SYMBOLIC DYNAMICS FOR NONUNIFORMLY HYPERBOLIC SYSTEMS

ICTP, 2021 LECTURE 1

GOALS: For nonuniformly hyperbolic systems:

1. Identify how to measure nonuniform hyperbolicity, and what are the good non-uniformly hyperbolic points.

2. Implement Bowen's approach on Markov partitions.

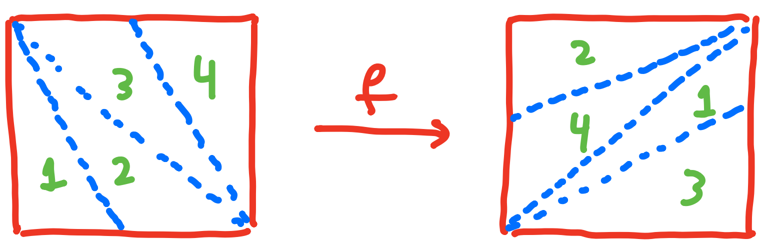
NOTATION: UH → uniformly hyperbolic

NUH → nonuniformly hyperbolic

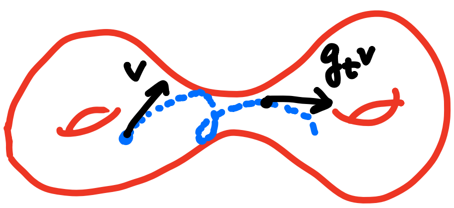
EXAMPLES

1. $f: \mathbb{T}^2 \to \mathbb{T}^2$, $f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

UH diffeo

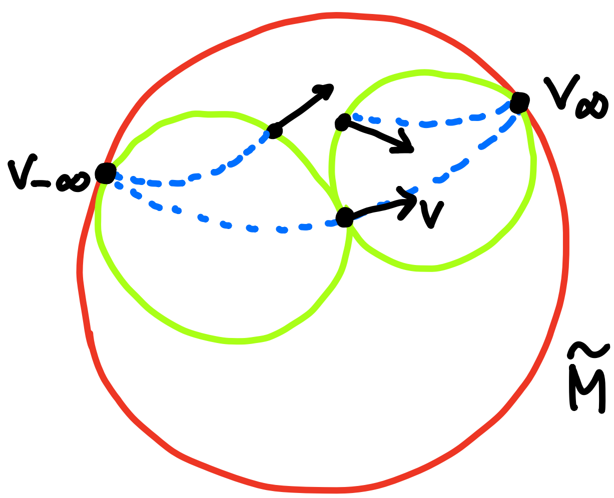


2. Geodesic flows in negative curvature



M = closed manifold with sectional curv. < 0

$$g = \{g_t\}_{t \in \mathbb{R}} : T^1 M \to T^1 M$$

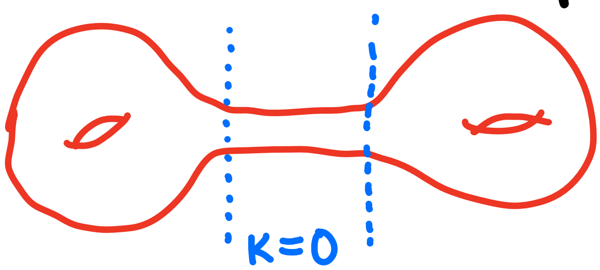


UH flow

3. $f : M^2 \to M^2$ diffeo with $h_{top}(f) > 0$.

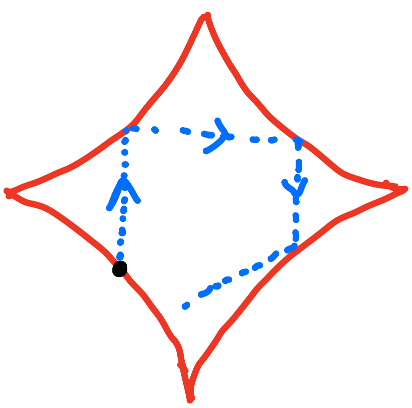
Ruelle inequality $\longrightarrow$ NUH diffeo

4. Geodesic flow in nonpositive curvature

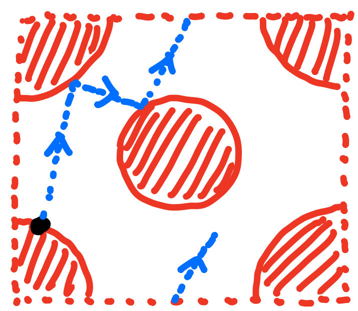


NUH flow

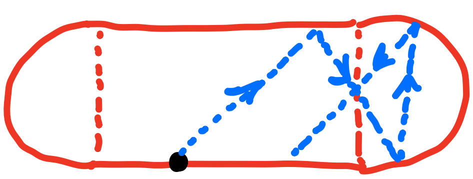
5. Billiards



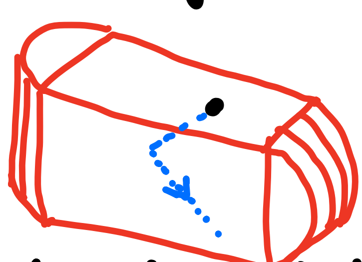
Dispersing (UH)



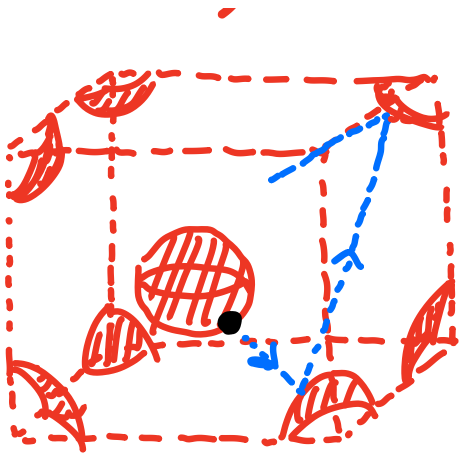
Dispersing (UH)



Bunimovich stadium (NUH)



Bunimovich stadium in dim $= 3$ (NUH)



Dispersing in dim = 3 (UH)

MAIN RESULTS: represent NUH models above by symbolic models

$$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

- (Σ, σ) = topological Markov shift

  $\Sigma = \Sigma(\mathcal{G})$, where $\mathcal{G}$ = oriented graph with countably many vertices

  $\sigma : \Sigma \to \Sigma$ left shift, $\sigma[\{v_n\}] = \{v_{n+1}\}$.

- $\pi : \Sigma \to M$ coding map $\begin{cases} \text{Hölder continuous} \\ \& \\ \text{"finite-to-one"} \end{cases}$

Important to preserve entropy

APPLICATIONS

1. Measures of maximal entropy (MME):
   (Sarig, Lima-Sarig, Ben Ovadia, Buzzi-Crovisier-Sarig, ...)

   - #MME at most countable

   - Transitive C^∞ surface diffeo with $h_{top}(f) > 0$: unique MME

2. Ergodic properties of MME:

   (Sarig, Ledrappier-Lima-Sarig)

   The MME is either Bernoulli or Bernoulli $\times$ rotation.

3. Periodic points: If $\exists$ MME, then:
(Sarig, Lima-Sarig, Buzzi, ...)

- Maps: $\mathrm{Per}_n(f) \geq \text{const.}\, e^{hn}$
- Flows: $\mathrm{Per}_T(\varphi) \geq \text{const.}\, \dfrac{e^{hT}}{T}$

4. Decay of correlations
(Buzzi-Crovisier-Sarig)

5. Hyperbolic SRB measures (Ben Ovadia)

UNIFORMLY HYPERBOLIC SYSTEMS

Let $\begin{cases} M^2 = \text{closed } C^\infty \text{ surface} \\ f: M \to M \quad C^{1+\beta} \text{ Anosov diffeo} \end{cases}$

- $TM = E^s \oplus E^u$, $E^\sigma_x = \langle e^\sigma_x \rangle$, $\sigma = s, u$
- $\langle \cdot, \cdot \rangle$ inner product of M, $\|e^\sigma_x\| = 1$.

Lyapunov inner product

We introduce a new inner product adapted to f:

λ<1 weaker than expansion/contraction

- $v_1^s, v_2^s \in E^s$:

$$⫴ v_1^s, v_2^s ⫴ = 2\sum_{n\geq 0} \lambda^{-2n} \langle df^n v_1^s, df^n v_2^s \rangle$$

- $v_1^u, v_2^u \in E^u$:

$$⫴ v_1^u, v_2^u ⫴ = 2\sum_{n\geq 0} \lambda^{-2n} \langle df^{-n} v_1^u, df^{-n} v_2^u \rangle$$

- $v^s \in E^s$, $v^u \in E^u$:

$$⫴ v^s, v^u ⫴ = 0.$$

Properties:

- $|||df\, v^s||| < \lambda |||v^s|||$

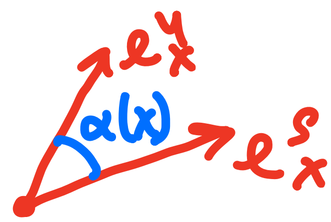
  $|||df^{-1} v^u||| < \lambda |||v^u|||$

(Adapted metric)

- $|||e_x^s|||, |||e_x^u||| \in [\mathcal{L}^{-1}, \mathcal{L}]$, $\forall x \in M$.

Hyperbolicity parameters $s(x), u(x), \alpha(x)$:

- $s(x) = |||e_x^s||| = \sqrt{2}\left(\sum_{n\geq 0} \lambda^{-2n} \|df^n e_x^s\|^2\right)^{1/2}$
- $u(x) = |||e_x^u||| = \sqrt{2}\left(\sum_{n\geq 0} \lambda^{-2n} \|df^{-n} e_x^u\|^2\right)^{1/2}$
- $\alpha(x) = \measuredangle(e_x^s, e_x^u)$



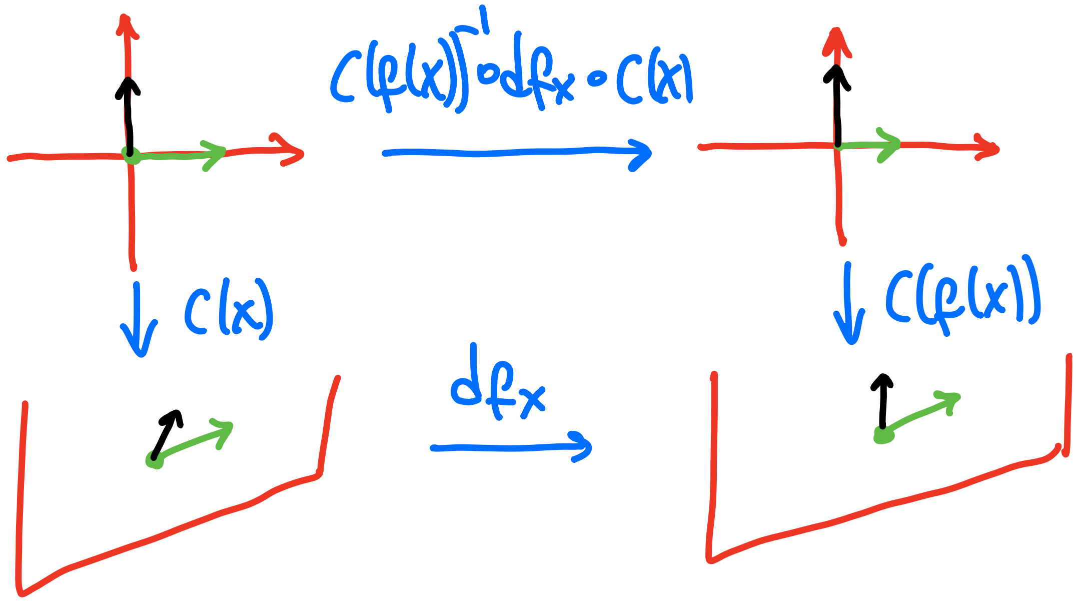
Diagonalizing df:

$C(x): \mathbb{R}^2 \to T_xM$ linear s.t.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \frac{e_x^s}{s(x)}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \frac{e_x^u}{u(x)}$$

Properties:

- $C(f(x))^{-1} \circ df_x \circ C(x) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $\quad |A| < \lambda$, $|B^{-1}| < \lambda$



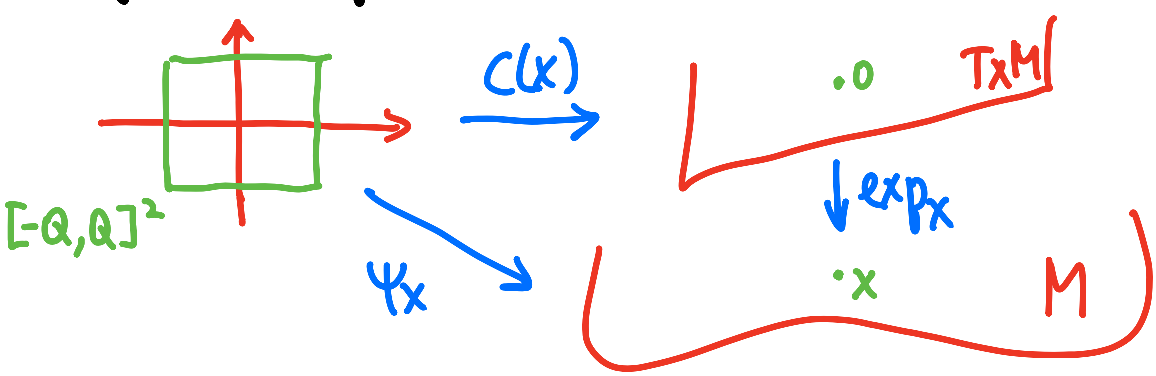
- $\|C^{\pm 1}(x)\| \in [\mathcal{L}^{-1}, \mathcal{L}]$, $\forall x \in M$.

Lyapunov charts:

Compose C with exponential map:

$\Psi_x : [-Q,Q]^2 \to M$,

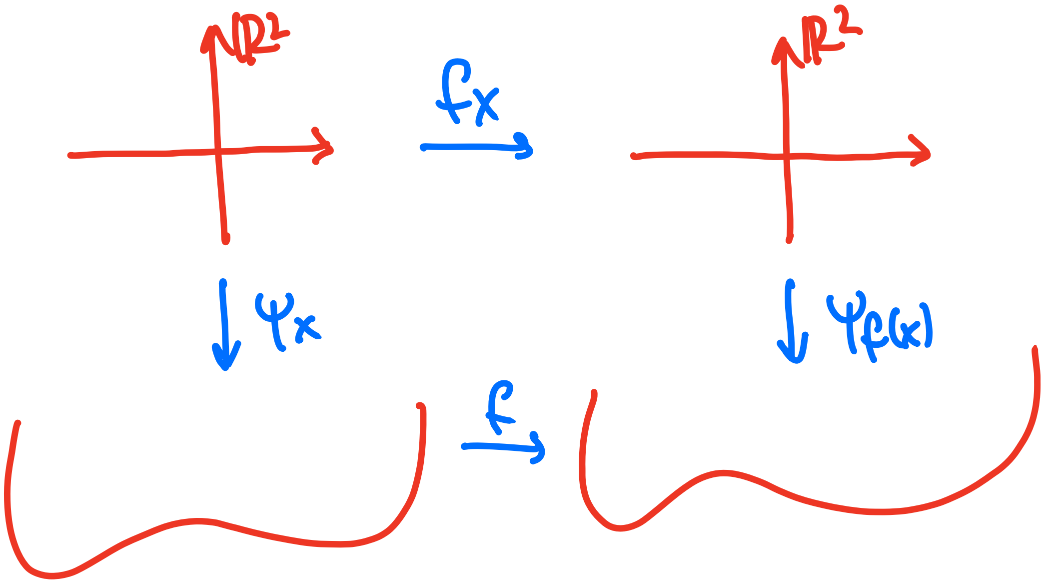
$\Psi_x = \exp_x \circ C(x)$



Above : $Q <$ injectivity radius (take: $Q = \varepsilon^{3/\beta}$)

Now represent f in Lyapunov charts:

$$f_x = \Psi_{f(x)}^{-1} \circ f \circ \Psi_x$$



<u>THM.</u>

$$f_x = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

where :

- $|A| < \lambda$, $|B^{-1}| < \lambda$ as above.
- $h_i(0,0) = 0$, $\nabla h_i(0,0) = 0$, $i = 1,2$.
- $\| h_i \|_{C^{1+\beta/2}} < \varepsilon$.

PROOF. Define h_1, h_2 s.t.

$$f_X - \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}.$$

Now write $(df_X)_{w_1} - (df_X)_{w_2}$ as:

$$\circledast = A_1 B_1 C_1 - A_2 B_2 C_2$$

where $A_1 \approx A_2$, $B_1 \approx B_2$, $C_1 \approx C_2$.

Lipschitz $\quad \|B_1 - B_2\| \leq \text{const.} \|w_1 - w_2\|^{\beta} \quad$ Lipschitz

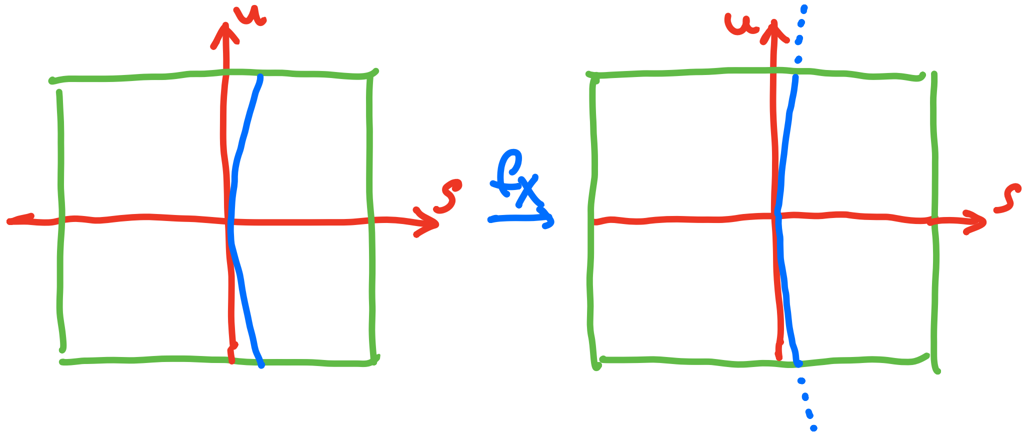
Then $\beta/2$ in the exponent kills the constants so that

$$\circledast \leq \varepsilon \cdot \|w_1 - w_2\|^{\beta/2}.$$

$\blacksquare$

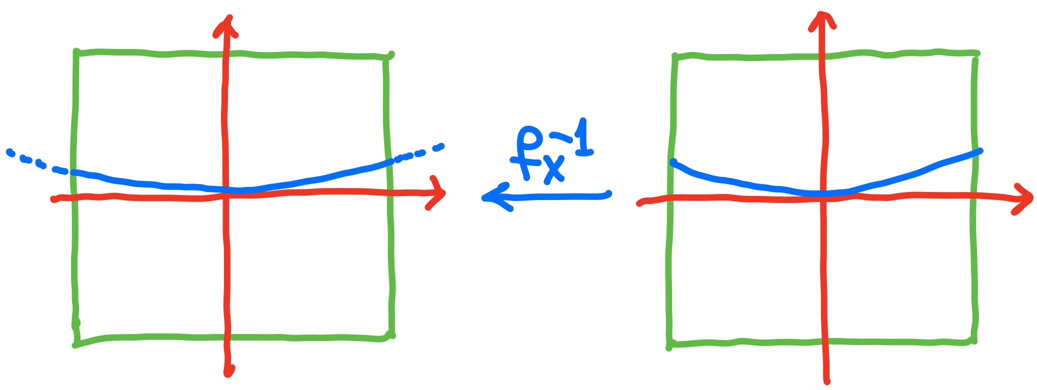
Graph transforms:

Look at the action of f_X in vertical graphs:



- f_x makes vertical graphs more vertical
- f_x approximates two vertical graphs

The action of f_x^{-1} in horizontal graphs is similar:



Formally, let:

$$\begin{cases} \mathcal{M}_x^u = \{\text{almost vertical graphs}\} \\ \mathcal{M}_x^s = \{\text{almost horizontal graphs}\} \end{cases}$$

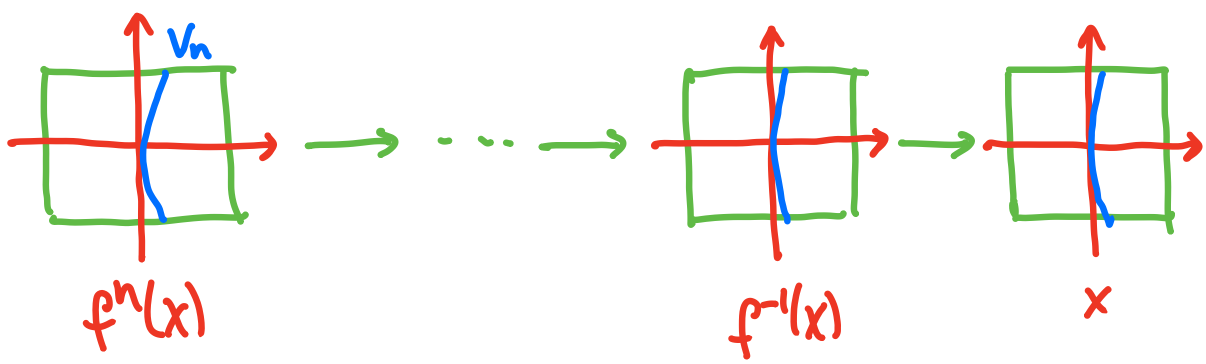
Put the sup norm in the functions.

$$\begin{cases} F^u_x : M^u_x \to M^u_{f(x)} & \text{action of } f_x + \text{restric.} \\ F^s_x : M^s_{f(x)} \to M^s_x & \text{"} \quad f_x^{-1} \quad \text{"} \end{cases}$$

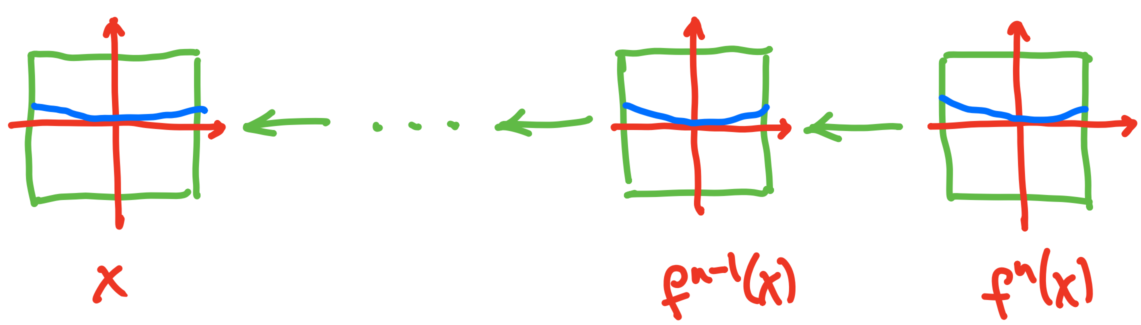
<u>THM.</u> F^s_x, F^u_x are contractions.

