COMBINATORICS OF SQUARE-TILED SURFACES AND GEOMETRY OF MODULI SPACES

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Abstract. This text corresponds to the lecture notes of a minicourse delivered from August 16 to 20, 2021 at the IMPA–ICTP online summer school "Aritmética, Grupos y Análisis (AGRA) IV". In particular, we discuss the same topics from our minicourse, namely, the basic theory of origamis and its connections to the calculation of Masur–Veech volumes of moduli spaces of translation surfaces.

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1. Square-tiled surfaces, Veech groups and arithmetic Teichmüller curves

1.1. **Basic definitions and some examples.** A *square-tiled surface* or *origami* is the surface determined by a pair of permutations $h, v \in S_n$ via the following construction: we take n copies $\mathrm{Sq}(j)$, $j = 1, \dots, n$, of the unit square $[0,1] \times [0,1]$, and we glue by *translations* the right, resp. top side of $Sq(j)$ with the left, resp. bottom side of $Sq(h(j))$, resp. $Sq(v(j))$.

Remark 1. The resulting square-tiled surface is *connected* if and only if the subgroup of S_n generated by h and v acts *transitively* on $\{1, \dots, n\}$.

Since we are mostly interested in origamis, we shall pay little attention to the particular way of labelling its squares by declaring that (h, v) is *equivalent* to $(\sigma h \sigma^{-1}, \sigma v \sigma^{-1})$.

Example 2. *The trivial permutations $h = (1) = v$ generate the square torus $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$, while the L-shaped origami in Figure 1 is obtained from the permutations $h = (1)(2,3)$ and $v = (1,2)(3)$.*

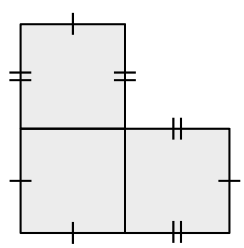

Figure 1. L-shaped origami

Example 3. *A finite group G generated by two elements r and u (e.g., $G = A_n$, S_n, $SL(2, \mathbb{F}_p)$, etc.) provides an origami because r and u act on G via the permutations $g \mapsto g \cdot r$ and $g \mapsto g \cdot u$.*

In particular, the quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ generates a famous origami called Eierlegende Wollmilchsau (cf. Figure 2).

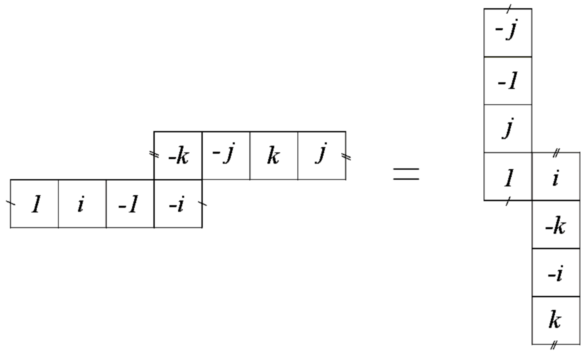

FIGURE 2. Eierlegende Wollmilchsau.

By definition, an origami is a finite cover $\pi : X \to \mathbb{T}^2 := \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ branched only at the origin $0 \in \mathbb{T}^2$. The ramification points of the origami are *conical singularities*. The total angle around a conical singularity is a multiple of 2π and it can be computed in combinatorial terms via the non-trivial cycles of the commutator $[h, v] = vhv^{-1}h^{-1}$:

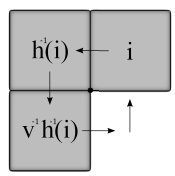

FIGURE 3. Turning by 2π around a corner.

In other terms, a non-trivial cycle c of $[h, v]$ is responsible for a conical singularity with total angle $2\pi \cdot$ length of c.

Therefore, the topology of the origami is determined by $[h, v]$: in fact, if $[h, v]$ has non-trivial cycles of lengths $k_1 + 1, \ldots, k_{\sigma+1}$, we can triangulate the origami by adding diagonals to each square to obtain $2n$ faces, $3n$ edges and $n - \sum_{j=1}^{\sigma} k_j$ vertices; by the Euler–Poincaré formula, the genus g of the origami satisfies

$$2 - 2g = 2n - 3n + \left(n - \sum_{j=1}^{\sigma} k_j\right) = -\sum_{j=1}^{\sigma} k_j.$$

The details of the derivations of these facts are left to the reader (cf. Exercise 7).

Definition 4. We say that an origami $\mathcal{O}$ belongs to a *stratum*[1] $\mathcal{H}(k_1, \ldots, k_\sigma)$ whenever the total angles of its conical singularities are $2\pi(k_j + 1)$, $j = 1, \ldots, \sigma$.

[1]Strictly speaking, one has to be a little bit careful here because a given origami might belong to multiple strata if its conical singularities are not numbered. For the time being, we will slightly abuse notation by keeping the labelling of conical singularities always implicit, and we postpone a serious discussion of this point to Section 3.

Remark 5. The nomenclature "stratum" will become clear later in Section 3.2. For now, let us just mention that origamis will play the role of *integral points* of strata, so that we will be able to compute volumes of certain moduli spaces by counting origamis (similarly to Gauss' idea of relating volumes of Euclidean balls to counting problems about integral vectors).

In the sequel, we will often avoid using "superfluous" squares by assuming that our origamis are *reduced*, i.e., the finite branched cover $\pi : X \to \mathbb{T}^2$ defining the origami doesn't factor through $\mathbb{C}/(n\mathbb{Z}\oplus im\mathbb{Z}) \to \mathbb{T}^2$ with $n\cdot m > 1$. Equivalently, the *period lattice* $\mathrm{Per}(\omega)$ spanned by $\int_\gamma \omega$, where $\omega = \pi^*(dz)$ and γ are paths joining conical singularities, of the origami coincides with $\mathbb{Z}\oplus i\mathbb{Z}$.

1.2. **Action of** $\mathrm{SL}(2,\mathbb{Z})$ **and Veech groups.** The group $\mathrm{SL}(2,\mathbb{Z})$ is generated by the parabolic matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(cf. Exercise 8 below).

Since T and S stabilize $\mathbb{T}^2$ (cf. Figure 4), it is not hard to check that the natural action

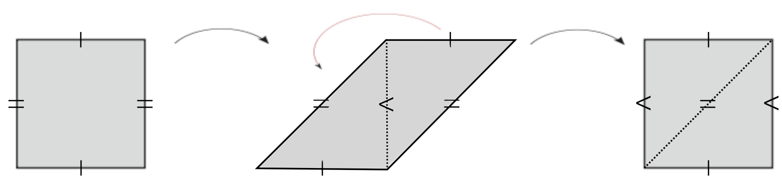

FIGURE 4. Cutting and pasting after shearing by T.

of $\mathrm{SL}(2,\mathbb{Z})$ transforms origamis into origamis. From the combinatorial point of view, $T(h,v) = (h, vh^{-1})$ and $S(h,v) = (hv^{-1}, v)$ (cf. Figure 5).

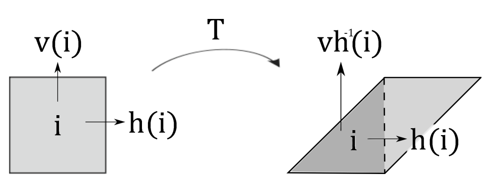

FIGURE 5. Action of T on pairs of permutations.

