$\mathcal{C} = \{ X \subset \mathbb{P}^4$ non-sing cubic 3fold $\}$ dim 10

$\mathcal{C} \hookrightarrow A_5$ Intermediate Jacobian

$X \longmapsto JX := H^{1,2}(X,\mathbb{C}) / H^3(X,\mathbb{Z})$

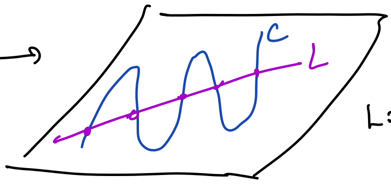
$(X, l)$ $X \in \mathcal{C}$, $l_0 \subset X \subset \mathbb{P}^4$ line.

$Bl_{l_0} X = \tilde{X}$ conic bundle. $\pi^{-1}(p) = \begin{cases} \text{conic} \\ 2 \text{ lines} \end{cases}$

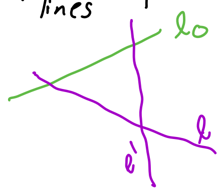
$\pi_l : X \dashrightarrow \mathbb{P}^2 \ni p$ (projection from $l$)

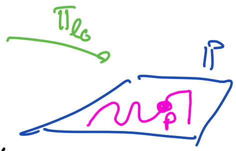
discriminant locus $:= \{ p \in \mathbb{P}^2 \mid \pi^{-1}(p) = \text{pair of lines} \} =: C$ plane quintic (smooth in gral)

$C \subset^{l} \!\rightarrow$




$L = \varphi^* \mathcal{O}_{\mathbb{P}^2}(1)$




$F(X) := \{ l \subset X \mid l \text{ line} \}$ Fano surface of lines

$\tilde{C} = \{ l \in F(X) \mid l \cap l_0 \neq \emptyset \}$ $\longleftrightarrow \eta \in JC[2]$ odd

étale $\pi \downarrow 2:1$

$C \leftarrow$ plane quintic $\quad ( h^0(C, \eta \otimes L) \equiv 1 )$

Prop. $\mathrm{Prym}(\tilde{C}/C) \simeq JX$

Obs. $\mathcal{P}_6^{-1}(JX) \simeq F(X)$ 2-dim'l

$$\begin{array}{ccccc} (C,\eta,L) \in \tilde{\mathcal{R}}_{\mathcal{C}} \subset \tilde{\mathcal{R}}_6 & \xrightarrow{\tilde{\mathcal{P}}_6} & \tilde{A}_5 \supset \tilde{\mathcal{C}} \ni (X,H) \\ \downarrow \mathbb{P}^2\text{-bdl} \qquad \downarrow & & \downarrow \qquad \downarrow \mathbb{P}^4\text{-bdl} \\ \underset{(\text{odd})}{\mathcal{R}_{\mathcal{C}}} \subset \underset{\text{codim } 3}{\mathcal{R}_6} & \xrightarrow{\mathcal{P}_6} & A_5 \supset \underset{\text{codim } 5}{\mathcal{C}} \ni X \end{array}$$

- $(C,\eta,L) \in \widetilde{R}_e$ $\quad$ $C \subset \mathbb{P}^2$ plane quintic. $\quad \eta \in JC[2] \smallsetminus \{0\}$

$\quad \big\downarrow \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad L \in (\mathbb{P}^2)^*$ line in $\mathbb{P}^2$

$\quad (C,\eta) \in R_C \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad h^0(L\otimes\eta) = 1$

- $(X,H) \in \widetilde{\mathcal{C}}$ $\quad X \subset \mathbb{P}^4,\ X \in \mathcal{C},\quad H \in (\mathbb{P}^4)^*$ Hyperplane

$$\widetilde{P}_e^{-1}(X,H) = \{\, \ell \in F(X) \mid \ell \in \underbrace{X \cap H}_{\text{smooth cubic surface for } X,H \text{ gral.}} \}$$

$*$ Every smooth cubic surface contains 27 lines.

