The hitchhiker's guide to condensed matter and statistical physics: Topological phenomena in matter

"Topological phenomena in frustrated magnetism"

Lecture plan:

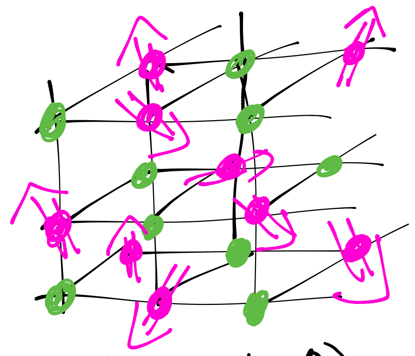
- conventional magnetism, local order and spontaneous symm. breaking

- competing interactions and frustration

- the classical 8-vertex model (degeneracy, topological order and fractionalized excitations)

- the (quantum) toric code and fractional statistics

Conventional magnetism: (localised moments)



(crystal)

local moments $\vec{\mu}_i$

simplest "toy" representation:
Ising moments $\sigma_i = \pm 1$

interactions $\longrightarrow$ Hamiltonian (energy) $H(\{\sigma_i\})$

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \quad (J>0 \text{ ferromag.})$$

nearest-neighbour only for simplicity

- energy favours $\sigma_i = \sigma_j$ (ferro. correlations)
- entropy favours random $\{\sigma_i\}$ configuration

low symm phase ~ 1 high symm. phase

T/J

ordered phase (local order parameter) critical point disordered phase

spontaneous symmetry breaking

(HEP paradigm)

Note: each individual energy term in H is minimised in the ordered phase

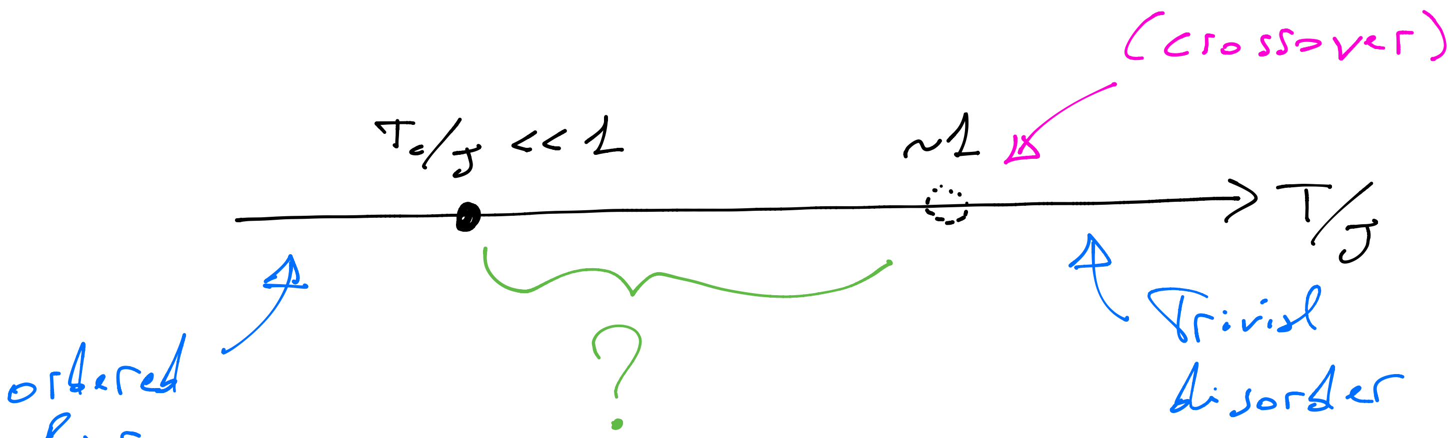
most interesting behaviour close to C.P.