We thus define the local invariant manifolds in charts:

$$V^u[x] = \lim_{n \to -\infty} \left(F^u_{f^{-1}(x)} \circ \cdots \circ F^u_{f^n(x)} \right)[V_n]$$



$$V^s[x] = \lim_{n \to +\infty} \left(F^s_x \circ \cdots \circ F^s_{f^{n-1}(x)} \right)[V_n]$$

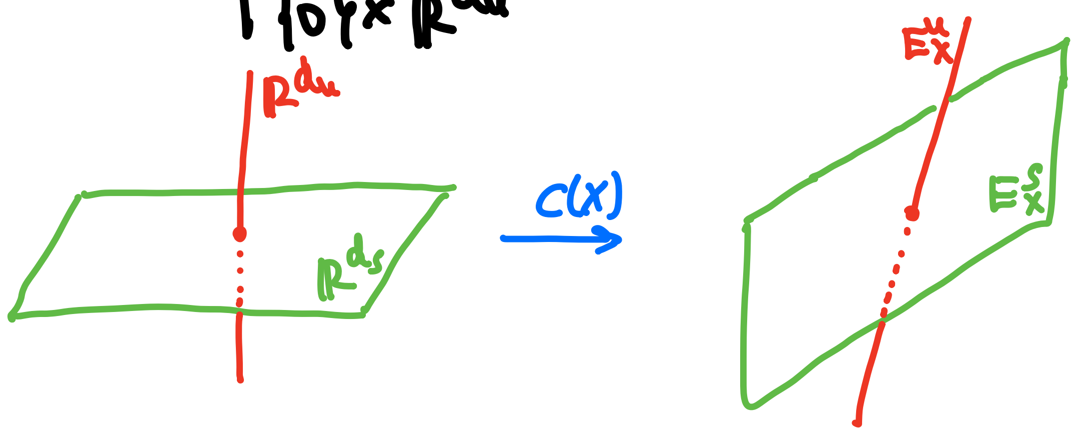


Adaptations for higher dimension:

Define $C(x)$ on subbundles:

$$C(x)\big|_{\mathbb{R}^{d_s}\times\{0\}} : \mathbb{R}^{d_s}\times\{0\} \to E^s_x$$

$$C(x)\big|_{\{0\}\times\mathbb{R}^{d_u}} : \{0\}\times\mathbb{R}^{d_u} \to E^u_x$$



s.t.

$$C(f(x))^{-1}\circ df_x \circ C(x) = \left[\begin{array}{c|c} D_s & 0 \\ \hline 0 & D_u \end{array}\right]$$

where:

- D_s is $d_s \times d_s$ with $\|D^s\| \le \lambda$
- D_u is $d_u \times d_u$ with $\|(D^u)^{-1}\| \le \lambda$

Then continue as above.

NONUNIFORMLY HYPERBOLIC SYSTEMS

Notation:

- Lyapunov exponent

$$\chi(v) = \lim_{n\to\pm\infty} \frac{1}{n} \log \|df^n v\|$$

- χ- hyperbolic measure: $|\chi(v)| > \chi$, $\forall v \neq 0$.

We do not work directly with measures but with a set of good NUH for a fixed parameter $\chi > 0$.

Nonuniformly hyperbolic locus NUH_χ:

The set of $x \in M^2$ s.t. $\exists\, e^s_x, e^u_x \in T_xM$ unitary and transverse s.t.

(NUH1) $\overline{\lim_{n\to+\infty}} \frac{1}{n} \log \|df^n e^s_x\| \leq -\chi$ (Contraction at least $-\chi$ in the future)

$\underline{\lim_{n\to+\infty}} \frac{1}{n} \log \|df^{-n} e^s_x\| > 0$. (Expansion in the past)

(NUH2) $\overline{\lim_{n\to+\infty}} \frac{1}{n}\log\|df^{-n}e^u_x\| \leq -\chi$

$$\underline{\lim_{n\to+\infty}} \frac{1}{n}\log\|df^{n}e^u_x\| > 0$$

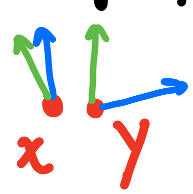
(NUH3) $s(x), u(x)$ are finite:

$$s(x) = \sqrt{2}\left(\sum_{n\geq 0} e^{2n\chi}\|df^{n}e^s_x\|^2\right)^{1/2}$$

$$u(x) = \sqrt{2}\left(\sum_{n\geq 0} e^{2n\chi}\|df^{-n}e^u_x\|^2\right)^{1/2}$$

Basic properties:

- NUH is invariant, usually not compact.
- μ χ-hyperbolic $\Longrightarrow \mu[\mathrm{NUH}_\chi]=1$.
- No continuity of s, u, α on NUH_χ.



"Inner product" on NUH_χ:

- $v_1^s, v_2^s \in E^s$:

$$\langle\langle\langle v_1^s, v_2^s \rangle\rangle\rangle = 2\sum_{n\geq 0} e^{2n\chi}\langle df^n v_1^s, df^n v_2^s\rangle$$

- $v_1^u, v_2^u \in E^u$:

$$\langle\langle\langle v_1^u, v_2^u \rangle\rangle\rangle = 2\sum_{n\geq 0} e^{2n\chi}\langle df^{-n} v_1^u, df^{-n} v_2^u\rangle$$

- $v^s \in E^s$, $v^u \in E^u$:

$$\langle\langle\langle v^s, v^u \rangle\rangle\rangle = 0.$$

Diagonalizing df:

Same as in UH case

Pesin chart:

$$\Psi_x : [-Q, Q]^2 \to M$$

$$\Psi_x = \exp_x \circ C(x).$$

Now, $f_x = \Psi_{f(x)}^{-1} \circ f \circ \Psi_x$ is hyperbolic-like only if we diminish Q.

New possibility: $\|C(f(x))^{-1}\|$ may be large

We have $\|(df_x)_{w_1} - (df_x)_{w_2}\| \leq \text{const.} \|C(f(x))^{-1}\| \cdot \|w_1 - w_2\|^{\beta}$

Parameter $Q(x)$:

$$Q(x) = \text{const.} \|C(f(x))^{-1}\|^{-\binom{\text{LARGE}}{\text{POWER}}}.$$

<u>THM.</u> (PESIN) On $[-Q(x), Q(x)]^2$,

$$f_x = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

where A, B, h_1, h_2 are as in the UH case.

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SYMBOLIC DYNAMICS FOR NONUNIFORMLY HYPERBOLIC SYSTEMS

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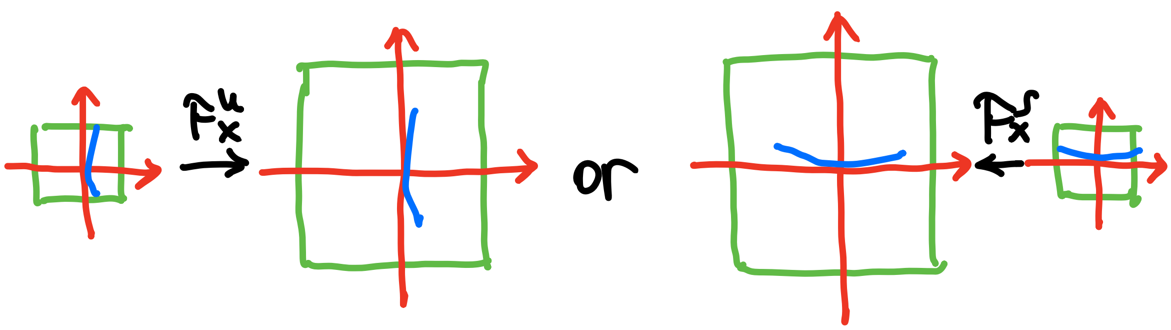
Next step: graph transforms in NUH_χ

Problem: $Q(x)$ depends on x



$Q(x) \ll Q(f(x))$ or $Q(x) \gg Q(f(x))$

Solution: define $q(x) \leq Q(x)$ with slow variation along orbits.

Fix $\varepsilon > 0$ small.

Parameter $q(x)$: $x \in NUH_\chi$

$$q(x) = \inf\{e^{\varepsilon|n|}\cdot Q(f^n(x)) : n \in \mathbb{Z}\}$$

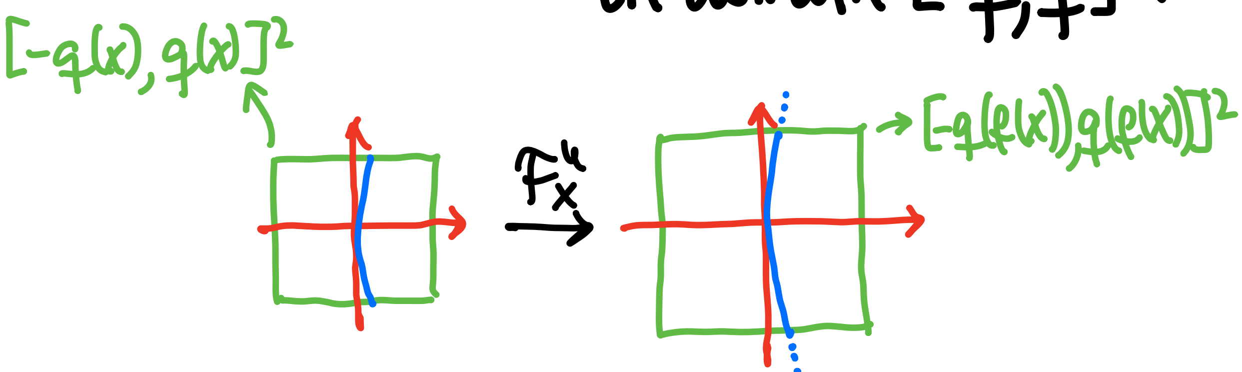
$\boxed{\text{New}}$ nonuniformly hyperbolic locus:

$$NUH_\chi^* = \{x \in NUH_\chi : q(x) > 0\}$$

Q does not decrease to zero exponentially fast along the orbit

Inside NUH_χ^*: F_x^s, F_x^u well-defined

in domain $[-q,q]^2$.



Indeed: $e^{-\varepsilon} \leq \dfrac{q(f(x))}{q(x)} \leq e^{\varepsilon}$ (slow varying)

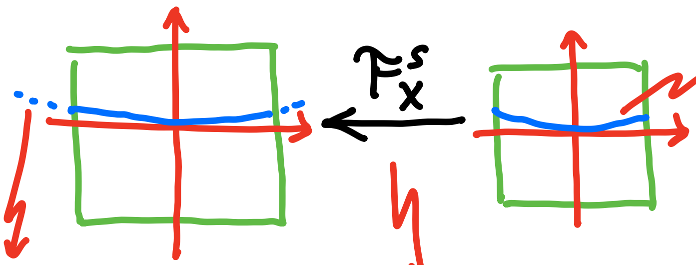
Even better...

