Nonperturbative fermion anomalies

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Abstract: Lecture notes on nonperturbative anomalies, focusing on fermions.

Contents

1 Introduction

Backgrounds :

- Basics of QFT, perturbative anomalies (E.g. Weinberg's QFT book, Vol.2)
- Basics of geometry and topology (E.g. Nakahara's book)

Ref. of these lectures: Witten-KY, 1909.08775

Anomalies : Gauge sym., Global sym.

Gauge. Perturbative anomaly cancellation in the standard model, string theory, etc.
Nontrivial at the nonperturbative level.
(I do not know the complete answer for string theory.)

Q : We are confident that string theory is consistent. Why do we care?
A : studies of anomalies = studies of topological structure

Global. 't Hooft anomalies : anomalies of global symmetries
Conserved in RG flows.

$$\text{UV theory} \xrightarrow{\text{RG flow}} \text{IR theory} \tag{1.1}$$

Either UV or IR may be difficult (strong coupling etc.).
UV anomaly = IR anomaly : constraints on dynamics

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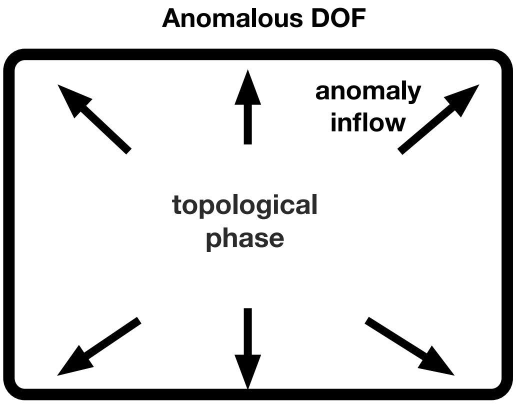
$$\text{Bulk topological phase} \longleftrightarrow \text{anomaly of boundary degrees of freedom} \tag{1.2}$$



This is essential for nonperturbative formulation of the concept of anomalies itself.

I do not distinguish gauge and global sym.

Gauge :

$$Z = \int [DA][D\psi]e^{-S[\psi,A]} = \int [DA]Z_\psi[A] \tag{1.3}$$

$$Z_\psi[A] = \int [D\psi]e^{-S[\psi,A]} \tag{1.4}$$

We consider only $Z_\psi[A]$.

Global : A_μ = Source fields of the symmetry.
$Z_\psi[A]$: generating functional (generalized to include nontrivial topology)

2 Topology and anomaly

Global anomalies (nonperturbative)

- Discrete symmetries
- Topologies of symmetry groups.

2.1 Topology of symmetry groups

Basic example:

$$\mathrm{SU}(2) \neq \mathrm{SO}(3) \neq \mathrm{O}(3) \tag{2.1}$$

Their Lie algebras are the same.

$$J_1, J_2, J_3, \qquad [J_i, J_j] = i\epsilon_{ijk} J_k. \tag{2.2}$$

Difference between SU(2) **and** SO(3)**:**

- "Spin" odd representations are possible in SU(2) but not in SO(3).
- Some fiber bundles which are impossible in SU(2) are possible in SO(3).

Example:
U(1) $\subset$ SO(3).

$$\mathrm{U}(1) = \{\exp(i\phi J_3)\}. \tag{2.3}$$

$\exp(2\pi i J_3) = 1$ in SO(3). This excludes representations with $J_3 = \frac{1}{2} + \mathbb{Z}$.
Fiber bundle on S^2 (polar coordinates (θ, ϕ))

$$A = \begin{cases} A_N = \frac{i}{2} J_3 (\cos\theta - 1) d\phi & \theta < \pi \\ A_S = \frac{i}{2} J_3 (\cos\theta + 1) d\phi & \theta > 0. \end{cases} \tag{2.4}$$

$$A_S = g^{-1} A_N g + g^{-1} dg, \qquad g = \exp(i J_3 \phi) \tag{2.5}$$

g: Single valued for SO(3) since $J_3 \in \mathbb{Z}$. But not for SU(2).

$$g : S^1 \to \mathrm{SO}(3). \tag{2.6}$$

g gives an element of $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$.
Topologically nontrivial bundle.

Difference between SO(3) and O(3):
Some new bundle for O(3).

