QFT in AdS from Hamiltonian Truncation

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based on 2104.10689 with Marco Meineri (CERN),
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QFT in AdS - why?

- Anti-de Sitter space is a fundamental spacetime in study of quantum gravity (strings/holography/SUGRA/…).
- In full-fledged quantum gravity on AdS_{d+1} there's matter + dynamical gravity (fluctuating $g_{\mu\nu}$) in the bulk.
  Exactly equivalent to local (including $T_{\mu\nu}$) CFT living on the boundary of spacetime.
- Simple version: put matter in the bulk, but freeze $g_{\mu\nu}$ to be empty AdS - no backreaction.
  Still maps to a CFT on the boundary, but without $T_{\mu\nu}$.
- This turns spacetime into a infinite box invariant under $SO(d,2)$ with a finite curvature radius R_{AdS}. Should be "large" compared to whatever you want to study.

QFT in AdS (2)

- Keeping everything else fixed, $R \to \infty$ is the flat-space limit of AdS physics, curvature effects unimportant (but boundary is always there!).
  When $R \to 0$ curvature dominates everything else
  (correlation length of system $\gg$ size of the universe).
- If QFT is controlled by relevant coupling λ of mass dimension $[\lambda] = y$, then physics depends on dim-less and tuneable coupling $\bar{\lambda} = \lambda R^y$.
  Same for several couplings $\bar{\lambda}_1, \bar{\lambda}_2, \ldots$
- Can think of boundary CFT as one/several-parameter family of consistent theories parametrized by the $\{\bar{\lambda}_\alpha\}$.
- Natural observable: spectrum = set of scaling dimensions Δ_i of the boundary CFT, $\Delta_i = f_i(\bar{\lambda}_\alpha)$. At large radius, they will scale as $\Delta_i \propto R$.

Current toolkit

- AdS perturbation theory is well-developed ("Witten diagrams")
- Exact solutions: integrable theories, Lagrangian theories with $N_f \to \infty$ flavors [Carmi-di Pietro-Komatsu 2018]
- Bootstrap methods: study boundary CFT via numerical bootstrap and deduce information about bulk physics [Paulos et al 2016]
- Latticize bulk QFT. Very non-trivial compared to $\mathbb{R}^4$!
  So far mostly construction of massive scalar in AdS_2.
  Work by Boston/Brown and Syracuse groups e.g. 1912.07606
- ...? Other nonperturbative methods undeveloped.

Quantizing AdS

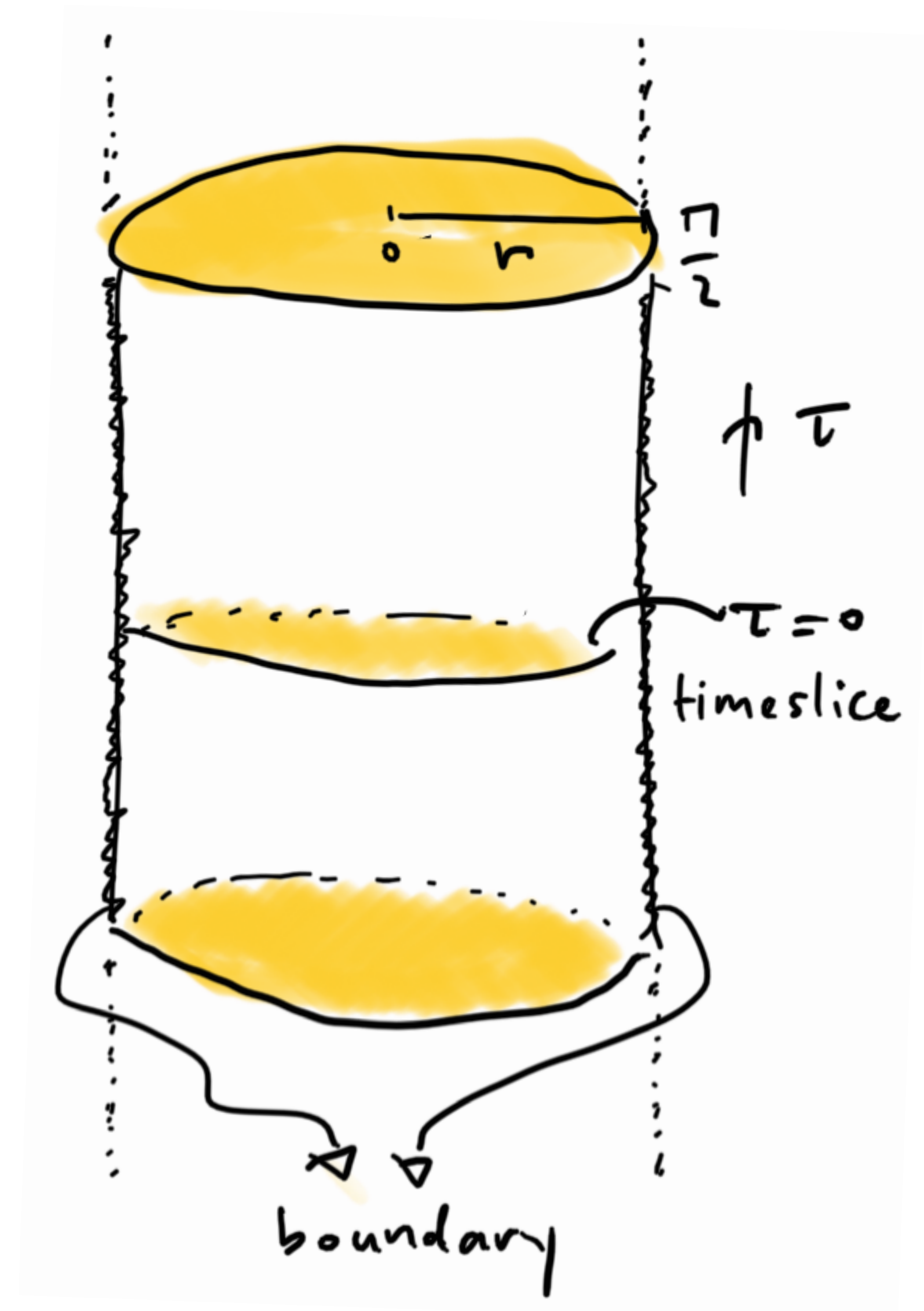
Will work in Hamiltonian picture; most convenient are global coordinates, where AdS is a foliation of solid disks (or closed intervals, in 1+1 dims):

$$ds^2 = \left(\frac{R}{\cos r}\right)^2 \left[d\tau^2 + dr^2\right]$$

with boundary at $r = \pm \pi/2$.

Energies are conserved. Hamiltonian $H = -\partial/\partial\tau$ is dilatation operator of the boundary CFT.

Isometry group is $SO(2,1) = SL(2,\mathbb{R})$. Beyond H, other two symmetries P, K mix τ and r.