In combinatorial group theory, the maps $(a,b) \mapsto (a, ba^{-1})$, $(a,b) \mapsto (ab^{-1}, b)$ are called *Nielsen transformations* because they played a prominent role in Nielsen's characterisation of pairs of generators of the free group F_2 on two generators.

Note that T and S preserve $[h,v]$. Hence, the natural action of $\mathrm{SL}(2,\mathbb{Z})$ permutes origamis in each stratum $\mathcal{H}(k_1,\ldots,k_\sigma)$.

The *Veech group* of a reduced origami (h, v) is its stabiliser in $\mathrm{SL}(2,\mathbb{Z})$. Note that any Veech group has finite index in $\mathrm{SL}(2,\mathbb{Z})$: for example, the Veech group of $\mathbb{T}^2$ is $\mathrm{SL}(2,\mathbb{Z})$ and the Veech group of the L-shaped origami $h = (1,2)(3)$, $v = (1,3)(2)$ has index 3 in $\mathrm{SL}(2,\mathbb{Z})$ (cf. Exercise 6).

1.3. **Arithmetic Teichmüller curves.** As we will see in details later, an origami is a particular example of *translation surface*, i.e., a surface obtained from a finite collection of polygons after gluing pairs of parallel sides by translations. As it turns out, a translation surface possesses a finite number of conical singularities whose total angles are multiples of 2π, and, hence, they can be organised into a stratum $\mathcal{H}(k_1, \dots, k_\sigma)$ (of the moduli space of translation surfaces). These strata are complex orbifolds (with at most 3 connected components) carrying a natural $\mathrm{SL}(2,\mathbb{R})$-action coming from the fact that the linear action of $A \in SL(2,\mathbb{R})$ maps a pair $v, v+a \in \mathbb{R}^2$ onto $A(v), A(v)+A(a) \in \mathbb{R}^2$.

In this context, the $\mathrm{SL}(2,\mathbb{R})$-orbit of an origami $\mathcal{O}$ is *closed* and it is isomorphic to the unit tangent bundle of a hyperbolic surface, namely, $\mathrm{SL}(\mathcal{O})\backslash \mathrm{SL}(2,\mathbb{R}) \simeq \mathrm{SL}(\mathcal{O})\backslash T^1\mathbb{H}$, where $\mathrm{SL}(\mathcal{O}) \subset \mathrm{SL}(2,\mathbb{Z})$ is the Veech group of $\mathcal{O}$.

The closed $\mathrm{SL}(2,\mathbb{R})$-orbits generated by origamis are called *arithmetic Teichmüller curves*, while other closed $\mathrm{SL}(2,\mathbb{R})$-orbits in strata are called *non-arithmetic Teichmüller curves* (because a theorem of Smillie asserts that they are isomorphic to $SL(2,\mathbb{R})/G$ where G is a lattice of $SL(2,\mathbb{R})$ which is *not* commensurable to $SL(2,\mathbb{Z})$).

The flat torus $\mathbb{T}^2$ has a $SL(2,\mathbb{R})$-orbit isomorphic to the unit tangent bundle of the *modular curve* $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$. In general, since any Veech group is a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$, every arithmetic Teichmüller curve is a finite cover of the modular curve.

Combinatorially, we can code an arithmetic Teichmüller curve generated by an origami $\mathcal{O}$ via the *graph* with vertices

$$\{\mathcal{O} = \mathcal{O}_1, \dots, \mathcal{O}_m\} = \mathrm{SL}(2,\mathbb{Z}) \cdot \mathcal{O}$$

and edges connecting $\mathcal{O}_k$ to $\mathcal{O}_l$ whenever $\mathcal{O}_k = T^{\pm 1}(\mathcal{O}_l)$ or $\mathcal{O}_k = S^{\pm 1}(\mathcal{O}_l)$ (for T and S generating $\mathrm{SL}(2,\mathbb{Z})$). In fact, this graph describes (in a certain sense) the "adjacencies" between the tiles of the tilling of $\mathbb{H}/\mathrm{SL}(\mathcal{O})$ obtained by lifting the usual fundamental domain

$$\left\{x + iy \in \mathbb{H} : |x| < \frac{1}{2}, x^2 + y^2 > 1\right\}$$

of the modular curve $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$.

1.4. **Exercises.**

Exercise 6. *Let us consider the two following origamis*

$$o_1 = (h_1, v_1) = ((1)(2,3),\ (1,2)(3)),$$
$$o_2 = (h_2, v_2) = ((1,2,3,4)(5,6,7,8),\ (1,5,3,7)(2,8,4,6)).$$

For each of them

(1) draw the flat representation of these origamis;

(2) *compute the angles of the conical singularities of the associated flat surface* $\mathcal{H}(k_1, \ldots, k_\sigma)$*;*
(3) *compute the orbit of* $\mathrm{SL}(2,\mathbb{Z})$ *on them.*
(4) *Compute the index of the Veech group of* o_1 *and* o_2 *in* $SL_2(\mathbb{Z})$*.*

Exercise 7. *Given an origami* $o = (h, v)$*.*

(1) *Show that the singularities of* o *are in bijection with the cycles of the commutator* $[h, v] = vhv^{-1}h^{-1}$*.*
(2) *Let* $k_i + 1$ *be the lengths of these cycles and* g *the genus of the origami* o*. Show that*

$$2g - 2 = \sum k_i$$

(3) *Compute the* k_i *and the genus* g *of* o_1 *and* o_2 *from Exercise 6.*

Exercise 8. *Recall that* $PSL_2(\mathbb{R})$ *acts on the upper plane* $\mathbb{R}$ *by homography and denote by* G *the subgroup of* $PSL_2(\mathbb{Z})$ *generated by*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(1) *Show that*

$$\{z : |z| \geq 1, -1/2 \leq Re(z) \leq 1/2\} \subset \mathbb{H}$$

is a fundamental region for G*.*
(2) *Show that* $G = PSL_2(\mathbb{Z})$*.*

2. Some properties of Veech groups

2.1. **Characteristic origamis.** Recall that an origami $\mathcal{O}$ is a finite cover of the flat torus $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ which is not ramified outside the origin $0 \in \mathbb{T}^2$. In particular, $\mathcal{O}$ determines a finite-index subgroup H of $\pi_1(\mathbb{T}^2 \setminus \{0\}) \simeq F_2$.

We say that $\mathcal{O}$ is a *characteristic origami* whenever the Galois group H is a characteristic subgroup of F_2, i.e., $\varphi(H) = H$ for all $\varphi \in \mathrm{Aut}(F_2)$.

The Eierlegende Wollmilchsau is a characteristic origami: cf. Exercise 14. In general, any origami is covered by a characteristic origami: indeed, $H_{char} := \bigcap_{\varphi\in\mathrm{Aut}(F_2)} \varphi(H)$ is a finite-index[2] characteristic subgroup of F_2 describing a characteristic origami $\mathcal{O}_{char}$ covering $\mathcal{O}$.