Example:
$S^1 = \{[0,1]$ with two ends glued$\}$. O(3) transition function g;

$$(0, v_0),\ (1, v_1) \in [0,1] \times V, \qquad v_1 = g v_0 \tag{2.7}$$

V: 3-dim. vector space. Take $\det g = -1$. Topologically nontrivial bundle on S^1.
g gives an element of $\pi_0(\mathrm{O}(3)) \cong \mathbb{Z}_2$.

2.2 Examples of global anomalies (traditional approach)

Anomalies related to $\pi_0(\mathrm{O}(N))$, $\pi_1(\mathrm{SO}(N))$.

$d = 1$ majorana fermions ψ_i $(i = 1, 2, \cdots N)$.

$$\mathcal{L} = \sum_{i=1}^{N} \frac{i}{2} \psi_i \frac{d}{dt} \psi_i. \tag{2.8}$$

O(N) symmetry

$$\psi_i \to M_{ij} \psi_j, \qquad M \in \mathrm{O}(N). \tag{2.9}$$

Partition function

$$\mathrm{Tr}\, e^{-\beta H} = \text{The path integral on } S^1. \tag{2.10}$$

$S^1 = [0, \beta]$ with 0 and β glued. Thermal boundary condition

$$\psi_i(\beta) = -\psi_i(0). \tag{2.11}$$

$\underline{\pi_0(\mathrm{O}(N))}$
Take $g \in \mathrm{O}(N)$ with $\det g = -1$.

$$g = \mathrm{diag}(-1, +1, \cdots, +1). \tag{2.12}$$

Consider

$$\mathrm{Tr}(e^{-\beta H} g) \tag{2.13}$$

The path integral on S^1 with

$$\psi_1(\beta) = +\psi_1(0), \qquad \psi_j(\beta) = -\psi_j(0)\ (j \neq 1). \tag{2.14}$$

Mode expansion

$$\psi_1(\tau) = A_0^{(1)} + \sum_{n \geq 1} (B_n^{(1)} e^{2\pi n i \tau/\beta} + C_n^{(1)} e^{-2\pi n i \tau/\beta}), \tag{2.15}$$

$$\psi_j(\tau) = \sum_{n \geq 1} (B_n^{(j)} e^{2\pi i (n-1/2)\tau/\beta} + C_n^{(j)} e^{-2\pi i (n-1/2)\tau/\beta}) \tag{2.16}$$

Nonzero modes appears in pairs (B, C).
$A_0^{(1)}$: zero mode. (The number of zero modes mod 2 : mod 2 index.)
The path integral measure

$$[D\psi] = dA_0^{(1)} \prod dB_n^{(i)} dC_n^{(i)} \tag{2.17}$$

$(-1)^F$ transformation

$$\psi_i(\tau) \to -\psi_i(\tau) \tag{2.18}$$

$$[D\psi] \to -[D\psi]. \tag{2.19}$$

The sign change due to the zero mode $A_0^{(1)}$.
$[D\psi]$ is not invariant under $(-1)^F$. An anomaly associated to $\pi_0(\mathrm{O}(N))$ $(\det g = -1)$.

$\underline{\pi_1(\mathrm{SO}(N))}$

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \oplus \mathrm{diag}(1, \cdots, 1). \tag{2.20}$$

We set

$$\psi = \frac{\psi_1 - i\psi_2}{\sqrt{2}}, \qquad \overline{\psi} = \frac{\psi_1 + i\psi_2}{\sqrt{2}}, \qquad \text{neglect } \psi_i \ (i \geq 3). \tag{2.21}$$

$$\mathcal{L} = i\overline{\psi}\frac{d}{dt}\psi \tag{2.22}$$

$$\mathrm{Tr}(e^{-\beta H} g(\theta)) \tag{2.23}$$

The path integral on S^1 with

$$\psi(\beta) = -e^{i\theta}\psi(0), \qquad \overline{\psi}(\beta) = -e^{-i\theta}\overline{\psi}(0) \tag{2.24}$$

$$\psi(\tau) = \sum_{n\in\mathbb{Z}} A_n \exp\left(2\pi i(n - \frac{1}{2} + \frac{\theta}{2\pi})\frac{\tau}{\beta}\right), \qquad \overline{\psi}(\tau) = \cdots . \tag{2.25}$$

$$\mathrm{Tr}(e^{-\beta H} g(\theta)) = \int [D\psi] e^{-S} \propto \prod_{n\in\mathbb{Z}} (n - \frac{1}{2} + \frac{\theta}{2\pi}) \tag{2.26}$$