- spin correlations decay as power law with distance
- system is "scale invariant"
- "critical scaling" of physical properties with distance from C.P. in parameter space
- <u>universality</u>

Frustrated magnetism:

$H = \sum_{ij} H_{ij}$ s.t. not all H_{ij} can be minimised at the same time

$\Rightarrow$ T_c is suppressed: $T_c/J \ll 1$



$(T > J)$

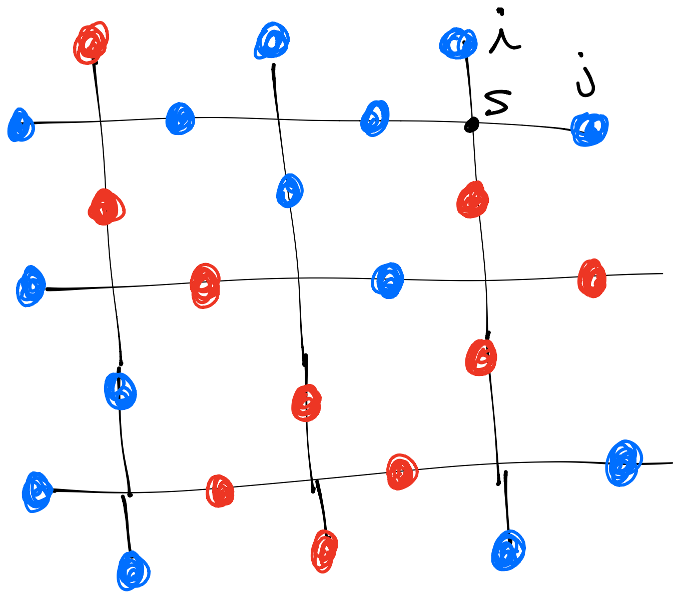
new phase:

- no long range order
- yet strongly correlated $(T < J)$

Spin liquid: large degeneracy / entanglement; emergent (not broken) symm.; topo order; fractionalised q.p.