Example 9. *The L-shaped origami $h = (1,2)(3)$, $v = (1,3)(2)$ is covered by a characteristic origami with 108 squares.*

In 1917, Nielsen showed that $\mathrm{Aut}(F_2)$ belongs to a short exact sequence

$$\{1\} \to \mathrm{Inn}(F_2) \to \mathrm{Aut}(F_2) \overset{\Phi}{\to} \mathrm{GL}(2,\mathbb{Z}) \to \{1\},$$

where $\mathrm{Inn}(F_2)$ are the *inner* automorphisms of $F_2 = \langle x, y\rangle$ and the map $\Phi : \mathrm{Aut}(F_2) \to \mathrm{GL}(2,\mathbb{Z})$ is

$$\Phi(\varphi) = \begin{pmatrix} \text{sum of exponents of } x \text{ in } \varphi(x) & \text{sum of exponents of } x \text{ in } \varphi(y) \\ \text{sum of exponents of } y \text{ in } \varphi(x) & \text{sum of exponents of } y \text{ in } \varphi(y) \end{pmatrix}$$

Remark 10. This result is a particular case of the Dehn–Nielsen–Baer theorem relating the outer automorphisms of fundamental groups of surfaces and the mapping class groups (of isotopy classes of homeomorphisms) of surfaces.

Schmithüsen proved in 2004 that the Veech group of an origami $\mathcal{O}$ is

$$\Phi(\{\varphi \in \mathrm{Aut}(F_2) : \varphi(H) = H\}) \cap \mathrm{SL}(2,\mathbb{Z}).$$

An immediate consequence of this result is the fact that any characteristic origami has Veech group equal to $\mathrm{SL}(2,\mathbb{Z})$. Moreover, Schmithüsen was able to use her theorem to obtain an *algorithm* for the computation of Veech groups. In particular, she derived that the origamis $\mathcal{O}_{2k}$ with $h = (1,2,\dots,2k)$ and $v = (1,2)(3,4)\dots(2k-1,2k)$ have Veech groups

$$\mathrm{SL}(\mathcal{O}_{2k}) \supset \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : a \equiv d \equiv 1,\ b \equiv c \equiv 0 \,(\mathrm{mod}\ 2k)\right\}.$$

[2]Because $\varphi(H)$ has the same index of H and F_2 has only finitely many subgroups of a given index (as F_2 is finitely generated).

2.2. **Origamis with prescribed Veech groups.** All concrete examples of Veech groups we met so far were *congruence subgroups* of $\mathrm{SL}(2,\mathbb{Z})$, that is, they contained

$$\Gamma(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : a \equiv d \equiv 1,\, b \equiv c \equiv 0 \,(\mathrm{mod}\ n) \right\}$$

for some $n \in \mathbb{N}$.

This observation leads us to the important open problem (going back to Thurston) of listing all finite-index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ which are the Veech groups of some origamis. An important partial progress towards this question was obtained in 2012:

Theorem 11 (Ellenberg–McReynolds)**.** *Any finite-index subgroup Γ of $\Gamma(2)$ containing $\{\pm Id\}$ is the Veech group of some origami.*

The basic idea behind the proof of this theorem is to explore the isomorphism $\mathbb{H}/\Gamma(2) \simeq \overline{\mathbb{C}} \setminus \{0,1,\infty\}$ and the fact that $\Gamma(2)$ is the group of affine homeomorphisms of the elliptic curve $E(2) = \mathbb{C}/2(\mathbb{Z} \oplus i\mathbb{Z})$ fixing its 2-torsion points $p = (0,0)$, $q = (1,0)$, $s = (0,1)$ in order to make a *fiber product* of the ramified coverings

$$E(2) \to E(2)/\{\pm \mathrm{Id}\} \quad \text{and} \quad \mathbb{H}/\Gamma \to \mathbb{H}/\Gamma(2)$$

with the purpose of getting an origami $\pi : Y \to E(2)$ such that

- π is ramified *exactly* at p, q, s,
- the degree of π is $[\Gamma(2) : \Gamma]$,
- all elements of $\Gamma(2)$ lift to affine homeomorphisms of Y, and
- the fiber $\pi^{-1}(r)$ of $r = (1,1)$ is "naturally isomorphic" to $\Gamma(2)/\Gamma$.

Next, one builds a finite cover $\mathcal{O} \to Y$ so that $\mathcal{O} \to Y \to E(2)$ ramifies in distinct ways above p, q, s, r, and $\mathcal{O} \to Y$ ramifies at $\mathrm{id} \cdot \Gamma(2) \in \Gamma(2)/\Gamma \simeq \pi^{-1}(r)$ differently from the other points of $\pi^{-1}(r)$. Finally, one uses these properties to check that $\mathcal{O}$ has Veech group Γ.

An interesting consequence of Ellenberg–McReynolds theorem is the existence of *non-congruence* Veech groups. In fact, Selberg famously proved that the first eigenvalue of the Laplacian of $\mathbb{H}/G$ is $\geq 3/16$ when G is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. On the other hand, we can produce cyclic covers $\mathbb{H}/\Gamma$ of the hyperbolic surface $\mathbb{H}/\Gamma(6)$ (with genus one and 12 cusps) whose Laplacians have arbitrarily small first eigenvalues because of the so-called *Cheeger–Buser inequality*. In this situation, we can employ Ellenberg–McReynolds' theorem to produce an origami $\mathcal{O}$ whose Veech group $\Gamma \subset \Gamma(6) \subset \Gamma(2)$ is not a congruence subgroup.

2.3. **Non-congruence Veech groups within $\mathcal{H}(2)$.** The non-congruence Veech groups are also present among origamis in $\mathcal{H}(2)$ with $n \geq 4$ squares.

Theorem 12 (Hubert–Lelièvre (2005))**.** *Each (reduced) origami $\mathcal{O} \in \mathcal{H}(2)$ with $n \geq 4$ has a non-congruence Veech group.*

Remark 13. If G is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and ℓ is the smallest integer with $\Gamma(\ell) \subset G$, then the reduction of G modulo ℓ has index $[\mathrm{SL}(2,\mathbb{Z}) : G]$. Since the indices of Veech groups of origamis in $\mathcal{H}(2)$ increase when their number of squares grow, the fact proved by Schmithüsen in 2015 that the reductions modulo m of these Veech groups have indices 1 or 3 for any $m \geq 2$ is a beautiful improvement of Hubert–Lelièvre's theorem.

In the sequel, we will sketch the proof of Hubert–Lelièvre's theorem when $n \geq 4$ is even and $n-2$ is not a power of 2.