Parameters $q^s(x), q^u(x)$: for $x \in NUH^*_\chi$, define

$$\begin{cases} q^s(x) = \inf\{e^{\varepsilon n} Q(f^n(x)) : n \geq 0\} \\ q^u(x) = \inf\{e^{\varepsilon n} Q(f^{-n}(x)) : n \geq 0\} \end{cases}$$

Main property:

$$\begin{cases} q^s(x) = \min\{e^{\varepsilon} q^s(f(x)), Q(x)\} \\ q^u(f(x)) = \min\{e^{\varepsilon} q^u(x), Q(f(x))\} \end{cases}$$



size $q^s(f(x))$

image has size $> q^s(x)$

F_x^s increases size by $\approx e^{\chi}$, hence new size is $> e^{\chi} \cdot q^s(f(x))$.

Then we can define F_x^s, F_x^u at different scales (in some sense, q^s and q^u are the largest scales)

Graph transforms F_x^s, F_x^u:

$$M_x^s = \left\{ \begin{array}{l} \text{graphs of } F : [-q^s(x), q^s(x)] \to \mathbb{R} \\ \text{s.t. } F(0) = F'(0) = 0, \text{Höl}_{\beta/2}(F') < 1/2 \end{array} \right\}$$

almost horizontal graphs

M_x^u similarly for almost vertical

$$\begin{cases} F_x^s : M^s_{f(x)} \to M^s_x \\ F_x^u : M^u_x \to M^u_{f(x)} \end{cases}$$

<u>THM.</u> F_x^s, F_x^u are contractions.

Local invariant manifolds:

Same as before

these are the Pesin local invariant manifolds

Adaptations for higher dimension:

Nonuniformly hyperbolic locus NUH_χ:

The set of $x \in M$ s.t. $\exists\ T_xM = E_x^s \oplus E_x^u$ s.t.:

(NUH1) $\overline{\lim_{n\to+\infty}} \frac{1}{n} \log \|df^n v\| \leq -\chi$

(Same)

$\underline{\lim_{n\to+\infty}} \frac{1}{n} \log \|df^{-n} v\| > 0.$

(NUH2) $\overline{\lim_{n\to+\infty}} \frac{1}{n} \log \|df^{-n} v\| \leq -\chi$

(Same)

$\underline{\lim_{n\to+\infty}} \frac{1}{n} \log \|df^{n} v\| > 0.$

(NUH3) $s(x) = \sup_{\substack{v\in E^s_x \\ \|v\|=1}} S(x,v)$, $u(x) = \sup_{\substack{w\in E^u_x \\ \|w\|=1}} U(x,w)$

are finite, where:

$$S(x,v) = \sqrt{2}\left(\sum_{n\geq 0} e^{2n\chi} \|df^n v\|^2\right)^{1/2}$$

$$U(x,w) = \sqrt{2}\left(\sum_{n\geq 0} e^{2n\chi} \|df^{-n} w\|^2\right)^{1/2}.$$

Then define $C(x)$ for $x \in NUH_\chi$ and continue as in dimension 2.

MAPS WITH DISCONTINUITIES AND BOUNDED DERIVATIVE

$\mathcal{S}$ = singular set = discontinuities

Setting: M^2 = surface, $\mathcal{S} \subset M$ closed, $f : M \setminus \mathcal{S} \to M$ $C^{1+\beta}$ with bounded df.

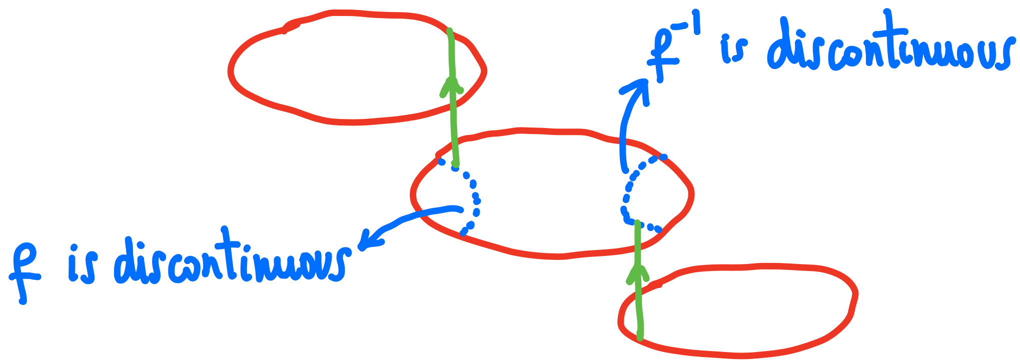
Problem: Orbits that approach $\mathcal{S}$ exponentially fast

If so, then NUH might not prevail over the effect of discontinuities

Example: $\varphi : N^3 \to N^3$ flow with positive speed

From flow to map

Construct M^2 = global Poincaré section and study $f: M \to M$ return map.



Redefining NUH$_\chi$: (NUH1)–(NUH3) and

(NUH4) $\lim \frac{1}{n} \log d(f^n(x), \mathfrak{S}) = 0$

(Subexponential convergence to $\mathfrak{S}$)

Redefining Pesin charts:

$$\Psi_x : [-\delta(x), \delta(x)]^2 \to M,$$

where $\delta(x) = \varepsilon^{3/\beta} \cdot \underbrace{d(x, \mathfrak{S})}_{\text{added term}}$.

Redefining $Q(x)$:

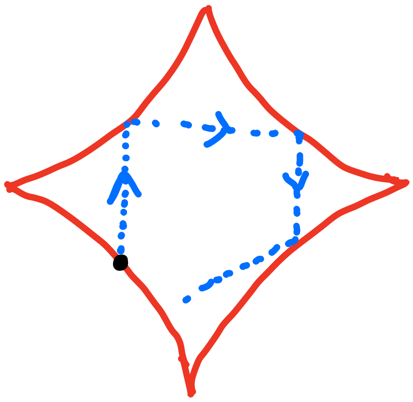
$$Q(x) = \text{const} \times \min\left\{ \|C(f(x))^{-1}\|^{-(\text{LARGE POWER})}, \underbrace{\rho(x)}_{\text{added term}} \right\},$$

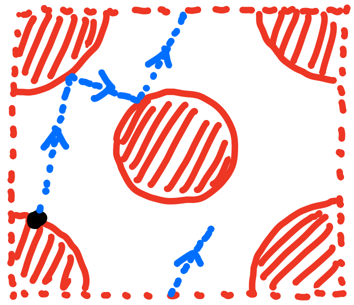
where $\rho(x) = d\left(\{f^{-1}(x), x, f(x)\}, \mathcal{S}\right)$.

MAPS WITH DISCONTINUITIES AND UNBOUNDED DERIVATIVE

Setting: M^2 = surface, $\mathcal{S} \subset M$ closed, $f : M \setminus \mathcal{S} \to M$ $C^{1+\beta}$ s.t. $\exists\, a > 1$ s.t. $d(x, \mathcal{S})^a \le \|df_x^{\pm 1}\| \le d(x, \mathcal{S})^{-a}$.

Example: billiards, e.g.





Redefining $Q(x)$:

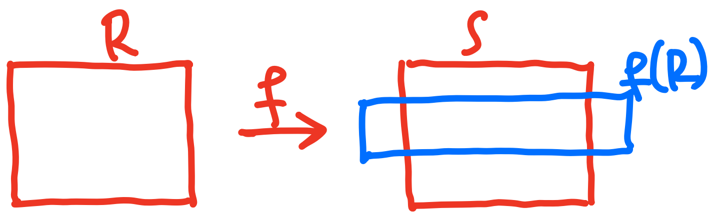
$$Q(x) = \text{const} \times \min\Big\{ \|C(x)^{-1}\|^{-(\text{LARGE POWER})}, \|C(f(x))^{-1}\|^{-(\text{LARGE POWER})}, \rho(x)^{(\text{LARGE POWER})} \Big\}$$

MARKOV PARTITIONS

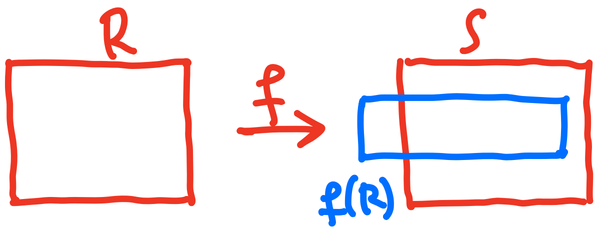
Here, we discuss Bowen's approach using pseudo-orbits. (1970's)

MARKOV PARTITION: family of sets (→ rectangles) with the Markov property

Markov property:



Good: intersection from one side to the other



Bad: intersection did not go from one side to the other

FACT: Markov partition $\Rightarrow \exists$ symbolic model

$$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

$\Sigma = \Sigma(V, E)$, where
$V = \{\text{rectangles}\}$
$E = \{R \to S : f(R) \cap S \neq \emptyset\}$

GOAL: Construct Markov partitions

MARKOV PARTITIONS FOR UH SYSTEMS

(Sinai, Adler–Weiss, Ratner, Bowen)

Pseudo-orbit: $\{x_n\} \subset M$ s.t.

$$d(f(x_n),x_{n+1})<\delta, \quad d(f^{-1}(x_{n+1}),x_n)<\delta$$

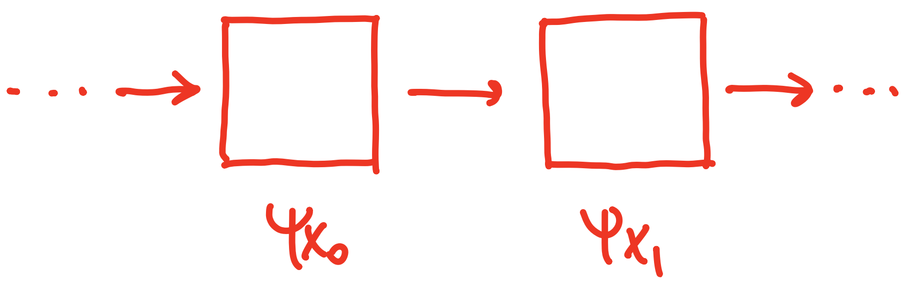
Rewriting definition using ε-overlap of Lyapunov charts:

- $\Psi_x \overset{\varepsilon}{\approx} \Psi_y$ <u>ε-overlap</u> if $d(x,y)<\delta$
- $\Psi_x \xrightarrow{\varepsilon} \Psi_y$ if $\Psi_{f(x)} \overset{\varepsilon}{\approx} \Psi_y$ and $\Psi_{f^{-1}(y)} \overset{\varepsilon}{\approx} \Psi_x$

- $\{\Psi_{x_n}\}_{n\in\mathbb{Z}}$ pseudo-orbit if

$$\Psi_{x_n} \xrightarrow{\varepsilon} \Psi_{x_{n+1}}, \quad \forall n \in \mathbb{Z}.$$

Idea: use pseudo-orbits to understand f in neighborhood of $\{x_n\}_{n\in\mathbb{Z}}$.