Truncation methodology: Rayleigh-Ritz for QFT

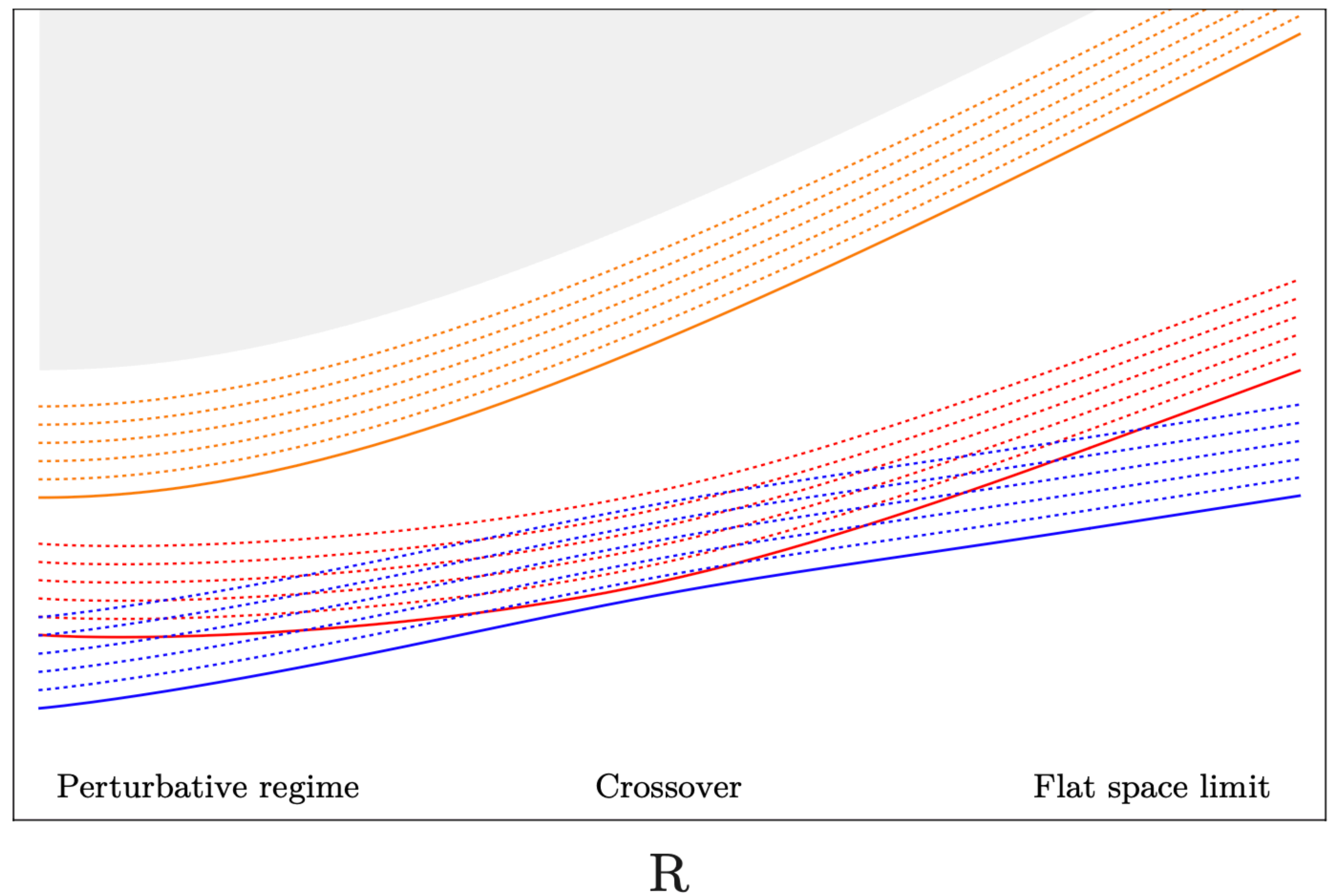
- Idea will be to split full Hamiltonian into a solvable (e.g. Gaussian) and a non-solvable part, $H = H_0 + \lambda V$. (Details to follow.)
- There's a by now well-tested recipe to compute QFT spectra:
  1. Fix cutoff Λ and find all states $|i\rangle$ with free energy $e_i \leq \Lambda$.
  2. Diagonalize $H(\lambda)$ in finite subspace of these low-energy states, yielding truncated energies $E_i(\lambda, \Lambda)$.
  3. Take $\lim_{\Lambda \to \infty} E_i(\lambda, \Lambda)$ to obtain the true energies $E_i(\lambda)$ of the interacting theory.
- Known in general as **Hamiltonian truncation**. Early variant described by Yurov & Al. Zamolodchikov in 1989, many developments since 2014. Common scheme known as Truncated Conformal Space Approach = TCSA.
- Hard cutoff breaks $SO(2,1)$. In the continuum limit $\Lambda \to \infty$ the full symmetry is supposed to be restored. Not obvious, needs to be checked.

Example: scalar field

- A massive particle in AdS corresponds to a boundary operator of dimension $$\Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2R^2}.$$
  Field $\phi(\tau, r)$ admits a mode decomposition
  $$\phi(\tau, r) = \sum_{n=0}^{\infty} a_n \, e^{-(\Delta+n)\tau} f_n(r) + \text{h.c.}$$
  and the free Hamiltonian is $H_0 = \sum_{n=0}^{\infty} (\Delta + n) a_n^\dagger a_n$.
- Fock space of states $\prod_i a_{n_i}^\dagger \, | \, \Omega \rangle$.
- Interactions are of the form $\lambda V = \lambda \int_{-\pi/2}^{\pi/2} \frac{dr}{(\cos r)^2} \mathscr{V}(\tau = 0, r)$
  with e.g. $\mathscr{V} = \phi^4$. Integral runs over $\tau = 0$ timeslice. Compute using CCR.

Expectations

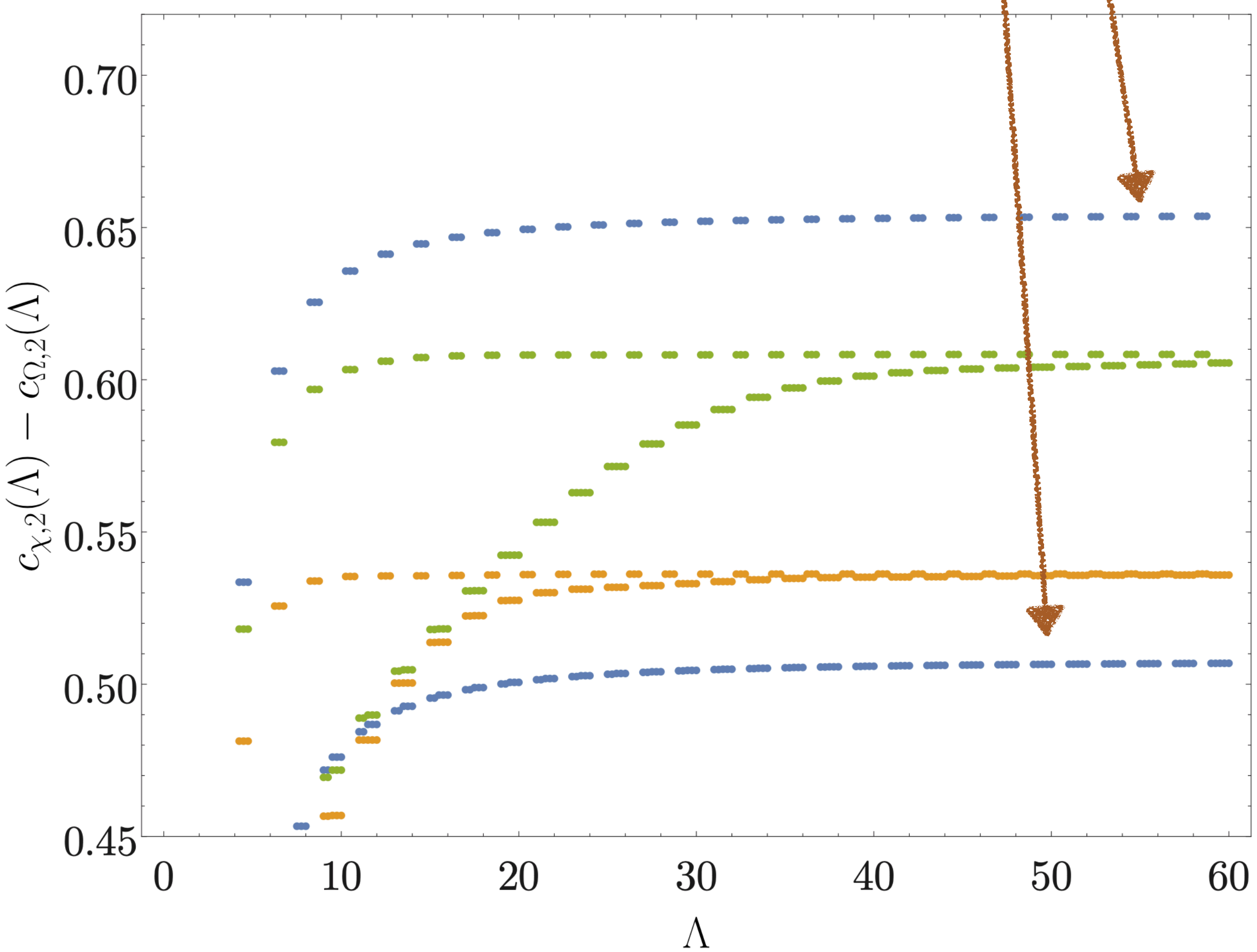
- Regardless of what one puts in, the spectrum should fall into multiplets of $SO(2,1)$: a "primary" state with energy $E_i(\lambda)$ and "descendants" with energies $E_i(\lambda) + 1$, $E_i(\lambda) + 2$, etc.