As we are going to see later, the reduced origamis in $\mathcal{H}(2)$ with an even number $n \geq 4$ of squares belong to a *single* $\mathrm{SL}(2,\mathbb{Z})$-orbit. In particular, their Veech groups Γ_n are mutually conjugated and, hence, they contain the conjugates of the matrices

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, k = 1, \ldots, n$$

because this $\mathrm{SL}(2,\mathbb{Z})$-orbit contains the origamis below:

[ADD FIGURE LATER]

By a lemma of Wohlfahrt, if Γ_n were congruence, then the smallest ℓ with $\Gamma(\ell) \subset \Gamma_n$ would be the least common multiple of the lengths of the cusps of Γ_n, that is,

$$\ell = \mathrm{lcm}(1, \ldots, n).$$

Next, we note that if m is a divisor of ℓ such that the reduction of Γ_n is $\mathrm{SL}(2,\mathbb{Z}/m\mathbb{Z})$, then

$$d_n := [\mathrm{SL}(2,\mathbb{Z}) : \Gamma_n] = [\Gamma(m) : \Gamma_n \cap \Gamma(m)].$$

Hence, if $\Gamma(\ell) \subset \Gamma_n \cap \Gamma(m)$, then d_n must divide $[\Gamma(m) : \Gamma(\ell)]$. Thus, we will reach a *contradiction* if there is a divisor m of ℓ such that the reduction of Γ_n modulo m is $\mathrm{SL}(2,\mathbb{Z}/m\mathbb{Z})$ *but* d_n does not divide $[\Gamma(m) : \Gamma(\ell)]$.

In this direction, let m be the largest divisor of ℓ which is coprime to n, that is,

$$m = \prod_{p \nmid n} p^{\lambda_p} = \ell / \prod_{p|n} p^{\lambda_p}$$

where $\ell := \prod p^{\lambda_p}$ and $n := \prod p^{\nu_p}$.

Since Γ_n contains *both* matrices

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

thanks to the origami below

[ADD FIGURE LATER]

we get that the reduction of Γ_n modulo m is $\mathrm{SL}(2,\mathbb{Z}/m\mathbb{Z})$.

Thus, our task is reduced to check that d_n does not divide $[\Gamma(m):\Gamma(\ell)]$. For this sake, we recall that it is known that

$$d_n = \frac{3}{8}(n-2)n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right) = \frac{3}{8}(n-2)\prod_{p|n}p^{2\nu_p-2}(p^2-1)$$

and

$$[\Gamma(m):\Gamma(\ell)] = \frac{\ell^3\prod_{p|\ell}\left(1-\frac{1}{p^2}\right)}{m^3\prod_{p|m}\left(1-\frac{1}{p^2}\right)} = \prod_{p|n}p^{3\lambda_p-2}(p^2-1),$$

so that if d_n divides $[\Gamma(m):\Gamma(\ell)]$, then $3(n-2)$ divides $8\prod_{p|n}p^{3\lambda_p-2\nu_p}$. However, this is *not* possible because $\gcd(n,n-2)=2$ and $n-2$ is not a power of 2 (so that $n-2$ is divided by an odd prime which does not divide n).

2.4. **Exercises.**

Exercise 14. *For each origami o_i from exercise 6 investigate if it is characteristic or not.*

Exercise 15. *In this exercise we denote by Γ the group $\Gamma(n)$ for some $n \geq 2$. Recall that a cusp (resp. an elliptic point) $z \in \mathbb{R}\cup\{\infty\}$ (resp. $z \in \mathbb{H}$) for Γ is the unique fix point of some $\gamma \in \Gamma$.*

1. *Show that there is no elliptic points for Γ.*
2. *Show that ∞ is a cup of Γ and its fixator is $\left\{\begin{pmatrix}1 & kn\\ 0 & 1\end{pmatrix} : k \in \mathbb{Z}\right\}$.*
3. *Compute the cardinal of $SL_2(\mathbb{Z}/n\mathbb{Z})$. In this count, the functions $\phi(n) = n\prod_{p|n}(1-\frac{1}{p})$ and $\psi(n) = n\prod_{p|n}(1+\frac{1}{p})$ can be used.*

Exercise 16. *Consider the origami o_3 given by the pair of permutations $(h_3, v_3) = ((1)(2,3,4),\ (1,2)(3)(4))$.*

1. *Show that the matrices*
$$\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix} \quad , \quad \begin{pmatrix}-1 & 3\\ -2 & 5\end{pmatrix} \quad , \quad \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix}3 & -5\\ 2 & -3\end{pmatrix}$$
are element of the Veech group Γ of o_3. In fact it can be shown that it is a basis of the Veech group.
2. *The group Γ has three cusps whose fixators are generated by T^3, ST^2S^-1 and $TST^4S^{-1}T^{-1}$. Show that if γ contains a $\Gamma(\ell)$, then it contains $\Gamma(12)$.*
3. *Let $p\colon PSL_2(\mathbb{Z}) \to PSL_2(\mathbb{Z}/3\mathbb{Z})$ be the natural projection. Show that $p(\overline{\Gamma})$ is equal to $PSL_2(\mathbb{Z}/3\mathbb{Z})$.*
4. *Let $N = \Gamma \cap \Gamma(3)$. Deduce from the fact that $[PSL_2(\mathbb{Z}):\overline{\Gamma}] = 9$ that $[\overline{\Gamma(3)}:N] = 9$.*
5. *Deduce from the fact that $[\overline{\Gamma(3)}:\overline{\Gamma}(12)] = 2^43$ that Γ is not a congruence subgroup.*

3. Translation surfaces, moduli spaces and Masur–Veech volumes

A flat torus is a quotient of $\mathbb{C}$ by a lattice Λ (a $\mathbb{Z}$-module of rank 2). We already encountered the square torus $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$. We recall that the moduli space of flat tori is identified with $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$. In this section, we introduce translation surfaces that generalize tori and origamis. Then we introduce their moduli spaces that are the projectivized strata $\mathbb{P}\mathcal{H}(k_1,\ldots,k_\sigma)$ which is, as the modular curve, a complex orbifold endowed with an $\mathrm{SL}(2,\mathbb{R})$-action.

3.1. Translation surfaces and $\mathrm{GL}(2,\mathbb{R})$-action.

Square-tiled surfaces are particular cases of translation surfaces that we define now.

Definition 17. An *(constructive) translation surface* is a compact surface S built from the following procedure. Pick a finite collection of Euclidean polygons P_1, P_2, ..., P_m in $\mathbb{R}^2$ and a pairing f of the edges of the polygons such that paired sides are parallel with opposite normal vectors. For each edge e there is a unique translation τ_e such that $\tau_e(e) = f(e)$. The surface S is the union of the polygons quotiented by the relation $x \sim \tau_e(x)$ for the points x on the edge e.

Definition 18. A *(geometric) translation surface* is a compact surface S and a finite $\Sigma \subset S$ and a translation structure defined on $S \setminus \Sigma$.

Definition 19. A *(analytic) translation surface* is a Riemann surface X, a finite set $\Sigma \subset X$ and an Abelian differential ω on X which is nowhere zero on $X \setminus \Sigma$.

Exercise 20. *Figure out the equivalence between the definitions.*

The singularity type of a translation surface is the tuple $\kappa = (k_1,\ldots,k_\sigma)$ such that at the point x_i we have an angle $2\pi(k_i+1)$.

Let M be the translation surface obtained from the polygons P_1, ..., P_m and side pairing σ. Let A be a matrix in $\mathrm{GL}(2,\mathbb{R})$. The translation surface $A\cdot M$ is the one obtained from the polygons $A\cdot P_1$, ..., $A\cdot P_m$ and where the side pairings is identical.

Exercise 21.
(1) Why is $A\cdot M$ a translation surface?
(2) What is its area in terms of the area of M ?

3.2. Strata of translation surfaces.

The "set of isomorphism classes" of translation surfaces can be turned into a geometric object called a *moduli space*. We define precisely this object in this section.