Sign choice : $\mathrm{Tr}(e^{-\beta H} g) > 0$ at $\theta = 0$.

$$\int [D\psi] e^{-S} = \prod_{n\geq 1} \left((n - \frac{1}{2})^2 - (\frac{\theta}{2\pi})^2\right). \tag{2.27}$$

Smoothly change θ from 0 to 2π.

$$\theta = 0: \text{ all factors positive} \tag{2.28}$$

$$\theta = 2\pi: \text{ one factor is negative, } \left((1-\frac{1}{2})^2 - (\frac{\theta}{2\pi})^2 \right) \tag{2.29}$$

Conclusion:

$$[D\psi]_{\theta=2\pi} = -[D\psi]_{\theta=0}. \tag{2.30}$$

But $g(\theta = 2\pi) = 1$ as an element of SO(N).
The path integral measure has a sign ambiguity: an anomaly.
An anomaly associated to $\pi_1(\mathrm{SO}(N))$. ($g(\theta) : S^1 \to \mathrm{SO}(N)$)

Interpretations:

1. We have the SO(N) symmetry which is anomalous.
   (Convenient for 't Hooft anomaly matching of global sym.)
2. We have the Spin(N) symmetry which is anomaly free since $\pi_1(\mathrm{Spin}(N)) = 0$.
   (Necessary for gauging the sym.)

3 Systematic description of anomalies

3.1 Chiral fermion as edge modes

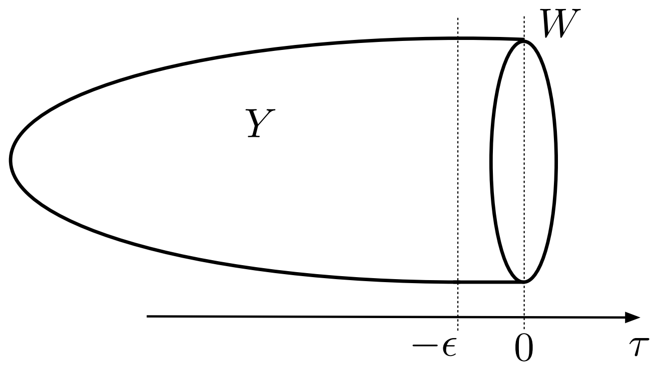
Setup:

$$W : d\text{-dim. manifold} \tag{3.1}$$
$$Y : d+1\text{-dim manifold with } \partial Y = W \tag{3.2}$$

Near the boundary ∂Y, assume

$$(-\epsilon, 0] \times W \subset Y \tag{3.3}$$

Coordinate normal to the boundary: $\tau \in (-\epsilon, 0]$.



Massive fermion in $(d+1)$-dim. Y

$$\mathcal{L} = \overline{\Psi}(\not{D}_Y + m)\Psi \tag{3.4}$$

Remark: Majorana fermion is also OK, $\overline{\Psi} \sim \Psi$.

γ^τ : gamma matrix in the direction τ. Eigenvalues $\gamma^\tau = \pm 1$.
We use a local boundary condition

$$\mathsf{L} : \ (1 - \gamma^\tau)\Psi|_{\tau=0} = 0 \tag{3.5}$$

Near ∂Y,

$$(\not{D}_Y + m)\Psi = \gamma^\tau(\partial_\tau + \mathcal{D}_W + m\gamma^\tau)\Psi \tag{3.6}$$
$$\mathcal{D}_W = \sum_{\mu \neq \tau} \gamma^\tau \gamma^\mu D_\mu. \tag{3.7}$$

Take $|m|$ very large. Localized solution of $(\not{D}_Y + m)\Psi = 0$ near ∂Y if $m < 0$:

$$\Psi = \chi \exp(-m\tau), \qquad (1 - \gamma^\tau)\chi = 0, \qquad \mathcal{D}_W \chi = 0. \tag{3.8}$$

$$(1-\gamma^\tau)\Psi|_{\tau=0} = (1-\gamma^\tau)\chi = 0. \tag{3.9}$$

$$(\partial_\tau + \mathcal{D}_W + m\gamma^\tau)\Psi = (-m\chi + \mathcal{D}_W\chi + m\gamma^\tau\chi)\exp(-m\tau) = 0. \tag{3.10}$$

$\tau < 0$ in our convention.
$\exp(-m\tau)$ localized near $\tau \sim 0$ if and only if $m < 0$.
No such mode for $m > 0$.