- The hard cutoff breaks spacetime symmetries, so instead probably $E_j - E_i \approx$ integer due to truncation error.
- Dim-less coupling $\bar{\lambda} = \lambda R^y$ can't be taken to be arbitrarily big due to cutoff effects.

A major issue involving UV divergences…

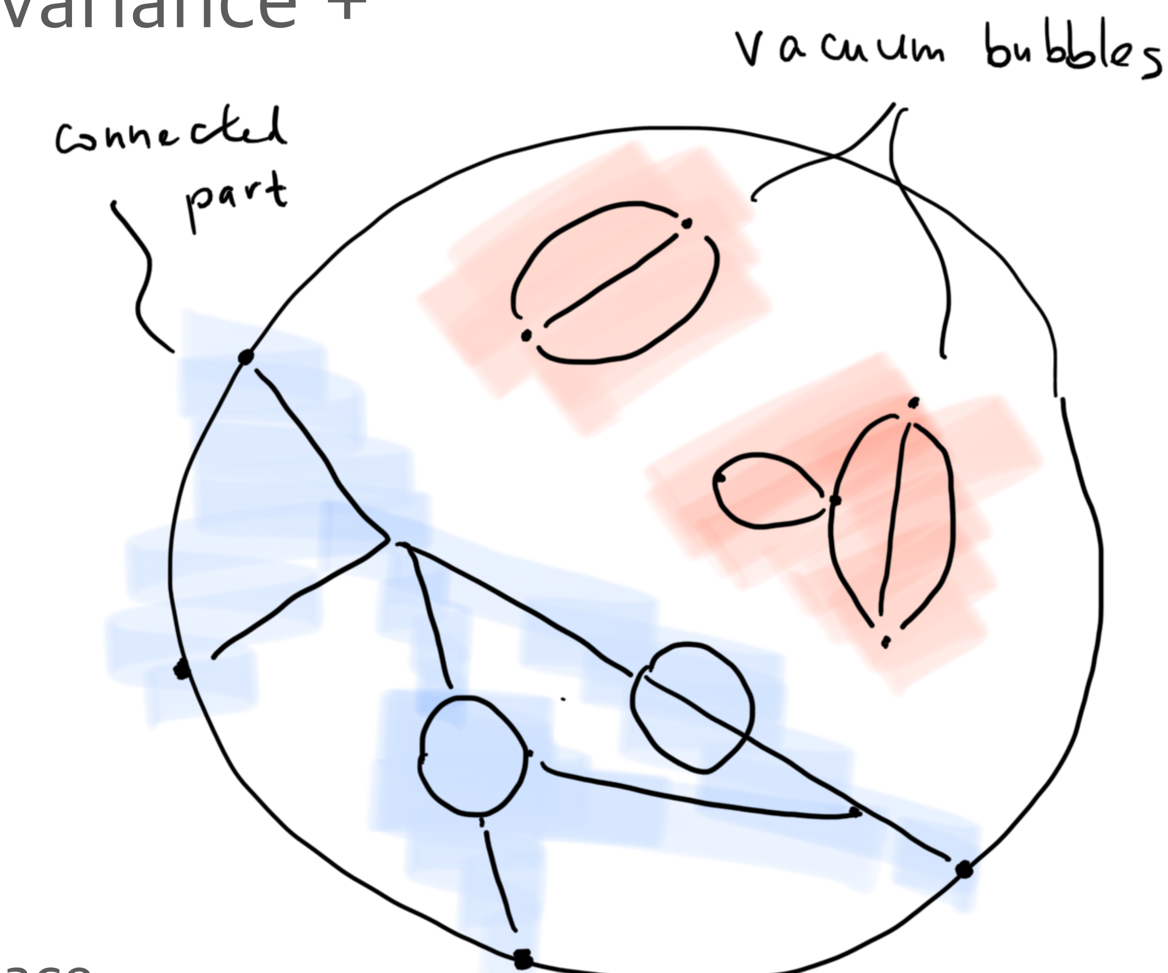
- Ground state energy $E_{\text{vac}}(\lambda, \Lambda)$ diverges (linearly) as $\Lambda \to \infty$.
  Happens for any QFT in AdS (proof: representation theory or spacetime behavior).
- … but confusingly, energy gaps don't seem to have a finite continuum limit either: $E_i(\lambda, \Lambda) - E_{\text{vac}}(\lambda, \Lambda)$ oscillates as $\Lambda \to \infty$. cf. blue lines
- Does this means that Hamiltonian truncation fails to construct a meaningful QFT in AdS?



…which is perhaps not that hard to solve.

In covariant perturbation theory, we know very well how to compute energies: simply subtract vacuum bubbles, just like in flat space.

Free energy has to diverge, due to covariance + infinite volume of AdS.
Meaning: free energy/Casimir energy just not a good observable.



(If there are intrinsic UV divergences like in flat space, need to treat those separately.)

Resolving the puzzle (2)

- Anomalous dimension from spacetime, e.g. at 2nd order in PT:

$$\delta E_i^{(2)} = -\int_0^\infty d\tau\,\left[g_i(\tau) - g_{\text{vac}}(\tau)\right], \quad g_i(\tau) = \langle i\,|\,V(\tau)V(0)\,|\,i\rangle_{\text{conn}}$$

where $V(\tau) = e^{H_0\tau}V(0)e^{-H_0\tau}$.

- Using Laplace transforms

package matrix elements $|\langle i|V|j\rangle|^2$

$$g_i(\tau) = \int_{\mathbb{R}} d\alpha\,\rho_i(\alpha)\,e^{-(\alpha-e_i)\tau}, \quad g_{\text{vac}}(\tau) = \int_{\mathbb{R}} d\alpha\,\rho_{\text{vac}}(\alpha)\,e^{-\alpha\tau}$$

this is instead

nb!

$$\delta E_i^{(2)} = -\int_{\mathbb{R}} \frac{d\alpha}{\alpha - e_i}\left[\rho_i(\alpha) - \rho_{\text{vac}}(\alpha - e_i)\right].$$

- Suggest that correct energies are obtained from following rule:

$$E_i^{\text{phys}}(\lambda) \equiv \lim_{\Lambda\to\infty} E_i(\lambda,\Lambda) - E_{\text{vac}}(\lambda, \Lambda - e_i)$$

- Need to put this prescription to the test.