Theorem 22. *Each stratum $\mathcal{H}(\mu)$ is a complex orbifold with a piecewise integral linear structure.*

Proof. (Sketch) Let S be a fixed topological surface of genus g where $2g-2 = \mu_1 + \ldots + \mu_\sigma$ and p_1, ..., p_σ distinct points in S. Each element in $\mathcal{H}(\mu)$ can be represented as a pair (X,ω) where X is a complex structure on S and ω is a one-form, holomorphic for X with zeros of order μ_i at

p_i and non-vanishing elsewhere. We can then consider the period map

$$\begin{array}{lcl} \mathcal{H}(\mu) & \to & H^1(S, \{p_1, \ldots, p_\sigma\}; \mathbb{C}) \\ (X, p_1, \ldots, p_\sigma, \omega) \mapsto [\omega] & & \end{array}$$

This map is locally injective (if $(X, p_1, \ldots, p_\sigma, \omega)$ admits an automorphism one needs to quotient the right hand side by $\mathrm{Aut}(X, p_1, \ldots, p_\sigma, \omega)$). □

From the "constructive" perspective of Definition 17 the period map defined in the proof above just associates to a polygon the vectors used for the sides.

3.3. **Masur-Veech volume and enumeration of square-tiled surfaces.** Each stratum $\mathcal{H}(\mu)$ is endowed with a canonical volume form called the *Masur-Veech volume form*. The projectivized stratum admits a natural normalization whose total mass is finite and whose value is directly related to the asymptotic enumeration of square-tiled surfaces.

Recall from Section 3.2 that the period map provides locally injective charts. The target $H^1(S, \{p_1, \ldots, p_\sigma\}; \mathbb{C})$ is a vector space of complex dimension $2g + \sigma - 1$. A vector space has a natural measure class, the *Lebesgue class*, which is the set of measures invariant by translations. It is a one-dimensional ray $\mathbb{R}\mathrm{Leb}$. Now, $H^1(S, \{p_1, \ldots, p_\sigma\}; \mathbb{C})$ admits a natural lattice $H^1(S, \{p_1, \ldots, p_\sigma\}; \mathbb{C})$, one can then pick Leb so that this lattice has covolume one.

Theorem 23. *We have*

$$\mathrm{Vol}(\mathbb{P}\mathcal{H}(\kappa)) = 2\,d \cdot \lim_{N \to \infty} \frac{1}{N^d} \# \mathcal{ST}_{\leq N}(\kappa).$$

where $d = \dim_{\mathbb{C}} \mathcal{H}\kappa)$ *and each square tiled surface is counted with a weight* $\frac{1}{\mathrm{Aut}(X,\omega)}$.

The proof will follow from the following equivalent definition of square-tiled surfaces.

Lemma 24. *A translation surface* (X, ω) *is square-tiled if and only if its image under the period map is an integral vector.*

Exercise 25. *Make a proof of Lemma 24.*

Proof of Theorem 23. Riemann integral : the number of rational points with denominator at most N in an open set U in $\mathbb{R}^d$ is proportional to the Lebesgue volume of U times N^d. □

3.4. **Reduced versus non-reduced origamis.** We explain how to switch between the count of reduced and non-reduced origamis. By mean of the Möbius formula one can pass freely from the count of reduced origamis to the count of non-reduced origamis.

Theorem 26. *Let* $\mathcal{H}(\mu)$ *be a stratum of translation surfaces and let* a_N *and* a'_N *be respectively the weighted count of all and reduced square-tiled surfaces in* $\mathcal{H}(\mu)$*. Then*

$$a_N = \sum_{d|N} \sigma\left(\frac{N}{d}\right)\ a'_d.$$

Hence

$$a'_N = \sum_{d|N} \sigma'\left(\frac{N}{d}\right)\ a_d$$

where σ' is the Dirichlet inverse of σ

N	1	2	3	4	5	6	7	8	9
$\sigma(N)$	1	3	4	7	6	12	8	15	13
$\sigma'(N)$	1	-3	-4	2	-6	12	-8	0	3

Proof. (Sketch) $\sigma_1(N)$ counts tori. □

3.5. **The Masur-Veech volume of $\mathcal{H}(2)$.** Here we show how to enumerate the origamis in the stratum $\mathcal{H}(2)$ and deduce its Masur-Veech volume. The computations follow the techniques of Zorich 2002.

The main result of this section is

Theorem 27. *The Masur-Veech volume of $\mathcal{H}(2)$ is $\frac{1}{120}\pi^4$.*

The way we prove this theorem is by enumerating square-tiled surfaces. To state the counting theorem we first need to introduce some functions. For $d \geq 0$, the *sum of divisors function* is

$$\sigma_d(N) := \sum_{k|N} k^d.$$

The function σ_0 simply counts the divisor of N. Let the associated generating series be

$$\tilde{E}_k(q) := \sum_{\ell>0} \ell^{k-1} \frac{q^k}{1-q^k} = \sum_{n>0} \sigma_{k-1}(n) q^n.$$

For even indices, $\tilde{E}_{2k}(q)$ coincide up to the constant term and the multiplicative factor with the so-called *normalized Eisenstein series* of weight $2k$

$$E_{2k}(q) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n.$$

The functions E_2, E_4 and E_6 are multiplicative generators of the quasi-modular forms for $\mathrm{SL}(2,\mathbb{Z})$ that will appear again in the next section.

Theorem 28. *The generating series of square tiled surfaces in $\mathcal{H}(2)$ is*

$$\frac{1}{4}\left(6\tilde{E}_2^2 - \tilde{E}_4(q) + \tilde{E}_2(q)\right) = 3\,q^3 + 9\,q^4 + 27\,q^5 + 45\,q^6 + O(q^7).$$

In particular up to the constant term it is a non-homogeneous quasimodular form of maximal weight four.

Proof. We decompose the square-tiled surfaces depending whether its horizontal direction is made of one or two cylinders, see Figure 6.

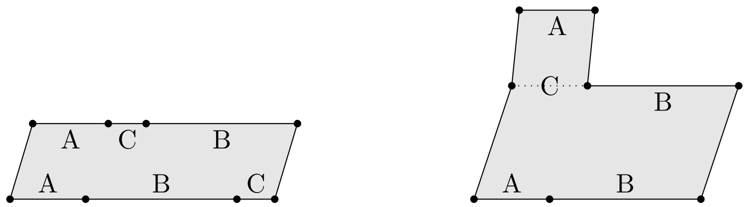

FIGURE 6. The two cylinder diagrams in $\mathcal{H}(2)$.