THM. If $\Psi_{f(x)} \overset{\varepsilon}{\approx} \Psi_y$, then

$$f_{xy} := \Psi_y^{-1} \circ f \circ \Psi_x = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

where:

- $|A| < \lambda$, $|B^{-1}| < \lambda$ as before.
- $\|h_i\|_{C^{1+\beta/3}} < \varepsilon$. Note: we decreased from $\beta/2$ to $\beta/3$.

Proof.

$$f_{xy} = \underbrace{\Psi_y^{-1} \circ \Psi_{f(x)}}_{\approx \mathrm{Id}} \circ \underbrace{\Psi_{f(x)}^{-1} \circ f \circ \Psi_x}_{= f_x}$$

$\blacksquare$

Graph transforms: If $\Psi_x \to \Psi_y$ then

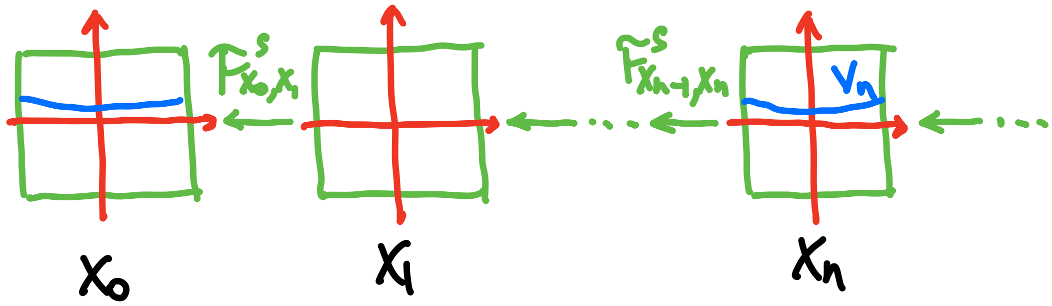
$$F_{xy}^s : \mathcal{M}_y^s \longrightarrow \mathcal{M}_x^s$$

$$F_{xy}^u : \mathcal{M}_x^u \longrightarrow \mathcal{M}_y^u$$

are defined similarly, and again they are <u>contractions</u>.

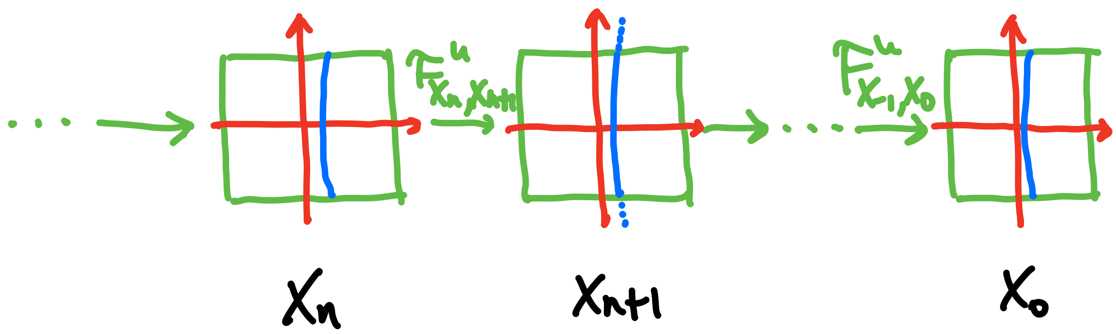
Stable/unstable manifolds of pseudo-orbits: If $\underline{v} = \{\Psi_{x_n}\}_{n\in\mathbb{Z}}$, then

$$V^s[\underline{v}] = \lim_{n\to+\infty} (F^s_{x_0,x_1} \circ \cdots \circ F^s_{x_{n-1},x_n})[V_n]$$



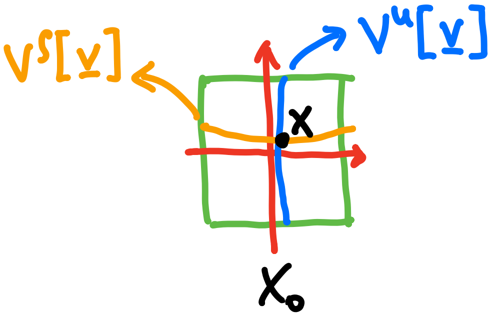
and

$$V^u[\underline{v}] = \lim_{n\to-\infty} \left(F^u_{x_{-1},x_0} \circ \cdots \circ F^u_{x_n,x_{n+1}}\right)[V_n]$$



<u>THM</u> (SHADOWING LEMMA) Every pseudo-orbit $\underline{v}$ shadows a unique

$$\{x\} = V^s[\underline{v}] \cap V^u[\underline{v}].$$



Shadowing:

$f^n(x) \in \Psi_{x_n}[-Q,Q]^2, \forall n \in \mathbb{Z}.$

Construction of Markov partition:

(3 steps)

Step 1 (Coarse graining):

- $X \subset M$ sufficienty dense
- $\mathcal{G} = (V, E)$, where $V = \{\Psi_x : x \in X\}$ and $E = \{\Psi_x \xrightarrow{\varepsilon} \Psi_y\}$
- $\Sigma = \Sigma(\mathcal{G})$: $\underline{v} \in \Sigma$ is pseudo-orbit

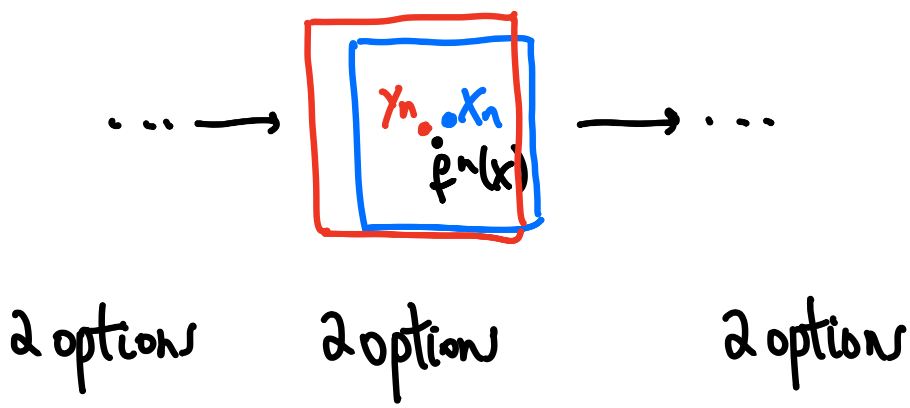
Step 2 (Infinite-to-one extension):

- Define $\pi : \Sigma \to M$,

$$\pi(\underline{v}) = V^s[\underline{v}] \cap V^u[\underline{v}]$$

- π is surjective: $x \in M \Rightarrow \exists\, x_n$ s.t. $x_n \approx f^n(x)$, $\forall n \in \mathbb{Z}$.

- $\pi\circ\sigma = f\circ\pi$: uniqueness of shadow.
- π usually ∞-to-one: if $\exists\, x_n, y_n \approx f^n(x)$, then any choice gives $\underline{v}\in\pi^{-1}(x)$.



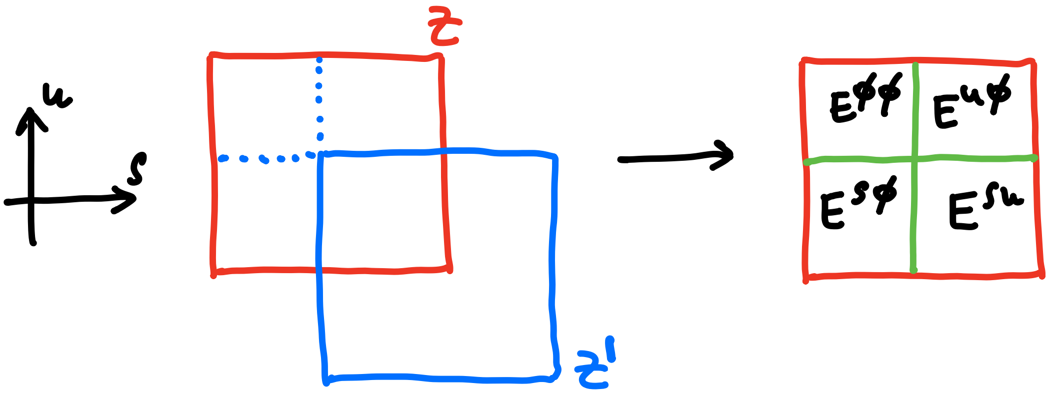
Step 3 (Bowen-Sinai refinement):

- Let $\mathcal{Z}=\{Z_v : v\in V\}$, where

$$Z_v=\{\pi(\underline{v}) : v_0=v\} = \pi([v_0]_0).$$

$\mathcal{Z}$ is a cover of M.

- Refine $\mathcal{Z}$, destroying intersections:



- $\mathcal{R}$ = refinement is a Markov partition.

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SYMBOLIC DYNAMICS FOR NONUNIFORMLY HYPERBOLIC SYSTEMS

ICTP, 2021 LECTURE 3

Goal: construct Markov partitions for NUH systems

Difficulties:

- Objects do not vary continuously

  (only measurably)

- NUH behaviour of points varies a lot

  we know how to measure: $s, u, \alpha, Q, q, q^s, q^u$

Previous result (KATOK): Katok horseshoes

Horseshoes with finitely many symbols and entropy $\approx$ topological entropy

(Restrict attention to Pesin sets – where continuity holds – and apply a more precise study of pseudo-orbits, using Bowen's approach)

New result (SARIG):

Horseshoe with **countably** many states and **full** topological entropy

$\boxed{\text{Even newer}}$ nonuniformly hyperbolic locus:

$$NUH_{\chi}^{\#} = \left\{ x \in NUH_{\chi}^{*} : \begin{array}{l} \overline{\lim\limits_{n\to+\infty}} \, q(f^n(x)) > 0 \text{ and} \\ \underline{\lim\limits_{n\to-\infty}} \, q(f^n(x)) > 0 \end{array} \right\}$$

Recall: NUH_{χ} : E^s, E^u with finite $s(x), u(x)$

NUH_{χ}^{*} : $q(x) > 0$ (subexponential Q)

$NUH_{\chi}^{\#}$: recurrence (Pliss times)

THM (SARIG) $f: M^2 \to M^2$ $C^{1+\beta}$ diffeo.

Given $\chi > 0$, $\exists (\Sigma, \sigma)$ and $\pi : \Sigma \to M$ Hölder continuous s.t.

(1)
$$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \Sigma \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

(2) $\pi[\Sigma^\#] = NUH_\chi^\#$

(3) $\pi|_{\Sigma^\#}$ is finite-to-one.

Above:

$$\Sigma^\# = \left\{ \underline{v} = \{v_n\}_{n \in \mathbb{Z}} \in \Sigma : \begin{array}{l} \exists\, v, w \text{ s.t. } v_n = v \text{ for} \\ \infty\text{'ly many } n > 0 \text{ and} \\ v_n = w \text{ for } \infty\text{'ly many} \\ n < 0 \end{array} \right\}$$

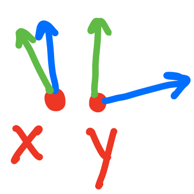
Compare it with $NUH_\chi^\#$.

Five main ingredients:

- ε-overlap
- ε-double charts
- Coarse graining
- Improvement lemma
- Inverse theorem

ε-overlap:

UH: $x \approx y \Rightarrow E^\sigma_x \approx E^\sigma_y$.