Examples: triangular Ising AFM; 8-vertex model; (quantum) toric code

Example: 8-vertex model

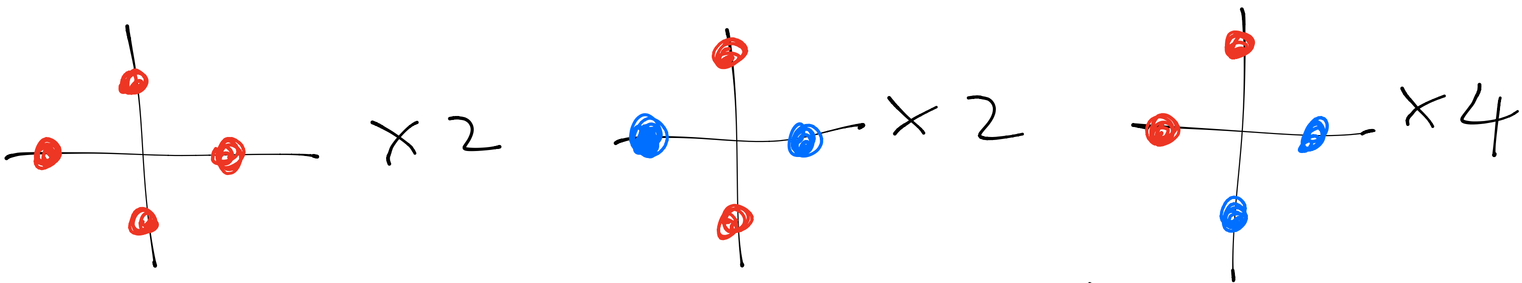


$$H = -\lambda \sum_{s} \prod_{i \in s} \sigma_i$$

$$\sigma_i = \pm 1$$

$$(\lambda > 0)$$

low energy states: $\prod_{i \in s} \sigma_i = +1$

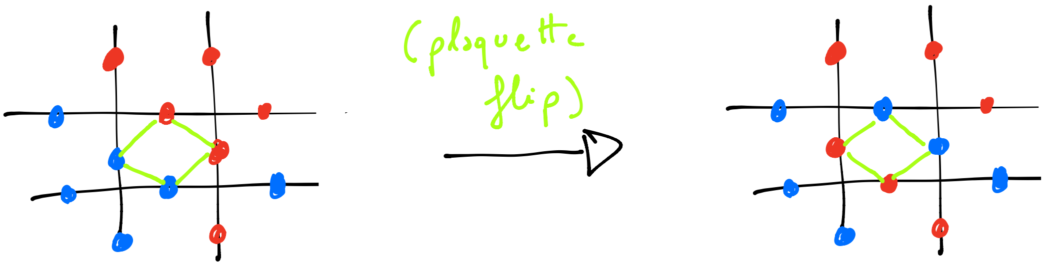


(hence: 8-vertex model)

exponentially many ways to arrange them on a lattice $\longrightarrow$ extensive degeneracy

$\Rightarrow$ finite entropy at zero temperature!

(toy model)



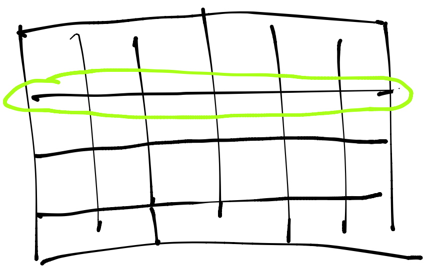
$\Rightarrow$ given any GS spin configuration, you can flip any plaquette and generate another GS config. : $2^{N/2}$ ($N/2$ = # of plaq.)

• not all independent (assuming periodic boundary conditions)

flip all but one plaquette $\Leftrightarrow$ flip that one plaq.

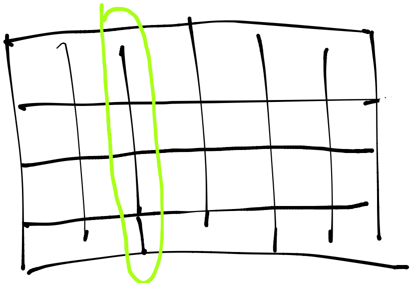
$\longrightarrow$ $2^{N/2 - 1}$

• there are other allowed operations (winding loop flips)



or

(p. b. c.)



Two more independent operations (and only 2)

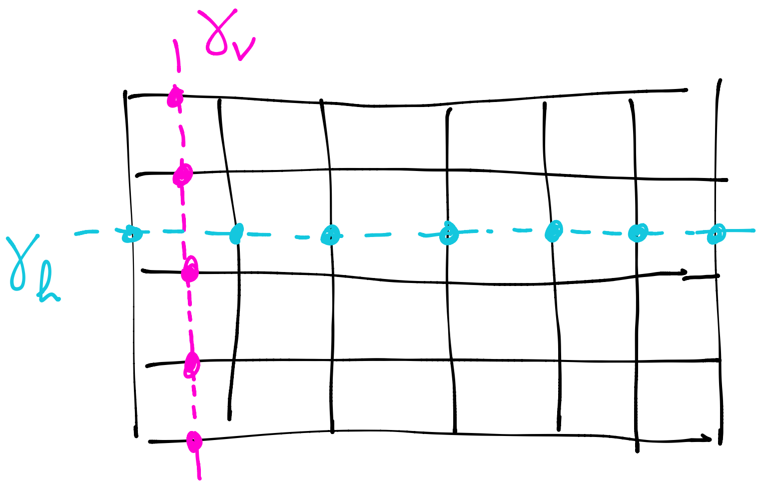
$\longrightarrow \quad 2^{N/2 + 1}$

exact counting gives GS entropy $\sim N \cdot \frac{1}{2} \ln 2$

4 topological sectors

(each containing $2^{N/2-1}$ configurations)

