that m divides $e(\overline{\mathcal{J}}^d(C))$ for all $m > 0$.

It follows by the scissor property of Euler characteristics that

$$e(\overline{\mathcal{J}}^g(X)) = \sum_{R_i \in \Pi} e(\overline{\mathcal{J}}^g(R_i))$$

where $R_i \in \Pi$ is a rational curve and $\pi^{-1}(R_i)$ is the fiber over R_i (i.e. the set of torsion free sheaves of rank 1 and degree g supported on R_i).

Problem 8. Show that

$$e(\overline{\mathcal{J}}^g(R_i)) = 1$$

if R_i is a nodal, irreducible rational curve. (Thus by a result of Xi Chen [3], if X is very general then

$$e(\overline{\mathcal{J}}^g(X)) = \#\{R_i \in \Pi\}.)$$

Hint: Locally at a nodal point $p \in R_i$ there are only 2 types of rank 1 torsion free sheaves (1) line bundles and (2) the ideal sheaf of a point. Show that if $p_1, \cdots, p_g \in R_i$ are the nodal points then $\overline{\mathcal{J}}^g(R_i)$ is stratified into loci $\overline{\mathcal{J}}^g_S \subset \overline{\mathcal{J}}^g(R_i)$ consisting of torsion-free sheaves that are not locally free exactly at the points in a subset $S \subset \{p_1, \cdots, p_g\}$. Conclude that the only stratum where $e(\overline{\mathcal{J}}^g_S) \neq 0$ is when $S = \{p_1, \cdots, p_g\}$ (a single point). See also [2, §3].

It remains to actually calculate the Euler characteristic of $\overline{\mathcal{J}}^g(X)$. This relies on
(1) the birational invariance of Euler characteristic for hyperkählers (see [5] or use the birational invariance of betti numbers of Calabi-Yaus [1]).

(2) the computation of the Euler characteristic of $X^{[n]}$ by Göttsche (see [4] for a nice write up of these results).

In particular, for a K3 surface, by (1) and (2) we have:

$$\sum(\#\text{ rational curves on a genus } g \text{ K3})q^g = \sum_{g\geq 0} e(\overline{\mathcal{J}}^g(X))q^g$$
$$= \sum_{g\geq 0} e(X^{[g]})q^g = \Pi_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right)^{e(X)}$$

where the sum over $g \geq 0$ is understood to take a very general genus g K3 surface.

Problem 9. Compute the Euler characteristic of $X^{[2]}$ for any complex surface using that (1) there is a birational map

$$h : X^{[2]} \to X^{(2)}$$

to the symmetric product $X^{(2)} := X^2/\Sigma_2$ which is given by blowing up the diagonal locus and (2) the exceptional divisor of h is a $\mathbb{P}^1$-bundle over X.

Problem 10. Find the number of bitangents to a very general plane sextic curve $C \subset \mathbb{P}^2$ using that a very general genus 2 K3 surface is a double cover of $\mathbb{P}^2$ branched at such a sextic.

References

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