NUH: $x \approx y$ and 

$\rightsquigarrow$ $C(x), C(y)$ are very different

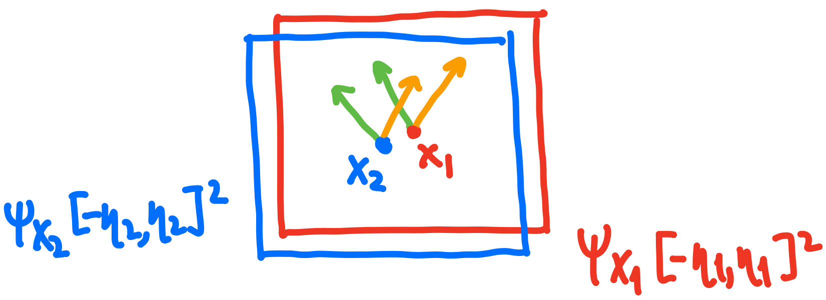
Pesin chart Ψ^η_x: restriction $\Psi_x : [-\eta,\eta]^2 \to M$

ε-overlap: $\Psi_{x_1}^{\eta_1} \overset{\varepsilon}{\approx} \Psi_{x_2}^{\eta_2}$ if

- $\dfrac{\eta_1}{\eta_2} = e^{\pm\varepsilon}$

very strong!

- $d(x_1,x_2) + \| C(x_1) - C(x_2) \| < (\eta_1 \eta_2)^4$



<u>THM</u> (SARIG) If $\Psi_{f(x)}^{\eta} \overset{\varepsilon}{\approx} \Psi_{y}^{\eta'}$ then

$$f_{x,y} = \Psi_y^{-1} \circ f \circ \Psi_x = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

s.t.:

- $|A|, |B^{-1}| < \lambda$.

- $\|h_i\|_{C^{1+\beta/3}} < \varepsilon$.

ε-overlap allows hyperbolicity!

ε- double charts:

$$\begin{cases} \text{UH: } \alpha(x) > 0 \text{ uniformly} \\ \text{NUH: } \alpha(x) \approx 0 \to \text{hard to measure hyperbolicity along } E^s \text{ and } E^u \text{ at same scales.} \end{cases}$$

Recall: for $x \in NUH_\chi^*$,

$$\begin{cases} q^s(x) = \inf\{e^{\varepsilon n} Q(f^n(x)) : n \geq 0\} \\ q^u(x) = \inf\{e^{\varepsilon n} Q(f^{-n}(x)) : n \geq 0\} \end{cases}$$

Two different scales for s, u directions.

ε-double chart: $\Psi_x^{p^s, p^u} = (\Psi_x^{p^s}, \Psi_x^{p^u})$

$\Psi_x^{p^s}$: behaviour of E^s analysed at scale p^s

$\Psi_x^{p^u}$: behaviour of E^u analysed at scale p^u

Write $v = \Psi_x^{p^s, p^u}$ and $w = \Psi_y^{q^s, q^u}$.

Edge $v \xrightarrow{\varepsilon} w$:

(GPO1) $\Psi_{f(x)}^{q^s \wedge q^u} \overset{\varepsilon}{\approx} \Psi_y^{q^s \wedge q^u}$ and

$\Psi_{f^{-1}(y)}^{p^s \wedge p^u} \overset{\varepsilon}{\approx} \Psi_x^{p^s \wedge p^u}$

(GPO2) $p^s = \min\{e^\varepsilon q^s, Q(x)\}$

$q^u = \min\{e^\varepsilon p^u, Q(y)\}$

(Exactly what $q^s(x), q^u(x)$ satisfy)

ε-generalized pseudo-orbit (ε-gpo):

$\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}$ s.t. $\Psi_{x_n}^{p_n^s, p_n^u} \xrightarrow{\varepsilon} \Psi_{x_{n+1}}^{p_{n+1}^s, p_{n+1}^u}$, $\forall n \in \mathbb{Z}$.

FACT: Every $x \in NUH_\chi^*$ generates an ε-gpo

$$\underline{v} = \{\Psi_{f^n(x)}^{q^s(f^n(x)), q^u(f^n(x))}\}_{n \in \mathbb{Z}}$$

Edges $v \xrightarrow{\varepsilon} w$ induce graph transforms $F^{s/u}_{v,w}$:

- $M^s_v = \{$almost horizontal graphs$\}$

$$= \left\{ \begin{array}{l} \text{graphs of } F:[-p^s,p^s]\to\mathbb{R} \text{ s.t.} \\ |F(0)| < \frac{1}{1000}(p^s\wedge p^u),\ |F'(0)| < \frac{1}{2}(p^s\wedge p^u)^{\beta/3}, \\ \|F'\|_{C^0} + \text{Höl}_{\beta/3}(F') \le 1/2 \end{array} \right\}$$

- M^u_v similarly
- $F^s_{v,w}: M^s_w \to M^s_v$ and $F^u_{v,w}: M^u_v \to M^u_w$

Stable/unstable manifolds of ε-gpo:

$$V^s[\underline{v}] = \lim_{n\to+\infty} \left(F^s_{v_0,v_1}\circ\cdots\circ F^s_{v_{n-1},v_n}\right)[V_n]$$

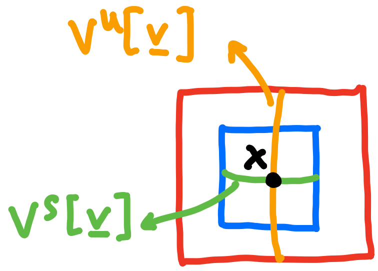
and

$$V^u[\underline{v}] = \lim_{n\to-\infty} \left(F^u_{v_{-1},v_0}\circ\cdots\circ F^u_{v_n,v_{n+1}}\right)[V_n].$$

These are genuine Pesin invariant manifolds

THM (SHADOWING LEMMA) Every ε-gpo $\underline{v}$ shadows a unique point

$$\{x\} = V^s[\underline{v}] \cap V^u[\underline{v}].$$



Shadowing:

$$f^n(x) \in \Psi_{x_n}[-p^s_n \wedge p^u_n, p^s_n \wedge p^u_n]^2, \ \forall n \in \mathbb{Z}.$$

Coarse graining:

Goal: find finitely/countably many ε-double charts s.t. ε-gpo's shadow $NUH^\#_\chi$.

Shadowing: arbitrary $x \in NUH^\#_\chi$

approximate $x, C(x), Q(x), q(x), q^s(x), q^u(x)$

Idea: Consider

$$\Gamma(x) = (\underline{x}, \underline{C}, \underline{Q}), \text{ where}$$

$$\begin{cases} \underline{x} = (f^{-1}(x), x, f(x)) \\ \underline{C}(x) = (C(f^{-1}(x)), C(x), C(f(x))) \\ \underline{Q} = Q(x). \end{cases}$$

Recall: their inverses can be huge!

For $\underline{\ell} = (\ell_{-1}, \ell_0, \ell_1)$, let

$$Y_{\underline{\ell}} = \{\Gamma(x) : e^{\ell_i} \leq \|C(f^i(x))^{-1}\| < e^{\ell_i+1}, |i| \leq 1\}$$

Then $\{\Gamma(x) : x \in NUH_\chi^\#\} = \bigcup_{\underline{\ell}} Y_{\underline{\ell}}$, with $Y_{\underline{\ell}}$ pre-compact.

$\Downarrow$

$\exists$ dense countable subset.

We obtain:

THM (SARIG) $\forall \varepsilon>0, \exists \mathcal{A}=$ countable family of ε-double charts s.t.:

(1) Discreteness: $\forall t>0$, $\{\Psi_x^{p^s,p^u} \in \mathcal{A} : p^s \wedge p^u > t\}$ is finite.

$p^s \wedge p^u > t$: Pesin set

(2) Sufficiency: $\forall x \in NUH_\chi^\#$, $\exists \varepsilon$-gpo $\underline{v} \in \mathcal{A}^{\mathbb{Z}}$ that shadows x.

Hence: $\Sigma, \sigma, \pi : \Sigma \to M$ as in UH case.

Improvement lemma:

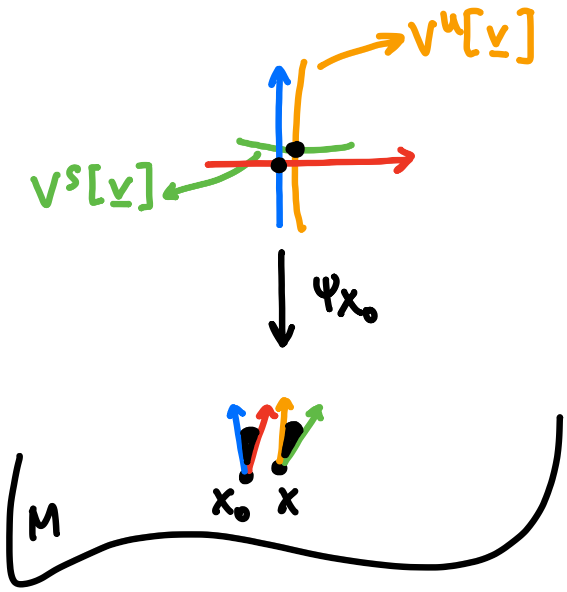
Goal: If $\underline{v} = \{\Psi_{x_n}^{p_n^s,p_n^u}\}_{n\in\mathbb{Z}}$ shadows x, relate hyperbolicity parameters of x with those of $\Psi_{x_n}^{p_n^s,p_n^u}$.

Angle (easy):

Angles in $\mathbb{R}^2$ are close

$\downarrow$

Angles at x_0, x are close



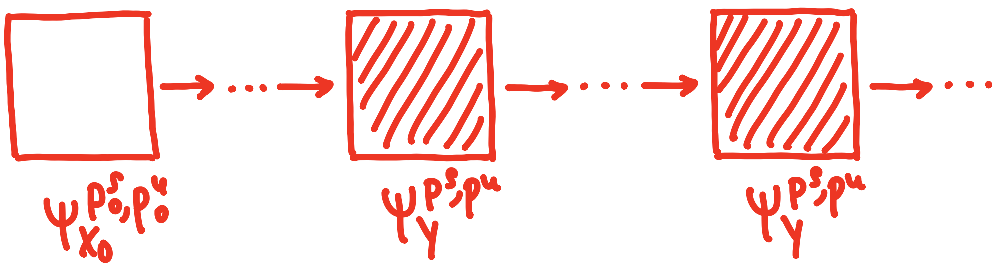
Problem: how to compare $s(x)$ and $s(x_0)$.

<u>LEMMA</u> (IMPROVEMENT LEMMA) If $\dfrac{s(f(x))}{s(x_1)}$ is big, then $\dfrac{s(x)}{s(x_0)}$ is smaller.

More specifically: for $\varepsilon \geq \sqrt{\epsilon}$, if $\dfrac{s(f(x))}{s(x_1)} = e^{\pm\varepsilon}$,

then $\dfrac{s(x)}{s(x_0)} = e^{\pm(\varepsilon - Q(x_0)^{\beta/4})}$.

Hence: if $\underline{v} \in \Sigma^{\#}$ then we have ∞'ly many improvements



COROLLARY. $\pi[\Sigma^{\#}] \subset NUH_{\chi}^{\#}$.

Proof of improvement lemma.