$\longrightarrow$ topological correlations

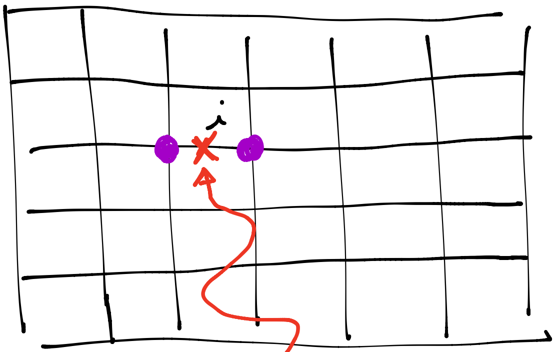


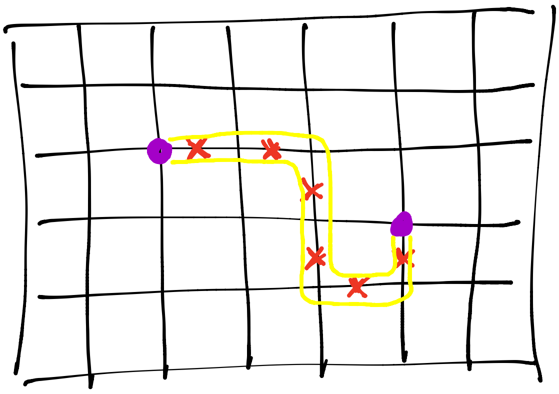
$$\Gamma_v = \prod_{i \in \gamma_v} \sigma_i = \pm 1$$

$$\Gamma_h = \prod_{i \in \gamma_h} \sigma_i = \pm 1$$

- invariant under plaquette operations
- divide config. space into 4 topo. sectors
- non-local order + only non-local ops. change it

Excitations :





creates two defects ⇐ flip spin at site i

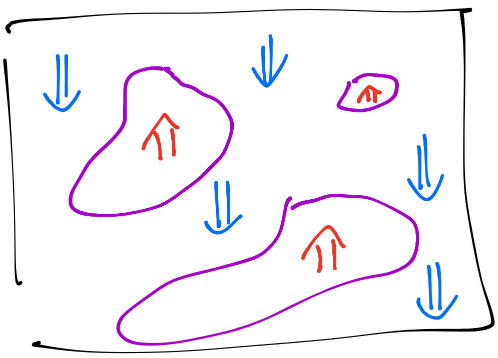
$\left(\prod_{j \in S} \sigma_j = -1\right)$

further flips separate the defects at will

same energy

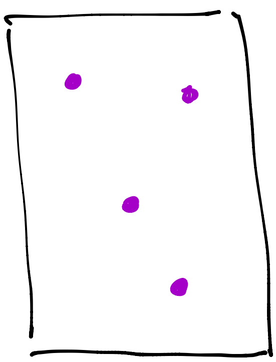
spin flip fractionalises into two (trivially) deconfined quasiparticle excitations

Conventional magnetism :



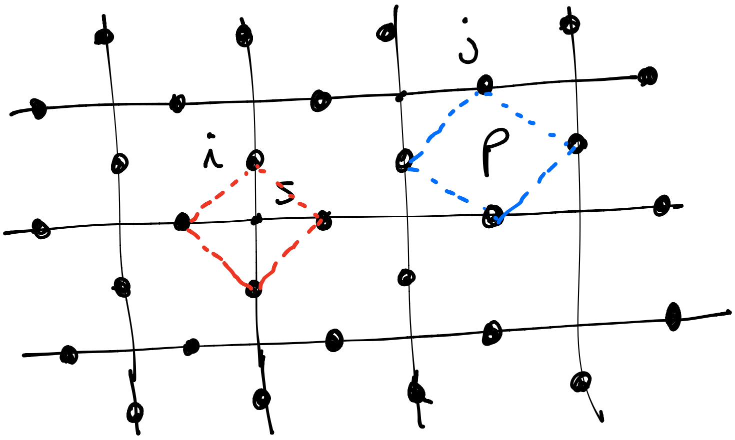
(excitations are domain walls)

frustrated magnetism:



- point-like q.p.
- random walk
- pair create / annihilate

Example: (quantum) Toric Code



- spin-1/2 d.o.f. on bonds (i,j) of square lattice
- p.b.c.
- $s \equiv$ sites or stars
- $p \equiv$ plaquettes

$$\hat{A}_s = \prod_{i \in s} \hat{\sigma}_i^z$$

(N spins → N/2 stars and N/2 plaquettes)

$$\hat{B}_p = \prod_{i \in p} \hat{\sigma}_i^x$$

$$\hat{H} = -\Delta_s \sum_s \hat{A}_s - \Delta_p \sum_p \hat{B}_p$$

$(\Delta_s > 0, \Delta_p > 0)$ (drop the hats...)

$$\underbrace{[A_s, B_p]}_{\text{they either share 2 spins or none at all}} = \underbrace{[A_s, A_{s'}] = [B_p, B_{p'}]}_{\text{trivial}} = 0$$

(and $\{\sigma_i^x, \sigma_i^z\} = 0$)

$\Rightarrow$ all A_s, B_p and H can be diagonalised simultaneously

ground state $|\psi_0\rangle$: $A_s|\psi_0\rangle = B_p|\psi_0\rangle = |\psi_0\rangle$

all star and plaquette ops. have eigenvalue $+1$

Explicit construction:

$$|\psi_0\rangle \propto \prod_p \underbrace{(1+B_p)}_{} \, |\underbrace{\{\sigma_i^z = +1, \forall i\}}_{}\rangle$$

B_p op. has eigenval. $= \pm 1$ $\rightarrow$ this term projects onto "+1" only

Tensor product state in σ^z basis

check: they commute

$$A_s|\psi_0\rangle = \frac{1}{N}\prod_p (1+B_p)\, A_s \underbrace{|\{\sigma_i^z = +1\}\rangle}_{= |\{\sigma_i^z = +1\}\rangle}$$

$$= |\psi_0\rangle \quad \checkmark$$

$$B_p|\psi_0\rangle = \frac{1}{N} \underbrace{B_p(1+B_p)}_{} \prod_{p'\neq p}(1+B_{p'})\,|\{\sigma_i^z = +1\}\rangle$$

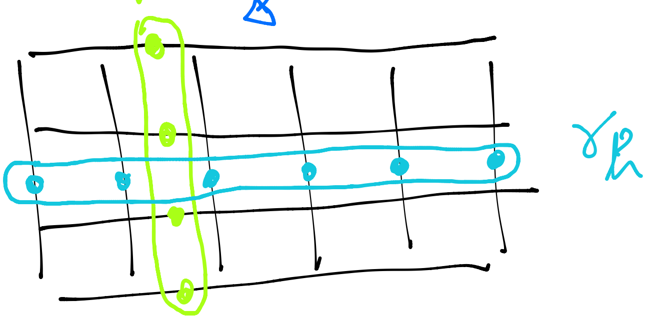
$B_p^2 = 1 \rightarrow B_p + 1$

$$= |\psi_0\rangle \quad \checkmark$$

The state is however <u>not unique</u>

(recall classical degeneracy : plaquette flip $\Leftrightarrow B_p$
and winding loop flips / topological winding observables)

(p.b.c. for simplicity)



$$\Gamma_v = \prod_{i \in \gamma_v} \sigma_i^z$$

$$\Gamma_h = \prod_{i \in \gamma_h} \sigma_i^z$$

$$[\Gamma_{h,v}, A_s] = [\Gamma_{h,v}, B_p] = 0$$

$\Rightarrow$ they can be diagonalised simultaneously

$\Gamma_{h,v}$ have eigenvalues $m_{h,v} = \pm 1$

$\longrightarrow$ $2^2 = 4$ ground states $\left(|\psi_0^{(m_h, m_v)}\rangle \right)$