Resolving the puzzle (3)

- Had to extend previous argument to all orders in perturbation theory. Logic always the same, and uses relation perturbed energies = integrated spectral densities of connected correlators in AdS.

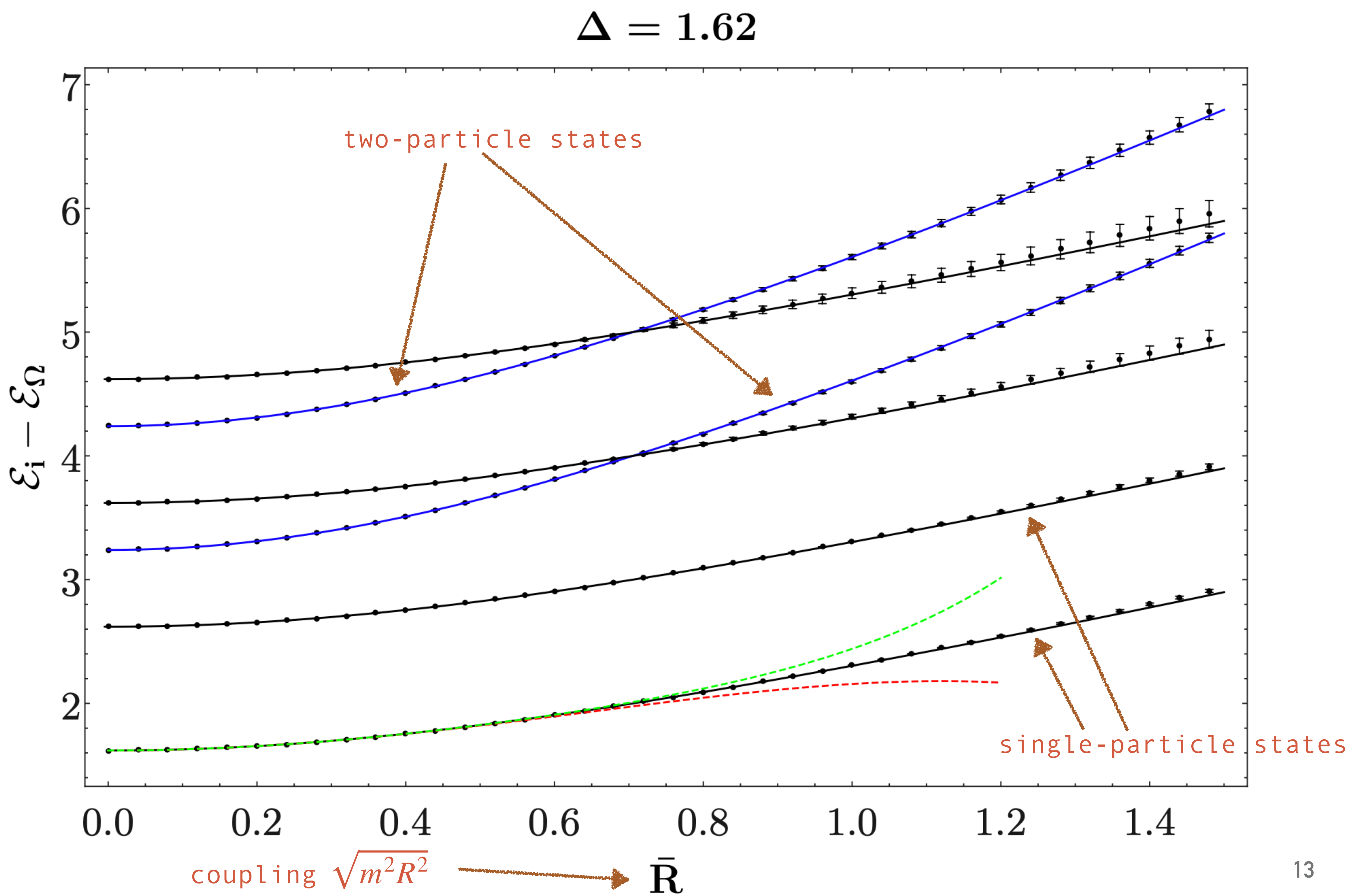
- Why did this shift $E_{\text{vac}}(\Lambda) \to E_{\text{vac}}(\Lambda - e_i)$ never show up before? Simply because here $E_{\text{vac}}(\Lambda) \sim \Lambda$ so very UV-sensitive. In UV-finite theories spectral densities $\rho_i(\alpha_1, \alpha_2, \ldots)$ decrease such that shift is immaterial:

$$E_{\text{vac}}(\Lambda) - E_{\text{vac}}(\Lambda - e_i) \approx e_i \frac{\partial}{\partial \Lambda} E_{\text{vac}}(\Lambda) \longrightarrow 0$$

- Can also derive prescription without appealing to covariant "wisdom". Introduce IR regulator $|r| \leq \pi/2 - \epsilon$ which makes theory manifestly finite. Then find unique way that yields good limit $\epsilon \to 0$.

- Recently [Elias-Miró & Hardy 2003.08405] found similar issue for specific model, ϕ^4 theory on $\mathbb{R} \times \mathbf{T}^2$. There resolved by adding specific counterterm to H. Is this the same prescription, in a very different-looking form?

First check: boson + $m^2\phi^2$ perturbation



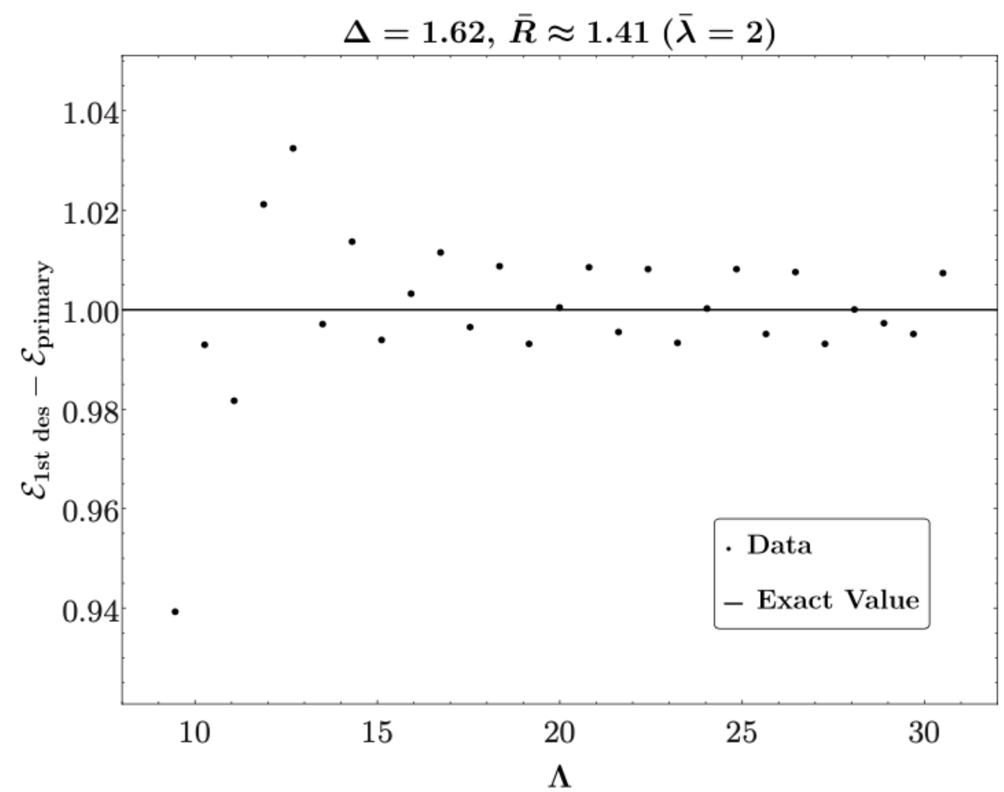
First check: boson + $m^2\phi^2$ perturbation (2)

- Numerical diagonalization in Mathematica.
- Truncation errors can be understood analytically, by looking at spacetime diagrams:

$$\text{truncation error of } E_i \approx \frac{c_i}{\Lambda^{\Delta_*-1}}$$

where c_i is a (computable) coefficient and Δ_* the dimension of the first boundary state turned on by perturbation V.