§ Tetragonal construction.

$$\begin{array}{c} \widetilde{C} \circlearrowleft \sigma \\ \text{étale } \pi \downarrow 2:1 \\ C \\ 4:1 \downarrow f \\ \mathbb{P}^1 \end{array}$$

$$\begin{tikzcd}
& \sigma \circlearrowright \widetilde{C}_0 \sqcup \widetilde{C}_1 \circlearrowleft \sigma = \widetilde{X} \arrow[ddl, "2:1"', "\pi_0"] \arrow[dd] \arrow[dr, "\pi_1", "2:1"'] \arrow[rr, hook] & & \widetilde{C}^{(4)} \circlearrowleft \sigma \arrow[dd, "2^4:1"', "\pi^{(4)}"] \\
& & C_1 \arrow[dl, "4:1", "f_1"'] & \\
C_0 \arrow[r, "f_0", "4:1"'] & g^1_4 \simeq \mathbb{P}^1 \arrow[rr, hook] & & C^{(4)}
\end{tikzcd}$$

$\widetilde{p}_i + \widetilde{p}_j + \widetilde{p}_k + \widetilde{p}_\ell$

$\quad\quad \longmapsto$

$C^{(4)} \ni p_1+p_2+p_3+p_4$

$\pi^{-1}(p) = \{\widetilde{p}_1, \widetilde{p}_2\}$

$D, D' \in \widetilde{X} \subset \widetilde{C}^{(4)}$ $\quad D \sim D' \iff$ push down to the same divisor in $C^{(4)}$ and share an even # of points

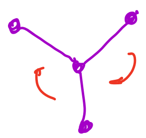
$(\widetilde{C}, C, f) \rightsquigarrow (\widetilde{C}_0, C_0, f_0),\ (\widetilde{C}_1, C_1, f_1)$

Tetragonal $\quad\quad\quad\quad\quad\quad\quad\quad$ "triality"

The tower corresponds to a repr.

$$\pi_1(\mathbb{P}^1 \smallsetminus \{\text{branch}\}) \longrightarrow WD_4$$

Dynkin diagram



$S_3 \cong$ Outer Autom.

Rmk The trigonal const. is a degeneration of the tetragonal.

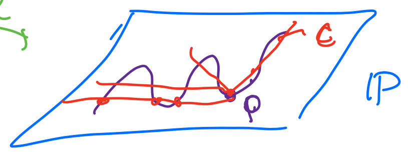
$X \subset \mathbb{P}^4$ smooth cubic 3-fold. $\quad \ell \subset X$ line.

$$\begin{array}{ccc} \widetilde{X} & & \widetilde{C} \\ & \searrow \pi_1 & \downarrow \pi \\ & & \mathbb{P}^2 \supset C \end{array}$$

Choose $A \simeq \mathbb{P}^2$, $A \subset \mathbb{P}^4$ $\quad A \cap X = \ell \cup \ell' \cup \ell''$

$\rightsquigarrow$ 3 plane quintics $C, C', C''$, $\rightsquigarrow$ $\tilde{C}, \tilde{C}', \tilde{C}''$

$$\begin{array}{ccc} \tilde{C} & \tilde{C}' & \tilde{C}'' \\ \downarrow & \downarrow & \downarrow \\ C & C' & C'' \\ {\scriptstyle 4:1}\downarrow f & f' \downarrow {\scriptstyle 4:1} & \downarrow f'' \\ \mathbb{P}^1 & \mathbb{P}^1 & \mathbb{P}^1 \end{array}$$


Projection from $p$ gives a $\rightsquigarrow g^1_4$ on $C$

$(\tilde{C}, C, f)$, $(\tilde{C}', C', f')$, $(\tilde{C}'', \tilde{C}', f'')$ are tetragonally related.

$A \simeq \mathbb{P}^2$ $\rightsquigarrow$ pencil of hyperplanes containing $A$ $\{T_\lambda\}_{\lambda \in \mathbb{P}^1}$ $T_\lambda \simeq \mathbb{P}^3$

$A \cap X \simeq \ell \cup \ell' \cup \ell''$

$T_\lambda \cap X \simeq Y_\lambda \leftarrow$ cubic surface

**Thm** (Donagi) $\quad \mathrm{Prym}(\tilde{C}/C) \simeq \mathrm{Prym}(\tilde{C}'/C') \simeq \mathrm{Prym}(\tilde{C}''/C'')$