γ^τ : generalized "chirality" operator of the boundary W.

$$\gamma^\tau \mathcal{D}_W + \mathcal{D}_W \gamma^\tau = 0 \tag{3.11}$$

χ : Massless "chiral" fermion on the boundary

$$\gamma^\tau \chi = +\chi, \qquad \mathcal{D}_W \chi = 0 \tag{3.12}$$

Example: $d+1=5$, $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5$.
Take $\gamma^\tau = \gamma^5$: chirality in $d=4$.

Take Pauli-Villars mass $M > 0$: no localized massless mode from unphysical PV field.

Remark:
Any d, any sym., any fermion with spin 1/2 is realized as a boundary mode.

Example:
$d+1=2$, Majorana massive fermion Ψ_i with O(N) symmetry $\Psi_i \to M_{ij}\Psi_j$

$$S = \frac{1}{2}\int_{\tau\le 0} d\tau d\sigma \sum_i \Psi_i^T \epsilon(\gamma^\tau\partial_\tau + \gamma^\sigma\partial_\sigma + m)\Psi_i, \tag{3.13}$$

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.14}$$

Localized mode

$$\Psi(\tau,\sigma) = \sqrt{2|m|}e^{-m\tau}\begin{pmatrix} \psi_i(\sigma) \\ 0 \end{pmatrix}. \tag{3.15}$$

$$(\gamma^\tau\partial_\tau + m)\,\Psi \to 0 \tag{3.16}$$

$$S \to \frac{1}{2}\int_{\tau\le 0} d\tau d\sigma \sum_i \Psi_i^T \epsilon(\gamma^\sigma\partial_\sigma)\Psi_i = \frac{1}{2}\int_{\tau\le 0} d\tau d\sigma \sum_i e^{-2m\tau}2|m|\psi_i^T \frac{d}{d\sigma}\psi_i \tag{3.17}$$

$$= \frac{1}{2}\int d\sigma\, \psi_i \frac{d}{d\sigma}\psi_i. \tag{3.18}$$

$d=1$ Majorana fermions studied before.

3.2 General anomaly

Partition function on $d+1$-dim. Y, $\partial Y = W$, the local boundary condition L:

$$Z(Y, \mathsf{L}) = \int [D\Psi] e^{-S(\Psi)}. \tag{3.19}$$

For $m < 0$,

- Bulk: large mass gap in the limit $|m| \to \infty$
- Edge: chiral fermion χ

We want to think

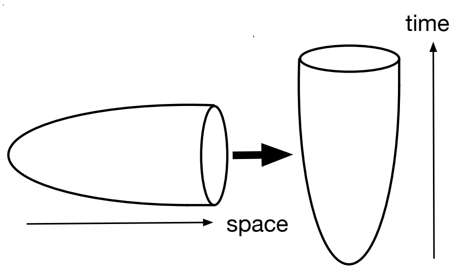
$$Z(Y, \mathsf{L}) \sim Z_\chi(W). \qquad \text{(chiral fermion partition function)} \tag{3.20}$$

Left hand side : completely gauge invariant, but may depend on Y.

Modern formulation of anomalies

- Everything is formulated in a completely gauge invariant way.
- Definition of the d-dim. theory depends on $(d+1)$-dim. : anomaly

Computation of $Z(Y, \mathsf{L})$.
Wick rotation



Path integral on $Y \longrightarrow |Y\rangle \in \mathcal{H}_W$. $\mathcal{H}_W$: Hilbert space on W.
The local b.c.L also gives a state vector $|\mathsf{L}\rangle$. (Compare $|x = x_0\rangle$ of QM.)

$$Z(Y, \mathsf{L}) = \langle \mathsf{L} | Y \rangle. \tag{3.21}$$

Time evolution near the boundary $(-\epsilon, 0] \times W$:

$$e^{-\epsilon H} \to |\Omega\rangle\langle\Omega| \qquad (|m|\epsilon \gg 1) \tag{3.22}$$