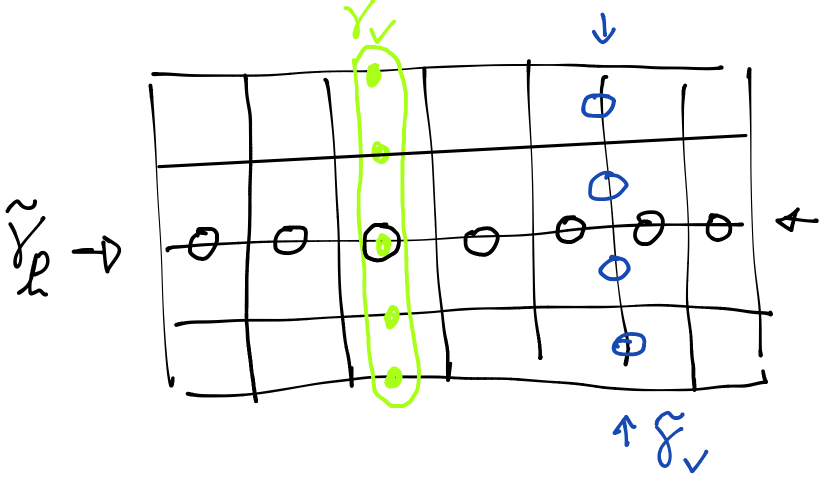
$$\Gamma_h |\psi_0^{(m_h,m_v)}\rangle = m_h |\psi_0^{(m_h,m_v)}\rangle$$

$$\Gamma_v |\psi_0^{(m_h,m_v)}\rangle = m_v |\psi_0^{(m_h,m_v)}\rangle$$

<u>topological</u> quantum numbers ! (cannot be measured locally)

Quantum information can be stored Topologically in the GS of the toric code (supersp. of $|\psi_0^{(m_h,m_v)}\rangle$)

+ topological quantum operations:



- $\langle \Gamma_v \rangle$ not affected by local operations within GS sector
- only changed by winding operations

$$\tilde{\Gamma}_{h,v} = \prod_{i \in \tilde{\gamma}_{h,v}} \sigma_i^x$$

also commute with A_s, B_p, H but not with $\Gamma_{h,v}$

(actually, $(\Gamma_h, \tilde{\Gamma}_v)$ and $(\Gamma_v, \tilde{\Gamma}_h)$ algebra $\Leftrightarrow$ spin-1/2 and Toric code GS encodes 2 topological qubits

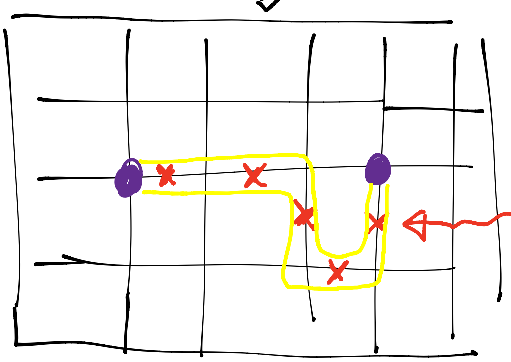
$\longrightarrow$ top quantum info and computing)

(entanglement...)

Toric code excitations : $A_s, B_p = -1$

(point-like quasiparticles)

"GS"



act with σ_i^x op. $\Leftrightarrow$ flip σ_i^z eigenval. in GS wavefunction

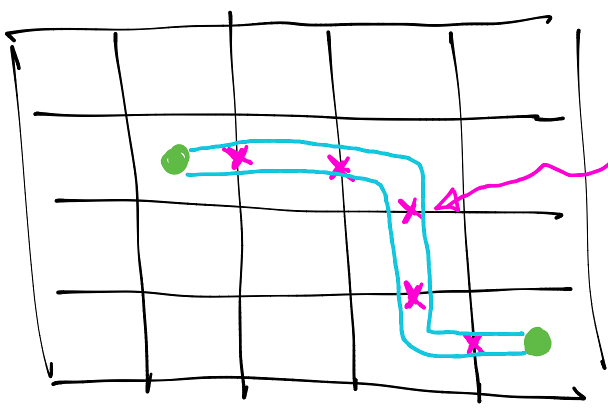
and commutes with B_p!