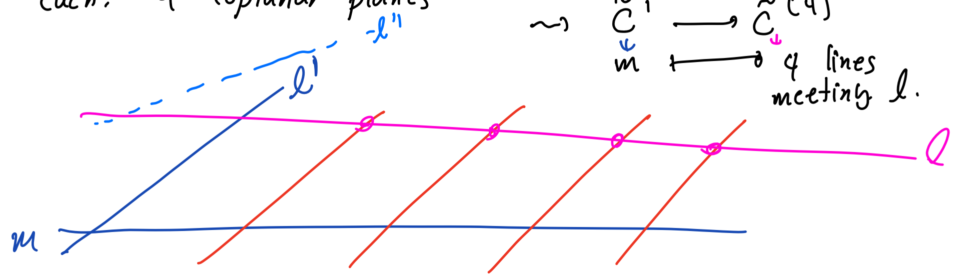
Proof.

$$\begin{tikzcd} \tilde{C}' \arrow[r, hook, "\varphi"] \arrow[d] & \tilde{C}^{(4)} \arrow[r] \arrow[d] & \mathrm{Pic}(\tilde{C}) \arrow[d] \\ \mathbb{P}^1 \arrow[r] & C^{(4)} \arrow[r] & \mathrm{Pic}(C) \end{tikzcd}$$

A line $m$ in $Y_\lambda$ meeting $\ell'$ (and not in $A$) also meets 4 of the 8 lines (in $Y_\lambda$ not in $A$) meeting $\ell$, one in each. 4 coplanar planes

$\rightsquigarrow$ $\tilde{C}' \longrightarrow \tilde{C}^{(4)}$

$m \longmapsto$ 4 lines meeting $\ell$.




$\rightsquigarrow$ Use universal property of Pryms, prove $[\varphi_*(\tilde{C}')] = 2\dfrac{[\Xi]^{g-2}}{(g-2)!}$

by degeneration

Let $\mathcal{M}_6^{Tet} = \{ [C, g^1_4] \mid C \in \mathcal{M}_6 \}$

$$\begin{array}{ccc} \mathcal{M}_6^{Tet} & & [C, g^1_4] \\ \Big\downarrow \text{deg } 5 & & \Big\downarrow \\ \mathcal{M}_6 & \ni & [C] \end{array}$$

By base change

$$\begin{array}{ccc} \mathcal{R}_6^{Tet} & & \mathcal{M}_6^{Tet} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{R}_6 & \longrightarrow & \mathcal{M}_6 \end{array}$$

Tetragonal const gives a correspondence $(2,2)$.

$$\mathcal{D} = \{ ([\tilde{C}, C, f], [\tilde{C}', C', f']) \in \mathcal{R}_6^T \times \mathcal{R}_6^T \mid \text{tetragonally-related} \}.$$

$$\swarrow_{2:1}^{pr_1} \qquad \searrow_{2:1}^{pr_2}$$

$\rightsquigarrow$ $(10, 10)$ - Correspondence

every line in a cubic surface intersect 10 others

$$\mathrm{Tet} \subset \mathcal{R}_6 \times \mathcal{R}_6$$
$$\{ [\tilde{C}, C], [\tilde{C}', C] \mid \text{tetragonally} \}$$

Thm (Donagi) The corresp. Tet. on $P_6^{-1}(A)$ for generic $A \in \mathcal{A}_5$ is isomorphic to the incidence corresp on the 27 lines on a non-sing cubic surface.

$\rightsquigarrow$ The monodromy group of $\mathcal{R}_6 \to \mathcal{A}_5$ is

$W(E_6) =$ symmetry group of the 27 lines.

10

- $\mathcal{C}$ cubic 3-folds. $\qquad W(E_6) \subset S_{27}$
- $\mathcal{J}_5$ Jacobians $\qquad WD_5 = $ symmetry of group of lines in the cubic surf with one marked one

12

$WD_5 = \mathrm{Stab}(\ell) \subset W(E_6)$.

- $\mathcal{B}$ Intermediate Jacobians of quartic double solids. $\qquad W(A_5) \simeq S_6$. lines on a nodal cubic surface

dim 14

Rmk $P_6$ fails to be finite over $\mathcal{J}_5$ and over $\mathcal{C}$

$B$ = Branch locus of $\mathcal{P}_6$ $\supset \mathcal{Y}_5 \cup \mathcal{C} \cup \{$Intermediate Jacobians of double solids$\}$

irred

degenerations.

§ Triple cyclic coverings.