When $|m| \to \infty$,

$$|Y\rangle \propto |\Omega\rangle \tag{3.23}$$

Hence

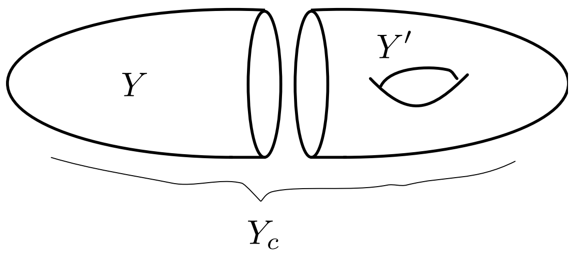
$$Z(Y,\mathsf{L}) = \langle \mathsf{L}|Y\rangle = Z(Y,\mathsf{L}) = \langle \mathsf{L}|\Omega\rangle\langle\Omega|Y\rangle \tag{3.24}$$

Let us assume $|\langle\Omega|Y\rangle| = 1$. (Detail omitted).

Y' : another manifold with $\partial Y' = W$.

$$\begin{aligned}
\frac{Z(Y,\mathsf{L})}{Z(Y',\mathsf{L})} &= \frac{\langle\Omega|Y\rangle}{\langle\Omega|Y'\rangle} && (3.25)\\
&= \langle Y'|\Omega\rangle\langle\Omega|Y\rangle \qquad (\because |\langle\Omega|Y\rangle| = 1) && (3.26)\\
&= \langle Y'|Y\rangle \qquad (\because |Y\rangle \propto |\Omega\rangle) && (3.27)\\
&= Z(Y_c) && (3.28)
\end{aligned}$$

$$Y_c = (Y \text{ and } Y' \text{ glued along } W \text{ with orientation reversal of } Y') \tag{3.29}$$



General anomaly formula:

$$\frac{Z(Y,\mathsf{L})}{Z(Y',\mathsf{L})} = Z(Y_c) \tag{3.30}$$

If $Z(Y_c) = 1$ for all closed manifolds, $Z(Y,\mathsf{L})$ can be used as a definition of the partition function of χ,

$$Z_\chi(W) := Z(Y,\mathsf{L}) \quad \text{if } Z(Y_c) = 1 \text{ for any } Y_c. \tag{3.31}$$

(Some important details omitted.)

3.3 Bulk theory: invertible field theory, SPT phase

Invertible field theory :

1. $\dim \mathcal{H}_W$=1 (only the ground state) in low energy limit on any closed W ($\partial W = \varnothing$).
2. $Z(Y_c)$ may be nontrivial.

Massive fermion $\to$ invertible theory when $|m| \to \infty$.

The bulk $d+1$-dim. partition function for Y_c, $\partial Y_c = \varnothing$.
For simplicity take $m = -M$ and $M \to +\infty$. Dirac fermion.

$$Z(Y_c) = \frac{\det(\not{D} + m)}{\det(\not{D} + M)} \tag{3.32}$$

$$= \prod_{\lambda \in \text{eigenvalue}(i\not{D})} \frac{(-i\lambda - M)}{(-i\lambda + M)} \tag{3.33}$$

$$= \prod_{\lambda} e^{-2\pi i s(\lambda)} \tag{3.34}$$

Here

$$s(\lambda) = -\frac{1}{2\pi} \arg\left(\frac{-i\lambda - M}{-i\lambda + M}\right) \tag{3.35}$$

$$-\pi \leq \arg < \pi. \tag{3.36}$$

Note

$$\lim_{M\to\infty} s(\lambda) = \frac{1}{2}\text{sign}(\lambda). \qquad (\text{sign}(\lambda = 0) \overset{\text{def}}{=} +1) \tag{3.37}$$

APS η-invariant

$$\eta = \left(\frac{1}{2}\sum_{\lambda} \text{sign}(\lambda)\right)_{\text{reg}} = \lim_{M\to\infty}\left(\sum_{\lambda} s(\lambda)\right) \tag{3.38}$$

The theory with $m < 0$ is nontrivial.

$$Z(Y_c) = \exp(-2\pi i \eta) \tag{3.39}$$

The theory with $m > 0$: trivial. Take $m = M$,

$$Z(Y_c) = \frac{\det(\not{D} + M)}{\det(\not{D} + M)} = 1. \tag{3.40}$$

	$m < 0$	$m > 0$
edge	localized χ	no localized mode
bulk	$Z(Y_c) = \exp(-2\pi i\eta)$	$Z(Y_c) = 1$

(3.41)

For Majorana fermion

$$Z(Y_c) = \frac{\text{Pf}(\not{D} - M)}{\text{Pf}(\not{D} + M)} = \exp(-\pi i \eta) \tag{3.42}$$

one Dirac = two Majorana.