$\Downarrow$

generates two deconfined defects at the ends of a string operator (i.e. flipped spins)

but also :

"GS"

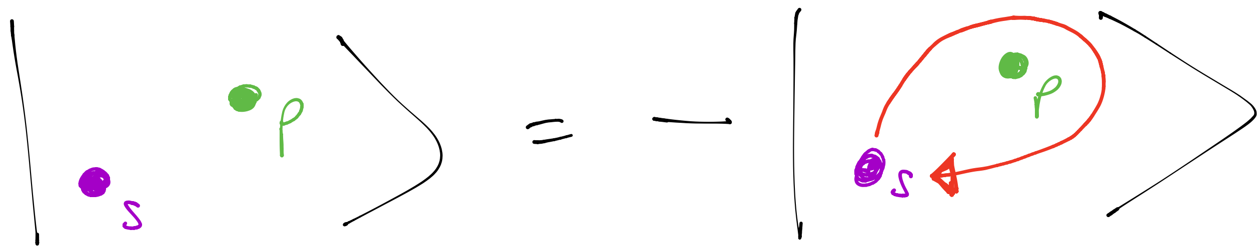


act with σ_i^z op. $\Leftrightarrow$ flip σ_i^x eigenval. in GS wavefunction

and commutes with A_s!

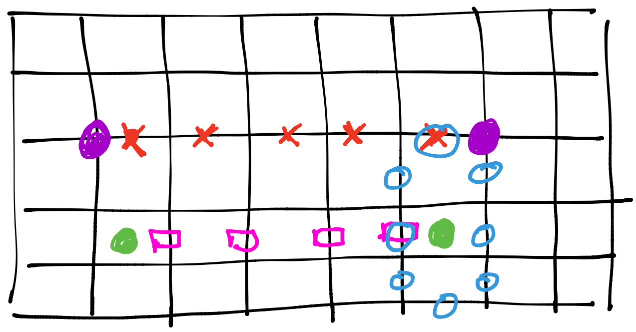
$\Rightarrow$ dual point-like quasiparticle excitations at the ends of dual string operator

Star and plaquette defects are bosons on their own, but they have mutual fractional statistics!



("half" of the behaviour of a fermion → semion; also equivalent to $s \equiv$ "electron" and $p \equiv$ "π magnetic flux", due to Aharonov-Bohm phase).

Explicitly:



× string γ_1

□ string γ_2

○ string γ_3 (closed loop)

$$|\psi_1\rangle = \left(\prod_{i \in \gamma_1} \sigma_i^x\right)\left(\prod_{i \in \gamma_2} \sigma_i^z\right)|\psi_0\rangle$$

→ state with excitations: ● and ●

$$|\psi_2\rangle = \left(\prod_{i\in\gamma_3} \sigma_i^x\right)|\psi_1\rangle$$

→ new state with excitations in the exact same positions!

(γ_3 has taken one star defect ● around a loop and back to the same place)

commute except for 1 site ($\boxed{+}$)

However,

$$|\psi_2\rangle = \left(\prod_{i\in\gamma_3} \sigma_i^x\right)\left(\prod_{i\in\gamma_2} \sigma_i^x\right)\left(\prod_{i\in\gamma_2} \sigma_i^z\right)|\psi_0\rangle$$