- Use this to extrapolate to $\Lambda \to \infty$ albeit with error bars.
- $SO(2,1)$ seems to be recovered:



ϕ^4 theory: what to expect?

- $m^2\phi^2 + \lambda\phi^4$ theory in flat space in $d = 2{,}3$ has two phases, depending on the ratio of couplings $\lambda/(m^2)^{\frac{1}{2}(4-d)}$:
  $\mathbb{Z}_2$-preserving with $\langle\phi\rangle = 0$ vs. $\mathbb{Z}_2$-broken with $\langle\phi\rangle \neq 0$.
  2nd order transition = Ising CFT in the middle.
  Studied both nonperturbatively and by resumming PT.

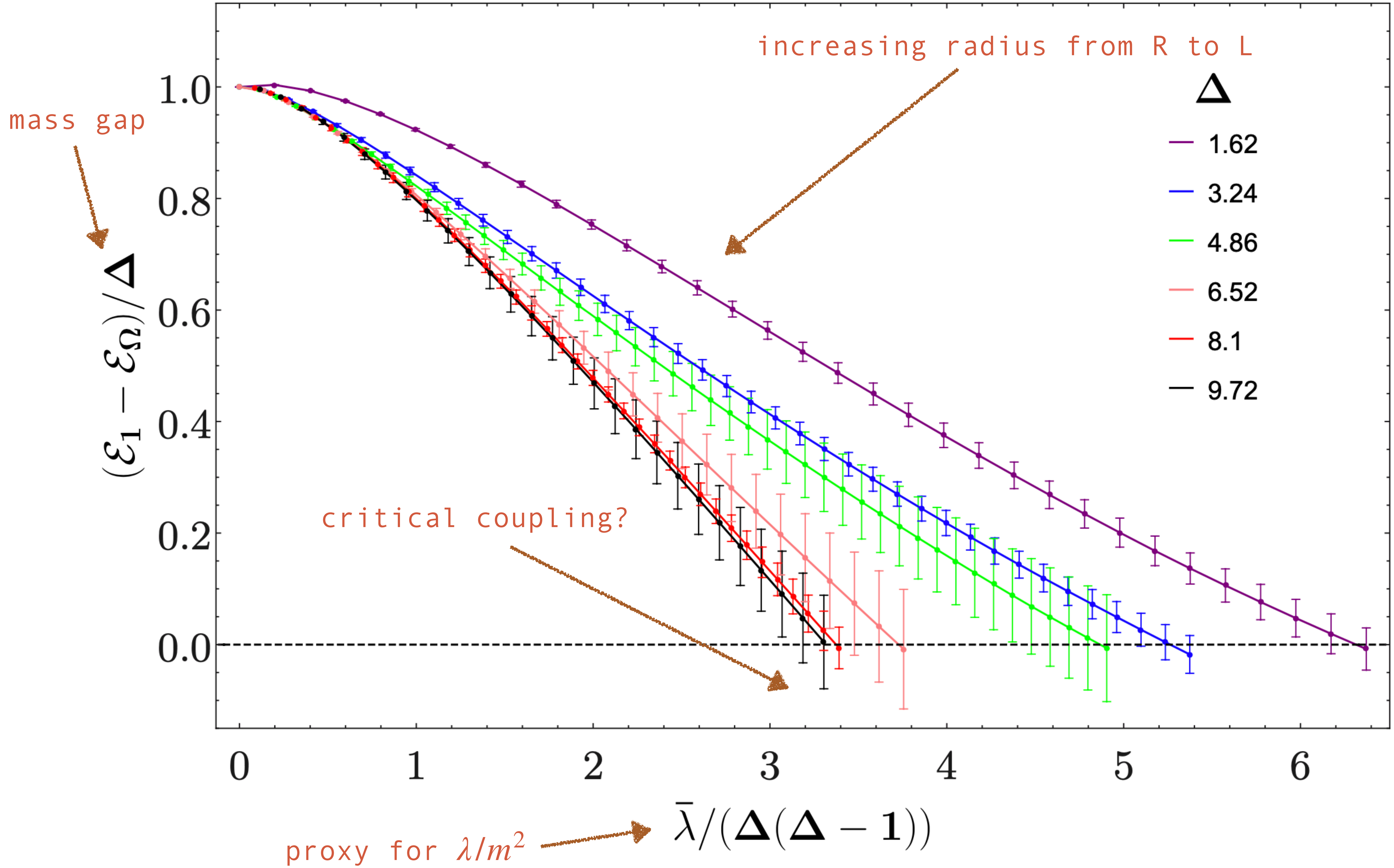
- Pheno of spontaneous symmetry breaking different in AdS$_{d+1}$ vs flat space. Semiclassics:

  - Flat space: end up in global minimum of potential $V(\phi)$, hence 2nd order.

  - AdS: can stay stuck in false vacuum due to curvature/boundary effects.
    Only certain to decay when $V''(0) < -\frac{1}{4}d^2$ (cf. BF bound).
    Favors 1st-order phase transitions.

- Interesting limit is $\lambda R^2 \to \infty$ and $m^2R^2 \to \infty$ with λ/m^2 fixed.
  Strong coupling limit! Can try to attack using HT…