A freedom to modify $Z(Y)$:

$$Z(Y) \to Z(Y) \exp(-\int \mathcal{L}_{\text{c.t.}}) \tag{3.43}$$

$$\mathcal{L}_{\text{c.t.}} : \text{manifestly gauge invariant counterterm} \tag{3.44}$$

$\int_Y \mathcal{L}_{\text{c.t.}}$ must be gauge invariant even if $\partial Y \neq \varnothing$.
E.g. Chern-Simons is not gauge invariant on manifolds with boundary, and should not be included in $\mathcal{L}_{\text{c.t.}}$.

Invertible field theory up to counterterms
= symmetry protected topological (SPT) phase with a given symmetry
= anomaly of the boundary theory.

4 Examples

4.1 $d = 3$, parity anomaly

$d = 3$, Lie group $\mathrm{U}(N)$ in the fundamental rep. $\mathbf{N}$.
Take e.g. $Y_c = T^4$ or S^4.

$$\eta = \left(\frac{1}{2} \sum_{\lambda} \mathrm{sign}(\lambda) \right)_{\mathrm{reg}} \tag{4.1}$$

$$= \text{ zero modes } \frac{1}{2}(n_+ + n_-) \text{ on } Y_c \tag{4.2}$$

$$\text{(Nonzero modes cancel in this example: Exercise)} \tag{4.3}$$

Here

$$n_+ \; : \text{the number of positive chirality zero modes} \tag{4.4}$$

$$n_- \; : \text{the number of negative chirality zero modes} \tag{4.5}$$

$$\eta = \frac{1}{2}(n_+ - n_-) \quad \mathrm{mod}\ \mathbb{Z} \tag{4.6}$$

$$= \frac{1}{2} \int_{Y_c} \frac{1}{8\pi^2} \mathrm{tr}\, F^2 \qquad \text{(Atiyah-Singer index theorem)} \tag{4.7}$$

- Without time-reversal, $-2\pi\eta$ is cancelled by counterterm $\mathcal{L}_{\mathrm{c.t.}} = \pi \frac{1}{8\pi^2} \mathrm{tr}\, F^2$.
- With time-reversal, $\pi \frac{1}{8\pi^2} \mathrm{tr}\, F^2$ is not manifestly gauge invariant:

$$\epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\, F_{\mu\nu} F_{\rho\sigma} \xrightarrow{\text{time-reversal}} -\epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\, F_{\mu\nu} F_{\rho\sigma} \tag{4.8}$$

  The counterterm cannot be used.
  Parity anomaly. (Topological insulator in $d = 3 + 1$ for $\mathrm{U}(1)$.)

4.2 $d = 4$, $\mathrm{SU}(2)$ Witten anomaly

$d = 4$ $\mathrm{SU}(2)$, Weyl in rep. R
No perturbative anomaly (exercise)

Take

$$Y_c = S^1 \times S^4 \tag{4.9}$$

$$S^4 : \text{ contains an SU(2) instanton} \tag{4.10}$$

$$S^1 : \text{ periodic b.c.} \tag{4.11}$$

$$\eta = \frac{1}{2}\sum \mathrm{sign}(\lambda) \qquad (\mathrm{sign}(0) = +1) \tag{4.12}$$

$$= \text{ zero modes } \frac{1}{2}(n_+ + n_-) \text{ on } S^4 \tag{4.13}$$

$$\text{(Nonzero modes do not contribute in this example: Exercise)} \tag{4.14}$$

$$= \frac{1}{2}(n_+ - n_-) \quad \mathrm{mod}\ 1 \tag{4.15}$$

$$= \frac{1}{2}N_R \qquad N_R : \text{ AS index of rep. } R \text{ on an instanton} \tag{4.16}$$

$$\exp(-2\pi i\eta) = (-1)^{N_R} : \text{ Witten SU(2) anomaly} \tag{4.17}$$

For $R = n$-dim. rep.

$$N_R = \frac{1}{6}n(n^2 - 1). \tag{4.18}$$

2-dim. rep. (fundamental of SU(2)) $N_R = 1$: anomalous.