$\sigma \in \mathrm{Aut}(P)$

$$\mathcal{R}_g(d) = \{ \overset{\circlearrowleft \sigma}{\tilde{C}} \xrightarrow[\pi]{d:1} C \mid \pi \text{ étale cyclic}, \ C \in \mathcal{M}_g \}$$

Faber's case.

type of pol.

$$\mathcal{P}_4(3) : \mathcal{R}_4(3) \longrightarrow \mathcal{B}_6 \subset \mathcal{A}_6^{(1\,1\,1\,3\,3\,3)}$$

$$[\underset{g=10}{\tilde{C}} \to \underset{g=4}{C}] \longmapsto (\underset{\dim 6}{P}, \Xi)$$

$$\mathcal{B}_6 = \{ (P,\Xi) \in \mathcal{A}_6^{(111333)} \mid \exists \sigma \in \mathrm{Aut}(P),\ \sigma^3 = id,\ \sigma^*\Xi \equiv \Xi \}.$$

Thm (Faber) $\mathcal{P}_4(3) : \mathcal{R}_4(3) \longrightarrow \mathcal{B}_6$ is generically finite of deg 16

dim 9

Computation on the fiber over: $X \in \mathcal{M}_3$ generic

$$JX \times JX \simeq P = \mathrm{Ker}(\overset{\circlearrowleft S_3}{JX \times JX \times JX} \xrightarrow{\text{sum}} JX)$$

not as polarized

principal

The restricted pol of the prin. on $(JX)^3$ is of type $(1\,1\,1\,3\,3\,3)$

$\rightsquigarrow$ $\exists \sigma \in \mathrm{Aut}\, P$, $\sigma^3 = id$

The structure of the fiber is the $(16_6)$-Kummer structure.

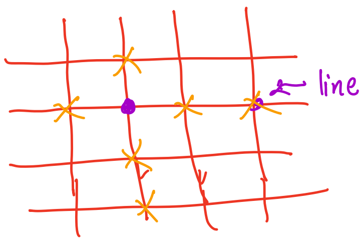
$A \in \mathcal{A}_2$

$S$ $\leftsquigarrow$ 16 lines

desingulari

$A/\langle \pm 1 \rangle$

16 $\rightsquigarrow$ nodes




(del Pezzo of deg 4.)
surface.

Symmetry group of $(16_6)$-str. is $(S_4 \times S_4) \rtimes (\mathbb{Z}/2\mathbb{Z})$

In gral

$$\begin{tikzcd} \tilde{C} \arrow[loop right, "\sigma"] \arrow[d, "\pi"', "d:1"] \\ C \end{tikzcd} \qquad \sigma^d = id$$

cyclic

$\deg(\pi) = d$ $\quad 2r \geq 0$ deg of ram. locus. (totally ramified).

$$\mathcal{B}_D = \left\{ (P, \Xi) \in \mathcal{A}^{\delta}_{\dim P} \;\middle|\; \exists\, \sigma \in Aut(P) \quad \sigma^d = id,\ \sigma^* \Xi \equiv \Xi \right\}$$

When is $\dim \mathcal{R}_{g,r}(d) = \dim \mathcal{B}_D$ ?

$\mathcal{R}_{g,r}(d) = \{ [\tilde{C} \xrightarrow[d:1]{\pi} C] \mid$ cyclic. deg ram. $2r \}$

| | $(g,d,r)$ | $\deg P_{g,r}(d)$ | $\dim P$ | Polarization. | deg = 9 − N (# P to blow up) |
|---|---|---|---|---|---|
| (Marcucci-Naranjo) | (1,2,3) | 1 | 3 | (1, 1, 2) | 8 |
| Nagaraj-Ramanan | (3, 2, 2) | 3 | 4 | (1, 2, 2, 2) | 7 |
| Bardelli-Ciliberto-Verra | (2, 3, 3) | ? $\varphi$ | 5 | (1 1 1 3 3) | 6 |
| Lange.-O. | (2, 7, 0) | 10 | 6 | (1 1 1 1 1 7) | 5 |
| Faber | (4, 3, 0) | 16 | 6 | (1 1 1 3 3 3) | 4 |
| Donagi-Smith | (6, 2, 0) | 27 | 5 | (2 2 2 2 2 2) | 3 |

(cyclic cases)

Ram. degree $= 2r$