ϕ^4 interaction - results



Playing with Virasoro CFTs

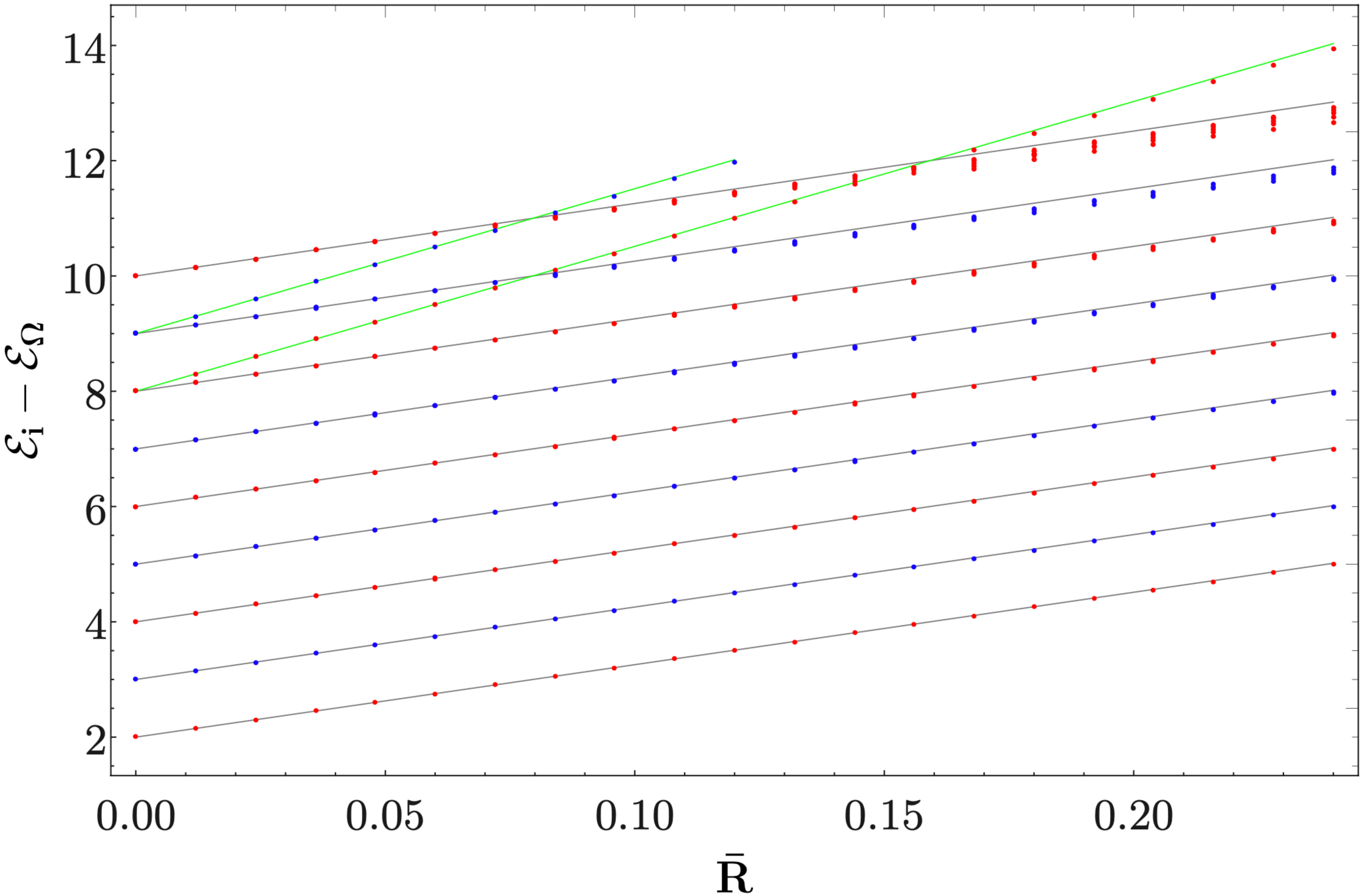
- Can play the same game by deforming a CFT in AdS_2:
$$S = S_{\text{CFT}} + \lambda \int_{\text{AdS}} \sqrt{g}\, d^2x\, \mathcal{O}(x)$$
so $[\lambda] = 2 - \Delta_{\mathcal{O}}$.
Source of many well-studied RG flows in flat space or on cylinder (Ising field theory, Lee-Yang flow, …).

- Related to boundary TCSA, but AdS is **not** the upper half plane/strip (different Weyl factor).

- Technologically different: BCFT Hilbert space with Virasoro generators $\{L_n\}$ instead of Fock space with $\{a_n^\dagger\}$.
Yet no fundamental differences with previous discussion.
Same recipe needed to regulate UV divergences!

2d Ising model: thermal deformation

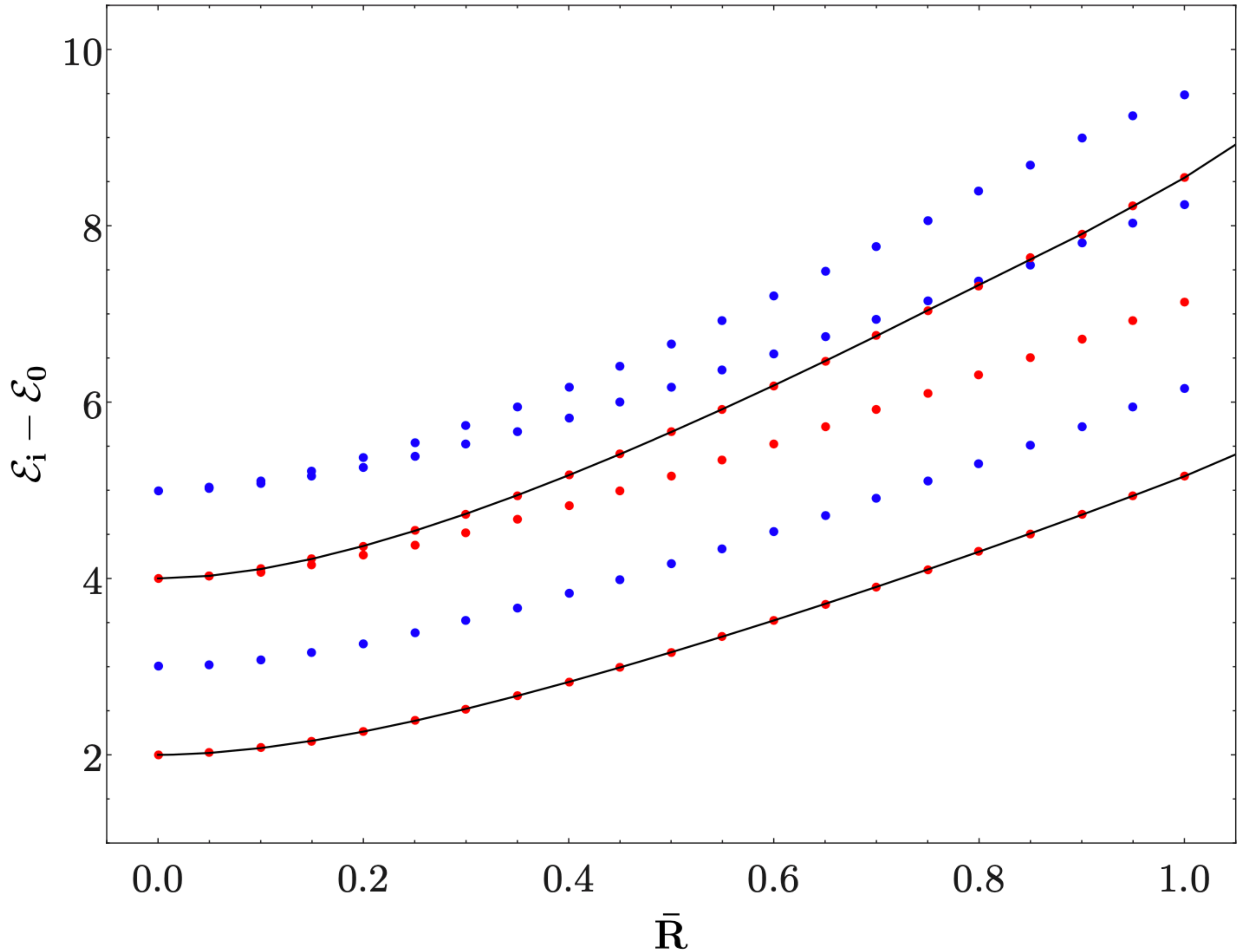
Here $S =$ Ising model $+ \lambda \int \epsilon$. Should be equal to free Majorana with $m = 2\pi\lambda$:



2d Ising model: magnetic deformation

Now $S =$ Ising model $+ \lambda \int \sigma$. Integrable on flat background:

8 massive particles with universal ratios m_{i+1}/m_i . HT data in the right "ballpark":



A speculation/conjecture (1)

- Finally, let's state a provocative conjecture inspired by old results and our new work:

  All CFT boundary conditions are connected by relevant deformations in AdS.