Applying f^{-1} along stable direction improves regularity:

Indeed:

$$\begin{cases} s(x)^2 = 2 + C \cdot s(f(x))^2 \\ s(x_0)^2 = 2 + C \cdot s(x_1)^2 \end{cases}$$

If $\dfrac{s(f(x))^2}{s(x_1)^2} = K \gg 1$, then

$$\frac{s(x)^2}{s(x_0)^2} \approx \frac{2 + K \cdot C s(x_1)^2}{2 + C s(x_1)^2} < K.$$

$\blacksquare$

Inverse theorem:

THM. (SARIG) If $\pi(\underline{v}) = x$ with $\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\} \in \Sigma^\#$, then:

(1) $x_n \approx f^n(x)$.

(2) $\dfrac{\sin\alpha(x_n)}{\sin\alpha(f^n(x))} \approx 1$.

(3) $\dfrac{s(x_n)}{s(f^n(x))} \approx 1$, $\dfrac{u(x_n)}{u(f^n(x))} \approx 1$.

(4) $\dfrac{p_n^s}{q^s(f^n(x))} \approx 1$, $\dfrac{p_n^u}{q^u(f^n(x))} \approx 1$.

These estimates play a crucial role for the Bowen-Sinai refinement.

Recall:

Step 1 (Coarse graining):

- $\mathfrak{A}$ = countable family of ε-double charts
- $\mathcal{G} = (V, E)$, where $V = \mathfrak{A}$ and

  $E = \{\Psi_x^{p^s,p^u} \xrightarrow{\varepsilon} \Psi_y^{q^s,q^u}\}$

- $\Sigma = \Sigma(\mathcal{G})$: $\underline{v} \in \Sigma$ is ε-gpo

Step 2 (Infinite-to-one extension):

- Define $\pi : \Sigma \to M$,

$$\pi(\underline{v}) = V^s[\underline{v}] \cap V^u[\underline{v}]$$

- π is surjective onto $NUH_\chi^\#$: $\pi[\Sigma^\#] = NUH_\chi^\#$.

- $\pi \circ \sigma = f \circ \pi$: same

- π is usually ∞-to-one : same

Step 3 (Bowen-Sinai refinement):

- Let $\mathcal{Z} = \{Z_v : v \in V\}$, where

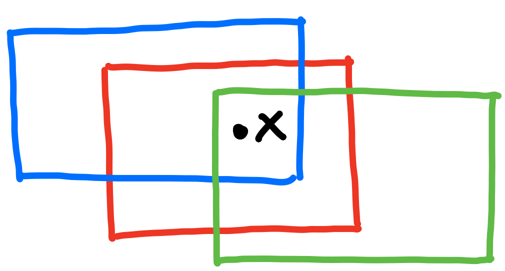
$$Z_v = \{\pi(\underline{v}) : \underline{v} \in NUH_\chi^\# \text{ and } v_0 = v\}$$
$$= \pi([v_0]_0 \cap \Sigma^\#).$$

$\mathcal{Z}$ is a countable cover of $NUH_\chi^\#$.

How to refine and still obtain countable?

Main property: $\mathcal{Z}$ is locally finite

$\forall x \in NUH_\chi^\#, \exists$ finitely many $Z \in \mathcal{Z}$ containing x.



Indeed:

$$x \in Z = \psi_x^{p^s, p^u} \Rightarrow \begin{cases} p^s \approx q^s(x) \\ p^u \approx q^u(x) \end{cases}$$

$$\rightarrow p^s \wedge p^u \approx q^s(x) \wedge q^u(x) = q(x)$$

$$\rightarrow \psi_x^{p^s, p^u} \in \{\psi_y^{q^s, q^u} : \underbrace{q^s \wedge q^u > q(x)}\}$$

finite, by coarse graining

Now refine as before.

Conclusion:

<u>THM</u> (SARIG) $f: M^2 \to M^2$ $C^{1+\beta}$ diffeo.

Given $\chi > 0$, $\exists\ (\Sigma, \sigma)$ and $\pi: \Sigma \to M$ Hölder continuous s.t.

(1)
$$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

(2) $\pi[\Sigma^\#] = NUH_\chi^\#$

(3) $\pi|_{\Sigma^\#}$ is finite-to-one.

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SYMBOLIC DYNAMICS FOR NONUNIFORMLY HYPERBOLIC SYSTEMS

ICTP, 2021 LECTURE 4

Goal: construct Markov partitions for more complicated NUH systems

- higher dimension diffeos
- flows
- billiards
- non-invertible maps

Five main ingredients:

- ε-overlap ✓
- ε-double charts ✓
- Coarse graining ?
- Improvement lemma ?

- Inverse theorem ?

Step 1 (Coarse graining): ?

Step 2 (Infinite-to-one extension): ✓

Step 3 (Bowen-Sinai refinement): ✓

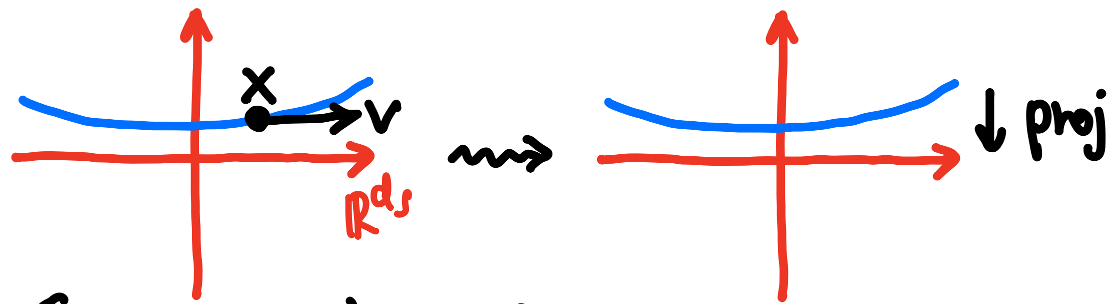
Higher dimension diffeomorphisms:

(Ben Ovadia)

- Coarse graining ✓
- Improvement lemma ?
- Inverse theorem ?

Improvement lemma: many $s(x,v), U(x,w)$.

Problem: compare $s(x,v)$ with $s(x_0,*)$?



$$\begin{cases} \mathrm{proj}(t, F(t)) = (t, 0) \\ d\,\mathrm{proj} : H^s_x \to \mathbb{R}^{d_s} \times \{0\} \cong H^s_{x_0} \end{cases}$$

$$\begin{tikzcd}
H^s_x \arrow[r, "d\,\mathrm{proj}"] \arrow[d, "d\psi_{x_0}"'] \arrow[dr, phantom, "\circlearrowright"] & H^s_{x_0} \arrow[d, "d\psi_{x_0}"] \\
E^s_x \arrow[r, "\Theta^s_{x,x_0}"'] & E^s_{x_0}
\end{tikzcd}$$

The diagram defines $\Theta^s_{x,x_0} : E^s_x \to E^s_{x_0}$,

which allows to compare $S(x,v)$ with $S(x_0, \Theta^s_{x,x_0}(v))$.

Then prove Improvement lemma and

Inverse theorem.

Many new technical difficulties

SURFACE MAPS WITH DISCONTINUITIES AND BOUNDED DERIVATIVE (Lima - Sarig)

Setting: M^2 = surface, $\mathscr{S} \subset M$ closed,

$f: M \setminus \mathscr{S} \to M$ $C^{1+\beta}$ with bounded df.

Already understood: invariant manifolds

Coarse graining: need to consider $\mathscr{S}$

loses compactness

(NUH1)-(NUH4)

Recall: for $x \in NUH_\chi^*$, let

$\Gamma(x) = (\underline{x}, \underline{C}, \underline{Q})$, where

$$\begin{cases} \underline{x} = (f^{-1}(x), x, f(x)) \\ \underline{C}(x) = (C(f^{-1}(x)), C(x), C(f(x))) \\ \underline{Q} = Q(x). \end{cases}$$

For $\underline{l} = (l_{-1}, l_0, l_1)$ and $\underline{k} = (k_{-1}, k_0, k_1)$, let

$$Y_{\underline{l},\underline{k}} = \left\{ \Gamma(x) : \begin{array}{l} e^{l_i} \leq \|C(f^i(x))^{-1}\| < e^{l_i+1} \\ e^{-k_i-1} \leq d(f^i(x), \mathcal{S}) < e^{k_i} \end{array}, |i| \leq 1 \right\}$$

Then $\{\Gamma(x) : x \in NUH^*_\chi\} = \bigcup\limits_{\underline{l},\underline{k}}^{\cdot} Y_{\underline{l},\underline{k}}$, with $Y_{\underline{l},\underline{k}}$ pre-compact.

SURFACE MAPS WITH DISCONTINUITIES AND UNBOUNDED DERIVATIVE

(Lima-Matheus)

Setting: M^2 = surface, $\mathcal{S} \subset M$ closed, $f : M \setminus \mathcal{S} \to M$ $C^{1+\beta}$ s.t. $\exists\, a > 1$ with

$$d(x, \mathcal{S})^a \leq \|df_x^{\pm 1}\| \leq d(x, \mathcal{S})^{-a}.$$

Already understood: invariant manifolds

On $B(x, r(x))$ for $r(x) = d(x, \mathcal{S})^{\left(\substack{\text{LARGE} \\ \text{POWER}}\right)}$,

f is well-behaved.

Coarse graining: fix $\mathcal{P} = \{D_i\}$ countable open cover of $M \setminus \mathcal{S}$ s.t.:

- $D_i = B(x_i, r(x_i))$
- Discreteness: $\forall t > 0$, $\{D \in \mathcal{P} : d(D, \mathcal{S}) \geq t\}$ is finite.

For $\underline{\ell} = (\ell_{-1}, \ell_0, \ell_1)$, $\underline{k} = (k_{-1}, k_0, k_1)$, $\underline{a} = (a_{-1}, a_0, a_1)$, let

$$Y_{\underline{k}, \underline{\ell}, \underline{a}} = \left\{ \Gamma(x) : \begin{array}{c} e^{\ell_i} \leq \|C(f^i(x))^{-1}\| < e^{\ell_i + 1} \\ e^{-k_i - 1} \leq d(f^i(x), \mathcal{S}) < e^{k_i} \\ f^i(x) \in D_{a_i} \end{array} : |i| \leq 1 \right\}$$

and repeat the preceding argument.

NON-INVERTIBLE MAPS IN HIGH DIMENSION WITH SINGULARITIES

(Araujo-Lima-Poletti)

Setting:

- M = Riemannian manifold with finite diameter

  possibly disconnected and/or with boundary

- $\mathcal{D} \subset M$ closed: discontinuity set

- Exponential map at x: $\exists\, a > 1$ s.t. $\forall x \in M \setminus \mathcal{D}$ $\exists\, \partial(x) > d(x, \mathcal{D})^a$ s.t. $\exp_x : B(0, \partial(x)) \to M$ is well-defined and regular

  $\| d\exp_x^{\pm 1} \| \leq 2$

- $f: M \setminus \mathcal{D} \to M$ map.
- $\mathcal{C} = \{x \in M \setminus \mathcal{D} : df_x$ is not invertible$\}$: critical set
- Singular set: $\mathcal{S} = \mathcal{C} \cup \mathcal{D}$.
- Regularity of f: for every $x \in M$ s.t. $x, f(x) \notin \mathcal{S}$, $\exists\, r(x) > \min\{d(x,\mathcal{S})^a, d(f(x),\mathcal{S})^a\}$ s.t.

inverse branch taking $f(x)$ to x

$f|_{B(x,r(x))}$, $g|_{B(f(x),r(x))}$ are diffeos with

$d(x,\mathcal{S})^a \leq \|df_y\|, \|dg_z\| \leq d(x,\mathcal{S})^{-a}$

and β-Hölder for df and dg.