One cylinder square-tiled surfaces with N squares in $\mathcal{H}(2)$ are in bijection with the 5-tuples

$$\left\{ (\ell_A, \ell_B, \ell_C, h, t) \in \mathbb{Z}^5_{\geq 0} : \begin{array}{l} \ell_A > 0,\, \ell_B > 0,\, \ell_C > 0 \\ (\ell_A + \ell_B + \ell_C) \cdot h = N \\ 0 \leq t < \ell_A + \ell_B + \ell_C \end{array} \right\}.$$

Their number is equal to

$$a_N := \frac{1}{3} \cdot \sum_{\ell | N} \ell \cdot \cdot \binom{\ell - 1}{2} = \frac{1}{6}\sigma_3(N) - \frac{1}{2}\sigma_2(N) + \frac{1}{3}\sigma_1(N).$$

(the factor $\frac{1}{3}$ accounts for the symmetry of the diagram). This can be equivalently expressed on the generating series

$$A(q) := \sum_{n \geq 1} a_N q^N = \frac{1}{6}\left(\tilde{E}_4(q) - 3\tilde{E}_3(q) + 2\tilde{E}_2(q)\right).$$

Now, two-cylinders square-tiled surfaces in $\mathcal{H}(2)$ are in bijection with

$$\left\{ (\ell_A, \ell_B, \ell_C, h_1, h_2, t_1, t_2) \in \mathbb{Z}^7_{\geq 0} : \begin{array}{l} \ell_A > 0,\, \ell_B > 0,\, \ell_C > 0 \\ \ell_A = \ell_C \\ (\ell_A + \ell_B) \cdot h_1 + \ell_C \cdot h_2 = N \\ 0 \leq t_1 < \ell_A + \ell_B \\ 0 \leq t_2 < \ell_C \end{array} \right\}$$

Their number is

$$\begin{aligned} b_n &:= \sum_{\substack{\ell_1, \ell_2, h_1, h_2 \\ h_1 > 0, h_2 > 0 \\ \ell_2 > \ell_1 > 0 \\ h_1 \ell_1 + h_2 \ell_2 = N}} \ell_1 \cdot \ell_2 \\ &= \frac{1}{2} \sum_{\substack{\ell_1, \ell_2, h_1, h_2 \\ h_1 > 0, h_2 > 0 \\ h_1 \ell_1 + h_2 \ell_2 = N}} \ell_1 \cdot \ell_2 - \sum_{\substack{\ell_1, h_1, h_2 \\ h_1, h_2 > 0 \\ (h_1 + h_2)(\ell_1) = N}} \ell_1^2. \end{aligned}$$

We will manipulate the generating series $B(q) := \sum_{n>0} b_n q^n$ and write it in terms of the $\tilde{E}_k$. We have

$$B = \frac{1}{2}\left(\tilde{E}_2^2 - q\frac{d}{dq}\tilde{E}_2 + \tilde{E}_3\right).$$

Now we use

Proposition 29 (Ramanujan identity)**.** *Let $D = q\frac{d}{dq}$. We have*

$$D\tilde{E}_2 = -2\tilde{E}_2^2 + \frac{1}{6}\tilde{E}_2 + \frac{5}{6}\tilde{E}_4$$

Hence

$$B = \frac{1}{12}\left(18\tilde{E}_2^2 - \tilde{E}_2 - 5\tilde{E}_4 + 6\tilde{E}_3\right).$$

□

N	3	4	5	6	7	8	9
a_N	1	4	10	21	35	60	85
b_N	2	5	17	24	55	75	116
$a_N + b_N$	3	9	27	45	90	135	201
a'_N	1	4	10	18	35	48	81
b'_N	2	5	17	18	55	60	108
$a'_N + b'_N$	3	9	27	36	90	108	189

FIGURE 7. The number of N-squares one-cylinder and two-cylinders square-tiled surfaces in $\mathcal{H}(2)$. Here a'_N and b'_N are the count of reduced origamis according to the formula of Theorem 26.

As can be noticed on Figure 7 the rough order of a_N and b_N are N^3 but there are deviations due to the wild behaviour of the number of divisor function $\sigma_3(N)$.

3.6. **Further readings.**

1. surveys
2. Zorich 2002
3. DGZZ : one cylinder contribution

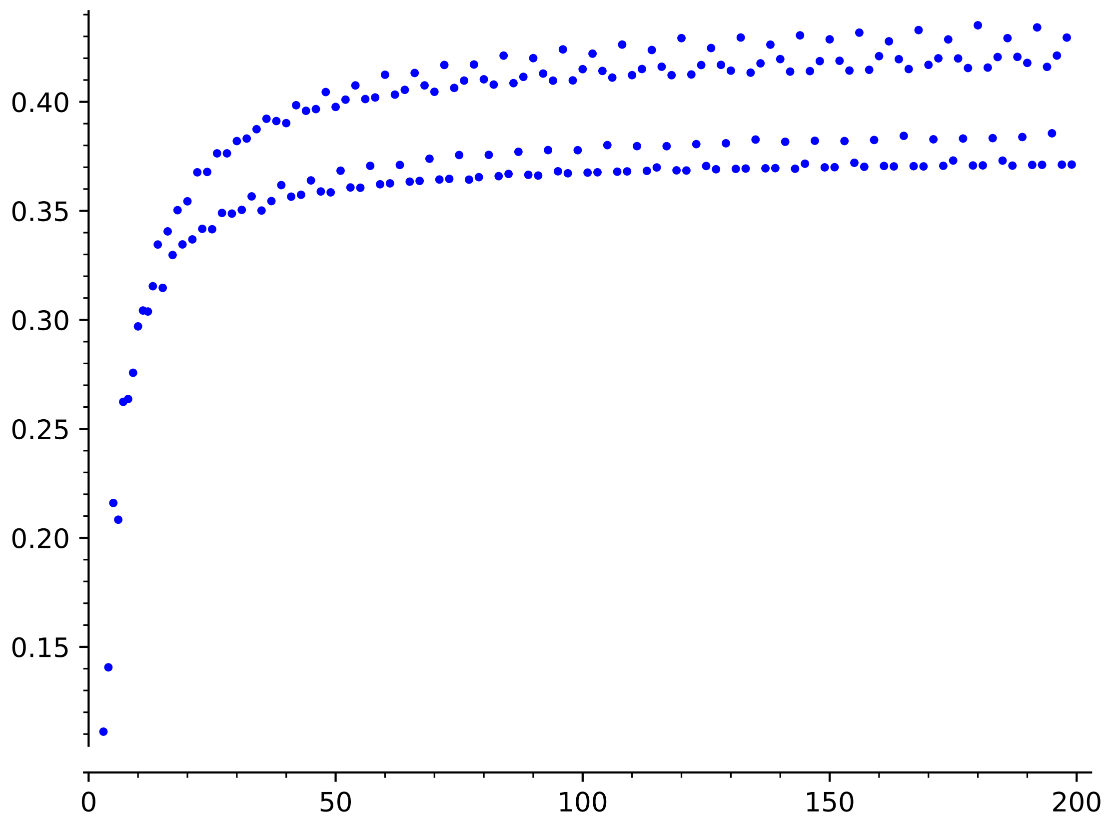

FIGURE 8. The sequence $\{(a_N + b_N)/N^3\}_{N\geq 3}$.

4. Enumeration of square-tiled surfaces via quasi-modular forms

We now count square-tiled surfaces in general strata. The technique was introduced in a famous article by A. Eskin and A. Okounkov.