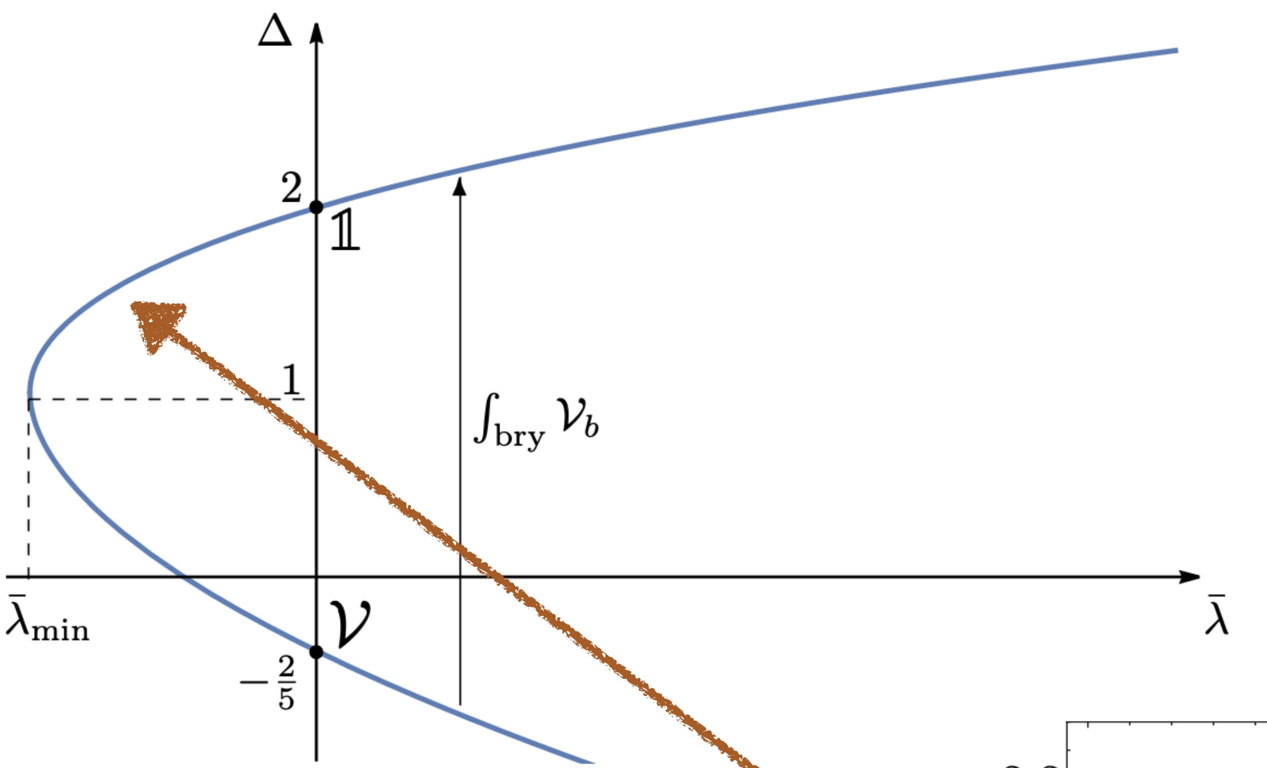
  This is a falsifiable conjecture, especially for CFTs with known set of BCs.

- Most basic example: massless scalar has two BCs (Dirichlet & Neumann). Can be extended in AdS to roots of $\Delta(\Delta - d) = m^2R^2$ (deforming by ϕ^2 in bulk). They meet when $\Delta = \frac{d}{2}$ that is to say $m^2R^2 = -\frac{d^2}{4} < 0$.

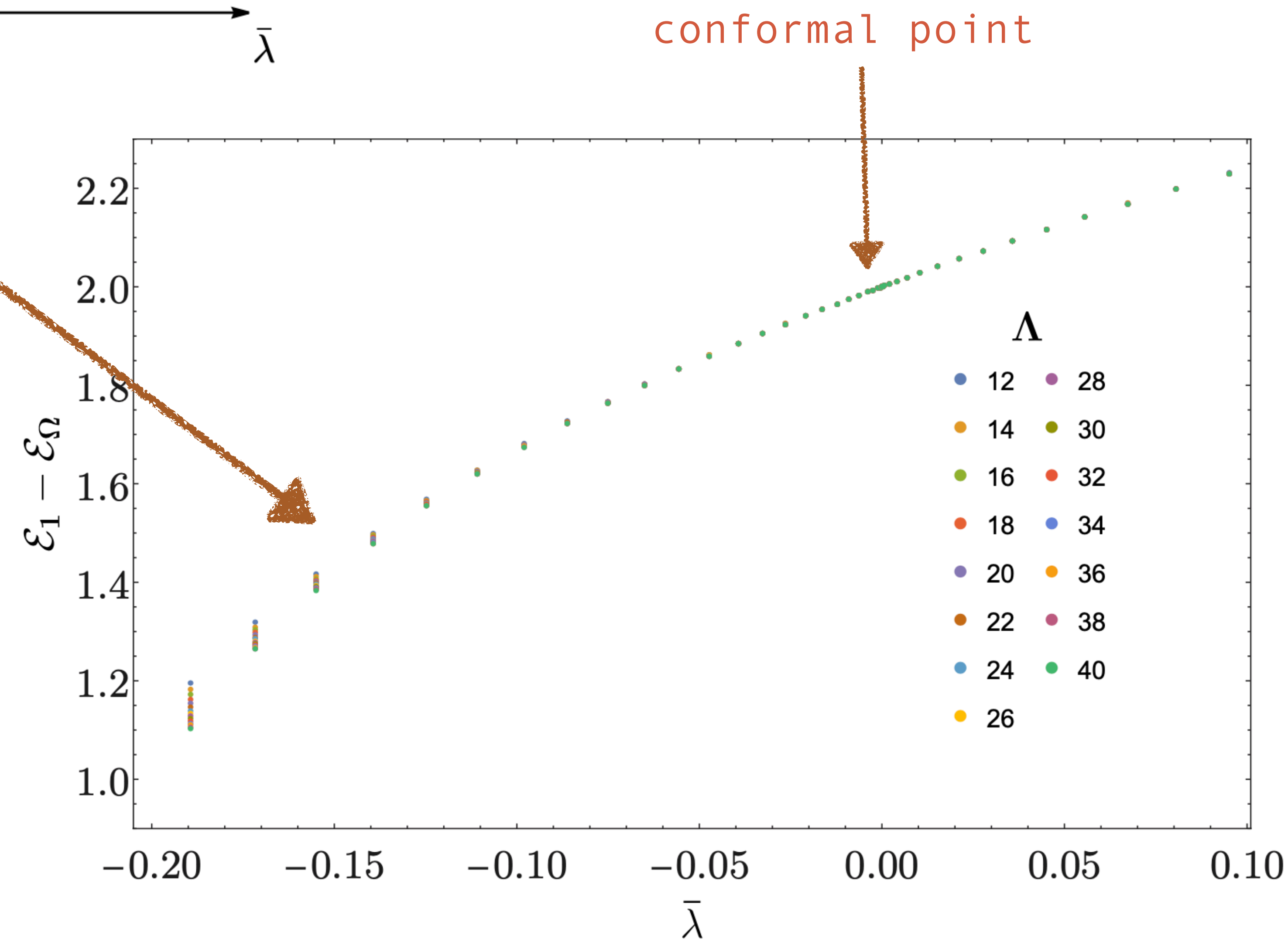
- When they meet the first singlet boundary operator has $\Delta = d$.
  This is natural: a singlet with $\Delta \leq d$ destabilizes one of the BCs.
  In our setup: expect two BCs to meet when spectrum contains $\Delta = 1$.