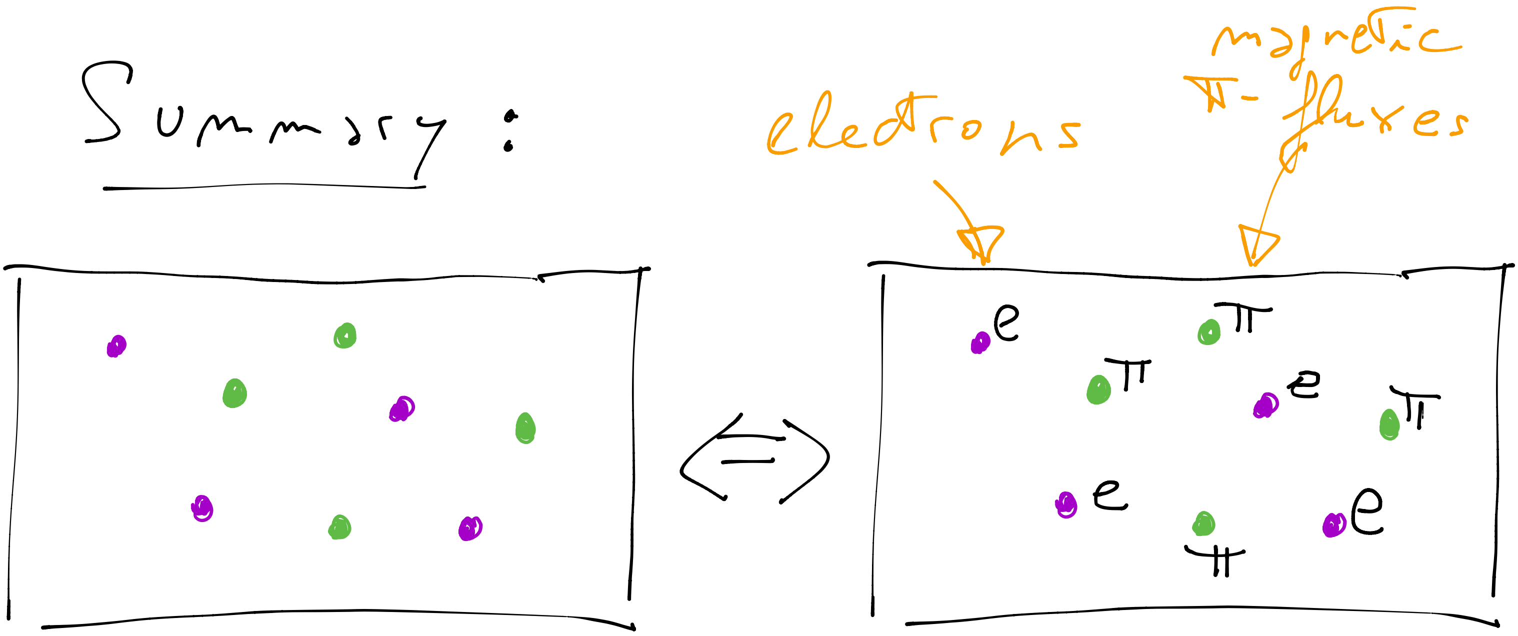
Commute

$$= -\left(\prod_{i\in\gamma_2} \sigma_i^x\right)\left(\prod_{i\in\gamma_2} \sigma_i^z\right)\underbrace{\left(\prod_{i\in\gamma_3} \sigma_i^x\right)}_{\text{closed loop}}|\psi_0\rangle$$

closed loop $= \prod B_p = +1$ on GS

$$= -|\psi_1\rangle \quad \checkmark$$

Summary:



- created and annihilated in pairs
- deconfined point-like q.p.
- mutual semionic statistics but completely static!

(all terms in Hamiltonian commute)

dynamics comes from additional perturbations (e.g., "transverse" fields)

$$H = -\Delta_s \sum_s A_s + h_s \sum_i \sigma_i^x - \Delta_p \sum_p B_p + h_p \sum_i \sigma_i^z$$

star defect cost

star defect hopping term

plaquette cost

plaquette hopping

(GS is stable so long as $h_s < \Delta_s$, $h_p < \Delta_p$)

arXiv:1909.08633

arXiv:1911.06331