4.3 $d = 1$ fermions revisited

$$\text{boundary: } S \to -\frac{1}{2}\int d\sigma\, \psi_i \frac{d}{d\sigma}\psi_i. \tag{4.19}$$

$$\text{bulk: } S = -\frac{1}{2}\int d\tau d\sigma \sum_i \Psi_i^T \epsilon(\gamma^\tau\partial_\tau + \gamma^\sigma\partial_\sigma + m)\Psi_i, \tag{4.20}$$

Let's compute bulk η on orientable spin manifolds.

$$i\not{D} = i\gamma^\mu D_\mu \qquad (x^1 = \tau, x^2 = \sigma). \tag{4.21}$$

Define

$$\overline{\gamma} = i^{-1}\gamma^1\gamma^2. \tag{4.22}$$

$$i\not{D}\Psi = \lambda\Psi \Longrightarrow i\not{D}\overline{\gamma}\Psi = -\lambda\Psi : \ (\lambda, -\lambda) \text{ pair} \tag{4.23}$$

$$\eta = \frac{1}{2}\sum_\lambda \mathrm{sign}(\lambda) \qquad (\mathrm{sign}(0) = +1) \tag{4.24}$$

$$= \frac{1}{2}(n_+ + n_-) \qquad (n_\pm : \text{ the numbers of positive, negative zero modes}) \tag{4.25}$$

Real basis:

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.26}$$

$$D_\mu : \text{ covariant derivative for O}(N) \text{ and spin connection, real } (D_\mu)^* = D_\mu \tag{4.27}$$

$$i\not{D}\Psi = 0 \Longrightarrow i\not{D}\Psi^* = 0 \tag{4.28}$$
$$\overline{\gamma}\Psi^* = (\overline{\gamma}^*\Psi)^* = -(\overline{\gamma}\Psi)^* \qquad (\overline{\gamma} = i^{-1}\gamma^1\gamma^2 \text{ : pure imaginary}) \tag{4.29}$$

$$\Psi \text{ positive chirality} \Longleftrightarrow \Psi^* \text{ negative chirality} \tag{4.30}$$

$n_+ = n_-$. Thus

$$\eta = n_+ \tag{4.31}$$

Anomaly for majorana:

$$\exp(-\pi i\eta) = (-1)^{n_+}. \tag{4.32}$$

$$n_+ \quad \text{mod } 2 \text{ : called mod 2 index.} \tag{4.33}$$

Example 1:

$$Y = T^2 \tag{4.34}$$
$$g = \mathrm{diag}(-1, 1, \cdots, 1) \in \mathrm{O}(N). \tag{4.35}$$
$$\Psi(\tau+1, \sigma) = -g\Psi(\tau, \sigma), \qquad \Psi(\tau, \sigma+1) = \Psi(\tau, \sigma). \tag{4.36}$$
$$\not{D} = \gamma^\mu\partial_\mu. \tag{4.37}$$

Zero modes:

$$\Psi_1(\tau, \sigma) = \text{const.}, \qquad \Psi_j(\tau, \sigma) = 0 \quad (j \geq 2). \tag{4.38}$$

Ψ_1 has two components $\overline{\gamma} = \pm 1 \Longrightarrow n_+ = n_- = 1$.

$$(-1)^{n_+} = -1 \text{ : anomaly of } \mathrm{O}(N) \tag{4.39}$$

Example 2:

$$Y = S^2 \tag{4.40}$$
$$\mathrm{U}(1) = \mathrm{SO}(2) \subset \mathrm{SO}(N), \quad \Psi_\pm := \Psi_1 \pm i\Psi_2 \text{ : charge } \pm 1, \text{ others } 0. \tag{4.41}$$
$$\mathrm{U}(1) \text{ gauge field } A = \frac{1}{2}i(\cos\theta \pm 1)d\phi, \quad \int_{S^2} \frac{iF}{2\pi} = 1 \tag{4.42}$$
$$\tag{4.43}$$

Ψ_+ has one zero mode with $\overline{\gamma} = +1$, Ψ_- has one zero mode with $\overline{\gamma} = -1$.
$\Longrightarrow n_+ = n_- = 1$.

$$(-1)^{n_+} = -1 \text{ : anomaly of } \mathrm{SO}(N) \tag{4.44}$$