- Another generic feature: need to go to negative coupling to see this phenomenon.
  Remember: slightly negative coupling not necessarily a problem in AdS.

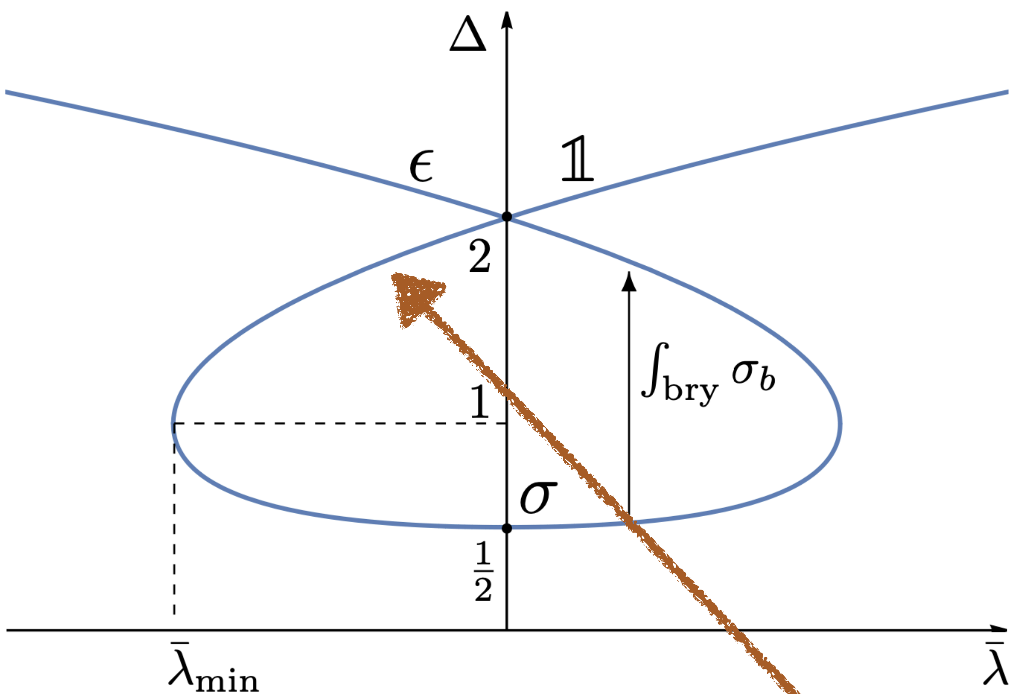
A speculation (2): Lee-Yang flow



Lee-Yang BCFT has two BCs: $\mathbb{1}$ and $\mathscr{V}$.
In HT computations, we used $\mathbb{1}$.
Impossible to go more negative:
hallmark of instabilities
(harmonic oscillator w/ negative mass)

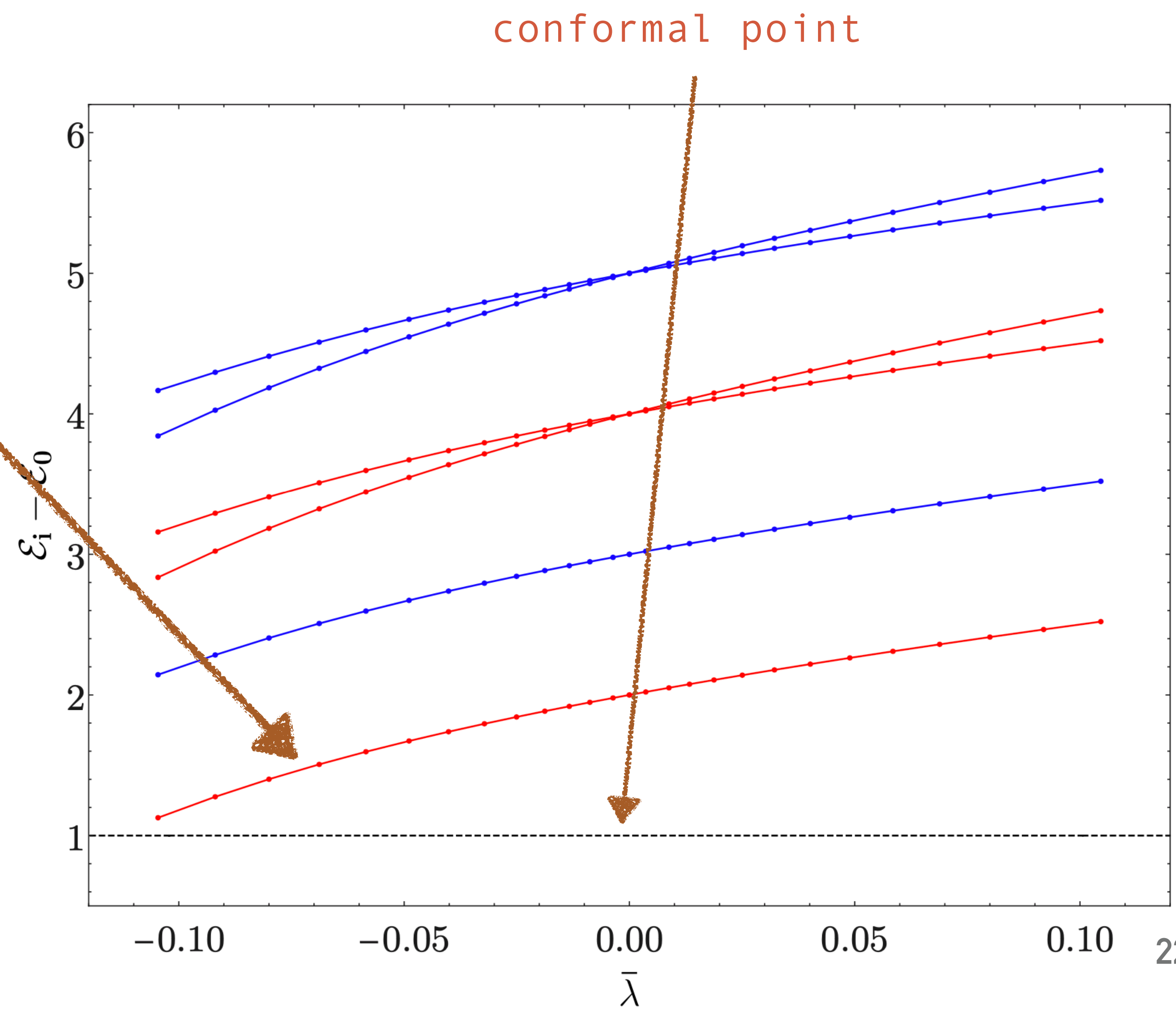


A speculation (3): Ising CFT + magnetic



Ising BCFT has 3 BCs: 1, σ and ϵ.

In HT computations, we used 1.



Summary/outlook

- Possible to study QFT in AdS directly using Hamiltonian methods. Good resolution with laptop/simple cluster runs.
- Explored for both deformations of massive boson and simple minimal model CFTs in AdS_2.
- Renormalization needed to extract finite physics from divergent raw data. Qualitatively different from same setup on cylinder $\mathbb{R} \times S^1$ or strip $\mathbb{R} \times [0,1]$. Prescription appears justified.
- Not yet in precision era. One has to add study large-energy asymptotics in more detail to reduce cutoff dependence.
- Formulated a speculation that should be easy to falsify! 🙂