Theorem 30 (Eskin-Okounkov)**.** *For any stratum $\mathcal{H}(\kappa)$, the generating series of square-tiled surfaces in $\mathcal{H}(\kappa)$ is a quasi-modular form of weight $2g$.*

Corollary 31. *The total mass of $\mathbb{P}\mathcal{H}(\kappa)$ with respect to the Masur-Veech volume form is a rational multiple of π^{2g}.*

4.1. **Frobenius formula and the generating function of square-tiled surfaces.**

4.2. **Bloch-Okounkov and Kerov-Olshanski theorems.**

Theorem 32 (Kerov-Olshanski)**.** *The $f_\mu(\lambda)$ are shifted symmetric functions.*

Let $f : \mathcal{P} \to \mathbb{R}$ be a function on integer partitions. Its q-bracket is the formal series in $\mathbb{Q}$

$$\langle f \rangle_q := \frac{\displaystyle\sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}}.$$

Theorem 33 (Bloch-Okounkov)**.** *Let f be a shifted symmetric function, then its q-bracket $\langle f \rangle_q$*

4.3. **Computations.**

5. Classification of $SL(2,\mathbb{Z})$-orbits of origamis

The minimal number of squares of an origami in a given stratum is described by the next proposition.

Proposition 34. *An origami in $\mathcal{H}(k_1,\ldots,k_\sigma)$ is tiled by at least $\sum_{j=1}^{\sigma}(k_j+1)$ squares. Moreover, the stratum $\mathcal{H}(k_1,\ldots,k_\sigma)$ contains an origami tiled by exactly $\sum_{j=1}^{\sigma}(k_j+1)$ squares.*

Proof. We saw that an origami in $\mathcal{H}(k_1,\ldots,k_\sigma)$ is given by a pair of permutations $(h,v)\in S_n\times S_n$ whose commutator $S_n\ni[h,v]$ has σ non-trivial cycles of lengths k_j+1, $j=1,\ldots,\sigma$. Therefore,

$$n\geq\sum_{n=1}^{\sigma}(k_n+1)=N.$$

Furthermore, a permutation $\mu\in S_N$ with non-trivial cycles of lengths k_j+1, $j=1,\ldots,\sigma$, is even because $\sum\limits_{j=1}^{\sigma}k_j=2g-2$. By a theorem of Gleason, $\mu\in A_N$ is the product of two N-cycles, say $\mu=v\rho$. Since N-cycles are conjugated, we can write $\rho=hv^{-1}h^{-1}$, so that $\mu=[h,v]$ and the origami associated to h and v belongs to $\mathcal{H}(k_1,\ldots,k_\sigma)$. □

The previous statement implies that the L-shaped origami with 3 squares in $\mathcal{H}(2)$. The $SL(2,\mathbb{Z})$-orbit of this origami was discussed before (cf. Exercise 6).

5.1. $SL(2,\mathbb{Z})$-orbits in $\mathcal{H}(2)$.

In what follows, we shall describe the $SL(2,\mathbb{Z})$-orbits of origamis in $\mathcal{H}(2)$ with $n\geq 4$ squares.

Theorem 35 (Hubert–Lelièvre, McMullen (2005))**.** *The (reduced) origamis in $\mathcal{H}(2)$ with $n\geq 4$ squares constitute:*

- *a single $SL(2,\mathbb{Z})$-orbit with $\frac{3}{8}(n-2)n^2\prod\limits_{p|n}(1-\frac{1}{p^2})$ elements when n is even,*
- *two $SL(2,\mathbb{Z})$-orbits with $\frac{3}{16}(n-1)n^2\prod\limits_{p|n}(1-\frac{1}{p^2})$ and $\frac{3}{16}(n-3)n^2\prod\limits_{p|n}(1-\frac{1}{p^2})$ elements when n is odd.*

The two $SL(2,\mathbb{Z})$-orbits mentioned above can be distinguished by their *monodromies*, that is, the conjugacy classes of the subgroups $\langle h,v\rangle$ of S_n generated by the permutations h, v associated to the corresponding origamis. Indeed, the monodromy is an *invariant* of a $SL(2,\mathbb{Z})$-orbit since $SL(2,\mathbb{Z})$ is generated by two elements acting on pairs of permutations by Nielsen transformations. Furthermore, for each $n\geq 5$ odd, there are *at least* two $SL(2,\mathbb{Z})$-orbits of origamis in $\mathcal{H}(2)$ with n squares because the origami with $h=(1,2,\ldots,n)$ and $v=(1,2,3)$ has monodromy A_n and the origami with $h=(1,2,\ldots,n)$ and $v=(1,2)$ has monodromy S_n.

Remark 36. The theorem by Hubert–Lelièvre and McMullen says that the monodromy is a *complete* invariant of $SL(2,\mathbb{Z})$-orbits of origamis in $\mathcal{H}(2)$. For $n\geq 5$ odd, the orbit with monodromy S_n, resp. A_n, is called of type A, resp. B.

We say that an origami $(h,v)\in S_n$ is *primitive* if it is not a non-trivial cover of another origami or, equivalently, if its monodromy $G=\langle h,v\rangle$ is a *primitive*[3] subgroup of S_n. The monodromy of a primitive origami with a large number of squares is always A_n or S_n:

Theorem 37 (Zmiaikou)**.** *A primitive origami $\mathcal{O}\in\mathcal{H}(k_1,\dots,k_\sigma)$ tiled by*

$$N\geq\left(2\sum_{n=1}^{\sigma}(k_n+1)\right)^2$$

squares has monodromy group A_N or S_N.

Proof. After some results obtained by Babai (in 1982) and Pyber (in 1991), a primitive subgroup G of S_m not containing the alternating group A_m satisfies

$$m<4\left(\min_{\alpha\in G\setminus\{id\}}\#\{1\leq v\leq n:\alpha(v)\neq v\}\right)^2.$$

In our context of primitive origamis $\mathcal{O}$, the desired theorem follows directly from the results of Babai and Pyber because the monodromy contains the commutator $[h,v]$ of a pair of permutations determining $\mathcal{O}$ and the support of $[h,v]$ has cardinality $\sum\limits_{n=1}^{\sigma}(k_n+1)$ for $\mathcal{O}\in\mathcal{H}(k_1,\dots,k_\sigma)$. □

Let us now close this subsection with a sketch of the proof of (a particular case of) the theorem of Hubert–Lelièvre and McMullen. A maximal collection of closed horizontal geodesics in an origami is called a *horizontal cylinder*. Any origami (h,v) is naturally *decomposed* into horizontal cylinders given by unions of certain cycles of h. In our current setting, it is possible to show that any origami in $\mathcal{H}(2)$ decomposes into one or two cylinders, so that any origami in $\mathcal{H}(2)$ is determined by 4 natural parameters (t,a,b,c), $0\leq t<n=a+b+c$ or 6 natural parameters $(t_1,t_2,w_1,w_2,h_1,h_2)$, $0\leq t_j<w_j$, $0<w_1<w_2<n=h_1w_1+h_2w_2$:

[ADD FIGURE LATER]

Assuming that $n>3$ is *prime*, Hubert–Lelièvre use a *descent* argument showing that the usual generators of $SL(2,\mathbb{Z})$ allow to decrease the heights of the cylinders: any origami in $\mathcal{H}(2)$ tiled by n squares and decomposed into two horizontal cylinders belongs to the $SL(2,\mathbb{Z})$-orbit of an origami with a single cylinder or two cylinders with heights $h_1=h_2=1$.

Next, Hubert–Lelièvre observe that the coprimality between w_1 and w_2 allows to set $t_1=1$ and $t_2=0$ after a certain number of applications of the Nielsen transformation $(h,v)\mapsto(h,vh^{-1})$. At this stage, if we look at the *vertical direction* (or, equivalently, apply the element $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ of $SL(2,\mathbb{Z})$), then we obtain an origami with a single cylinder.

In summary, we showed that the $SL(2,\mathbb{Z})$-orbit of any origami in $\mathcal{H}(2)$ tiled by a prime number $n>3$ squares contains an origami consisting of a single horizontal cylinder.

[3] A block Δ for G is a subset of $\{1,\dots,n\}$ such that $\alpha(\Delta)=\Delta$ or $\alpha(\Delta)\cap\Delta=\emptyset$ for all $\alpha\in G$. A primitive subgroup of S_n is a subgroup without blocks of sizes between 2 and $n-1$.

Remark 38. The $SL(2,\mathbb{Z})$-orbits of certain origamis of genus 3 might not contain origamis consisting of a single horizontal cylinder: for example, this is the case of the Eierlegende Wollmilchsau. In particular, this indicates that the strategy used by Hubert–Lelièvre doesn't easily generalize to origamis with genus ≥ 3.

Hence, our task is reduced to show that all one-cylinder origamis with the same monodromy belong to the same $SL(2,\mathbb{Z})$-orbit.

The first step towards this goal is to establish that an origami with parameters (a,b,c) belongs to the $SL(2,\mathbb{Z})$-orbit of an origami with parameters $(1,d,e)$. For this sake, it suffices to connect (a,b,c) with $(\delta,k\delta,\gamma)$ for $\delta \mid \gcd(a,b)$ (because n prime forces $\gcd(\delta,\gamma)=1$, so that we can repeat the same argument to reach $(1,d,e)$). In this direction, note that the origami with parameters $(0,a,b,c)$ has two cylinders in the vertical direction: one of them has height c and the other has a certain twist t. In particular, we obtain a one-cylinder origami with parameters $(\delta,k\delta,\gamma)$, where $\delta=\gcd(1+t,\ell)$ and $\ell=\gcd(a,b)$, by looking at the direction $(1+t,\ell)$.

The second step towards our goal is to apply the usual generators of $SL(2,\mathbb{Z})$ to connect $(1,d,e)$ and $(1,1,n-2)$ or $(1,2,n-3)$ depending on whether d and e are both odd or both even. For example, if d and e are both odd, then the vertical direction gives a L-shaped origami with width $1+e$ and height $1+d$. By applying $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and looking at the vertical direction, we get an origami with two cylinders of heights 1. Using the Nielsen transformation $(h,v)\mapsto(h,vh^{-1})$ to set the twist parameters to $t_1=0=t_2$, we get in the diagonal direction $(1,1)$ an origami with a single cylinder and parameters $(1,1,n-2)$.

5.2. **HLK invariant and Delecroix–Lelièvre conjecture.** In their original article, the $SL(2,\mathbb{Z})$-orbits of origamis in $\mathcal{H}(2)$ were distinguished using an invariant called nowadays *Hubert–Lelièvre–Kani (HLK) invariant*.

This invariant concerns the positions of the *Weierstrass points* of the origami. In plain terms, any origami $\mathcal{O}\in\mathcal{H}(2)$ is *hyperelliptic*, that is, $\mathcal{O}$ is a branched cover of $\mathbb{T}^2$ admitting an involution ι with 6 fixed points such that ι lifts the map $\iota_0(z)=-z$ of $\mathbb{T}^2$. By definition, the 6 fixed points of ι project to the 2-torsion points of $\mathbb{T}^2$ (that is, the 4 fixed points of ι_0), so that we can write a list $(l_0,[l_1,l_2,l_3])$ where l_0 is the number of fixed points of ι projecting to $0\in\mathbb{T}^2$ and l_1, l_2 and l_3 are the numbers of fixed points of ι projecting to $1/2$, $i/2$ and $(1+i)/2$. Since $SL(2,\mathbb{Z})$ fixes the origin and permutes the other 2-torsion points of $\mathbb{T}^2$, the list $(l_0,[l_1,l_2,l_3])$ is an invariant (HLK) of the $SL(2,\mathbb{Z})$-orbit of $\mathcal{O}$ when $[l_1,l_2,l_3]$ is taken modulo permutations.

A quick calculation shows that the possible values of the HLK invariant of origamis in $\mathcal{H}(2)$ are $(1,[3,1,1])$ and $(3,[1,1,1])$ (cf. Exercise 39).

Unfortunately, the classification of $SL(2,\mathbb{Z})$-orbits of origamis is not known beyond $\mathcal{H}(2)$. Nonetheless, we dispose of precise conjectures about such classifications in many strata thanks to the numerical investigations by several authors. For instance, Delecroix and Lelièvre conjectured that the monodromy and the HLK invariant (in the presence of "anti-automorphisms") constitute *complete* invariants of arithmetic Teichmüller curves in many strata including $\mathcal{H}(1,1)$ and $\mathcal{H}(4)$.

In the case of $\mathcal{H}(1,1)$, the Delecroix–Lelièvre conjecture predicts the existence of two $SL(2,\mathbb{Z})$-orbits of reduced origamis with $n > 6$ squares. In the case of $\mathcal{H}(4)$, the Delecroix–Lelièvre conjecture implies that there are seven $SL(2,\mathbb{Z})$-orbits of reduced origamis with an odd number $n > 8$ of squares and there are six or seven $SL(2,\mathbb{Z})$-orbits of reduced origamis with a number $n > 8$ squares which is 0 or 2 modulo 4.

5.3. **McMullen's expansion conjecture in $\mathcal{H}(2)$.** We saw that the arithmetic Teichmüller curves are coded by graphs whose vertices are $SL(2,\mathbb{Z})$-orbits of origamis and whose edges connect origamis deduced from each other by a fixed set of generators of $SL(2,\mathbb{Z})$ (e.g., two parabolic matrices).

It was conjectured by McMullen that the corresponding family of graphs associated to arithmetic Teichmüller curves in $\mathcal{H}(2)$ is *expander*, i.e., the adjacency matrices of these graphs possess an *uniform* spectral gap. This conjecture is still open despite some evidences towards it: for example, these graphs are associated to Teichmüller curves whose genera tend to infinity, they are not planar, their diameters seem to grow slowly (e.g., the graph associated to origamis in $\mathcal{H}(2)$ with 66 squares has 69120 vertices and diameter $23 \approx 2\log(69120)$), etc.

As it turns out, McMullen's conjecture admits a geometric version asserting that the first eigenvalues of the Laplacians of the arithmetic Teichmüller curves in $\mathcal{H}(2)$ are uniformly bounded away from zero.

5.4. **Exercises.**

Exercise 39. *[Relate values of HLK and monodromies in H(2)]*

6. Some extra topics

6.1. Kontsevich–Zorich cocycle and its Lyapunov exponents.
???????

6.2. Geometric version of McMullen's expansion conjecture.
????????

6.3. Duryev's partial classification of $SL(2,\mathbb{Z})$-orbits in $\mathcal{H}(1,1)$.
???????????????????

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