Problem: f not invertible

$\rightsquigarrow$

no symmetry between future and past

Idea: Code the natural extension of f

Let $(\hat{M}, \hat{f})$ be the natural extension

we will soon define

<u>THM</u> (Araujo–Lima–Poletti) Let M, f as above. For $\chi > 0$, $\exists\, NUH_\chi^\# \subset \hat{M}$, (Σ, σ) with countable states and $\pi: \Sigma \to \hat{M}$ Hölder continuous s.t.:

(1)
$$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma \arrow[d, "\pi"] \\ \hat{M} \arrow[r, "\hat{f}"'] & \hat{M} \end{tikzcd}$$

(2) $\pi[\Sigma^\#] = NUH_\chi^\#$

(3) $\pi|_{\Sigma^\#}$ is finite-to-one.

<u>Obs.</u>: As before, the oriented graph has finite degree (but usually not uniformly bounded).

Natural extension:

- $\hat{M} = \{ \hat{x} = (x_n)_{n\in\mathbb{Z}} : f(x_n) = x_{n+1}, \forall n \in \mathbb{Z} \}$,

  $\hat{x} = (\dots, x_{-1}, x_0; x_1, \dots)$

- Define $\hat{f} : \hat{M} \to \hat{M}$ by "left shift":

  $\hat{f}(\dots, x_{-1}, x_0; x_1, \dots) = (\dots, x_0, x_1; x_2, \dots)$

- Canonical projection $\vartheta : \hat{M} \to M$, $\hat{x} \mapsto x_0$.

- Lift $\mathcal{S}$ to $\hat{M}$: $\mathcal{S} \mapsto \bigcup_{n\in\mathbb{Z}} \hat{f}^n(\vartheta^{-1}[\mathcal{S}])$.

- On the complement $\hat{M} \setminus \bigcup_{n\in\mathbb{Z}} \hat{f}^n(\vartheta^{-1}[\mathcal{S}])$,

  define bundle

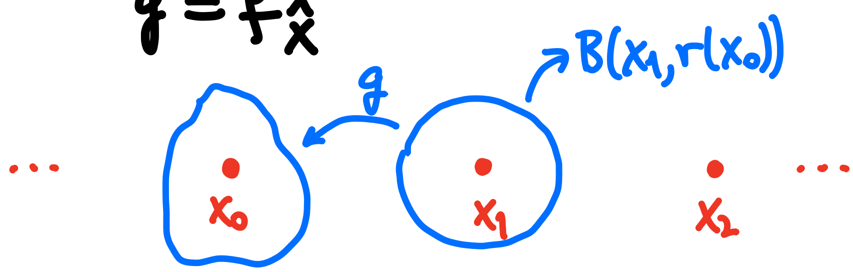
  $$\widehat{TM} = \bigsqcup_{\hat{x}} \widehat{TM}_{\hat{x}}$$

  where $\widehat{TM}_{\hat{x}} = TM_{x_0}$

and lift df to invertible cocycle $(\widehat{df}^n_{\hat{x}})_{n\in\mathbb{Z}}$.

- Inverse branch taking $f(x)$ to x:

$$g = f^{-1}_{\hat{x}}$$



Nonuniformly hyperbolic locus NUH_χ:

The set of $\hat{x} \in \hat{M} \setminus \bigcup_{n\in\mathbb{Z}} \hat{f}^n(\vartheta^{-1}[\mathcal{S}])$ s.t.

$\exists\ \widehat{TM}_{\hat{x}} = E^s_{\hat{x}} \oplus E^u_{\hat{x}}$ s.t. (NUH1)–(NUH3) with respect to $\widehat{df}$.

Then introduce:

- "Inner product" on NUH_χ.
- $C(\hat{x})$, diagonalization of $\widehat{df}$.

- Pesin chart $\Psi_{\hat{x}}$.
- Parameters $Q(\hat{x}), q(\hat{x}), q^s(\hat{x}), q^u(\hat{x})$.
- f and inverse branches $g = f_{\hat{x}}^{-1}$ in Pesin charts:

$$\begin{cases} F_{\hat{x}} := \Psi_{\hat{f}(\hat{x})}^{-1} \circ f \circ \Psi_{\hat{x}} \\ F_{\hat{x}}^{-1} := \Psi_{\hat{x}}^{-1} \circ f_{\hat{x}}^{-1} \circ \Psi_{\hat{f}(\hat{x})} \end{cases}$$

They are small perturbations of hyp. matrices.

- ε-overlap
- Maps $F_{\hat{x},\hat{y}}$ and $F_{\hat{x},\hat{y}}^{-1}$: if $\Psi_{\hat{f}(\hat{x})}^{\eta} \overset{\varepsilon}{\approx} \Psi_{\hat{y}}^{\eta'}$, let

$$F_{\hat{x},\hat{y}} := \Psi_{\hat{y}}^{-1} \circ f \circ \Psi_{\hat{x}}.$$

If $\Psi_{\hat{x}}^{\eta} \overset{\varepsilon}{\approx} \Psi_{\hat{f}^{-1}(\hat{y})}^{\eta'}$, let

$$F_{\hat{x},\hat{y}}^{-1} := \Psi_{\hat{x}}^{-1} \circ f_{\hat{x}}^{-1} \circ \Psi_{\hat{y}}.$$

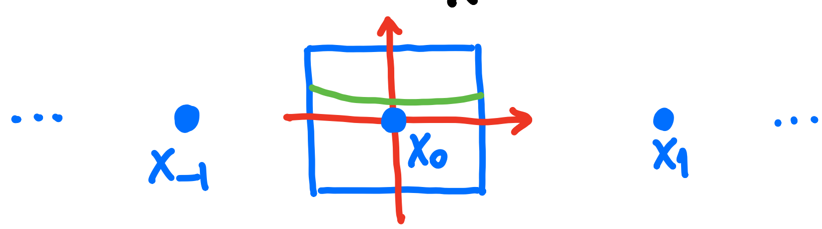
Again, they are small perturbations of hyp. matrices.

- ε-double chart $\psi_{\hat{x}}^{p^s,p^u}$.   use p^s even in nonuniformly expanding context.
- Edge $\psi_{\hat{x}}^{p^s,p^u} \xrightarrow{\varepsilon} \psi_{\hat{y}}^{q^s,q^u}$.
- Graph transforms: if $v \xrightarrow{\varepsilon} w$, define

  $\mathcal{F}^s_{v,w}: M^s_w \to M^s_v$ and $\mathcal{F}^u_{v,w}: M^u_v \to M^u_w$,

  where $M_v^{s/u}$ are graphs at the zeroth

  position x_0 of $v = \psi_{\hat{x}}^{p^s,p^u}$.



- ε-gpo $\underline{v} = \{\psi_{\hat{x}_n}^{p_n^s,p_n^u}\}_{n\in\mathbb{Z}}$

Problem: no smooth structure on $\hat{M}$

Recall: graph transforms are

- geometrically defined and

- provide shadowing.

Now: use them as before to define invariant

manifolds $\leftrightarrow$ identification of 0^{th} position

Stable/unstable manifolds of ε-gpo:

$$V^s[\underline{v}] = \lim_{n \to +\infty} \left(F^s_{v_0, v_1} \circ \cdots \circ F^s_{v_{n-1}, v_n} \right)[V_n]$$

and

$$V^u[\underline{v}] = \lim_{n \to -\infty} \left(F^u_{v_{-1}, v_0} \circ \cdots \circ F^u_{v_n, v_{n+1}} \right)[V_n].$$

Same definition as before

Stable/unstable sets of ε-gpo:

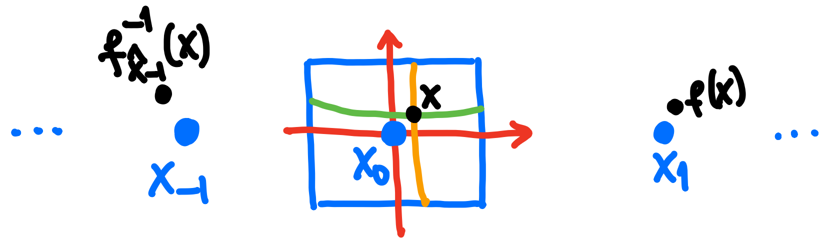
From 0^{th} position determined by $V^{s/u}[\underline{v}]$, recover other positions:

- Positive positions: just apply f
- Negative positions: each edge

$$\Psi_{\hat{x}_n}^{p_n^s, p_n^u} \xrightarrow{\varepsilon} \Psi_{\hat{x}_{n+1}}^{p_{n+1}^s, p_{n+1}^u}$$

is associated with a single inverse branch $f_{\hat{x}_n}^{-1}$.

Then negative positions are uniquely defined.



This defines invariant sets $\hat{V}^s[\underline{v}]$ and $\hat{V}^u[\underline{v}]$, which are subsets of $\hat{M}$.

Next:

- Coarse graining: $\mathcal{A}=\{\Psi_x^{p^s,p^u}\}$ countable
- $\pi:\Sigma\to\widehat{M}$ infinite-to-one extension:

$$
\begin{tikzcd}
\Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma \arrow[d, "\pi"] \\
\widehat{M} \arrow[r, "\widehat{f}"'] & \widehat{M}
\end{tikzcd}
$$

- $\mathcal{Z}$ = Markov cover of $NUH_\chi^\# \subset \widehat{M}$
- $\mathcal{R}$ = Bowen-Sinai refinement

set-theoretical –
no smoothness needed

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SYMBOLIC DYNAMICS FOR NONUNIFORMLY HYPERBOLIC SYSTEMS

ICTP, 2021 LECTURE 5

Goal: present applications of existence of Markov partitions for NUH systems

Idea: understand properties that live in $NUH_\chi^\#$ and easier to study at a symbolic level

inside Σ or $\Sigma^\#$

APPLICATIONS

1. Measures of maximal entropy (MME)

   - # MME

   - Uniqueness of MME

- Exponential decay of correlations

2. Ergodic properties of MME: the Bernoulli property

3. Counting periodic trajectories

4. Hyperbolic SRB measures

Measures of maximal entropy:

Setup: $f : M^2 \to M^2$ $C^{1+\beta}$ diffeo with $h_{top}(f) > 0$.

- μ MME (they exist, e.g. when f is C^∞, by Newhouse)

- $\chi < h_{top}(f) \xRightarrow[\text{Ruelle}]{} \mu[NUH_\chi] = 1 \xRightarrow[\text{Recurrence}]{} \mu[NUH_\chi^\#] = 1$

- $$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma \arrow[d, "\pi"] & \ni \nu \arrow[d] \\ M \arrow[r, "f"'] & M & \ni \mu \end{tikzcd}$$

  Lift μ to ν s.t.

  $h_\mu(f) = h_\nu(\sigma)$.

- $\pi|_{\Sigma^\#} : \Sigma^\# \to \mathrm{NUH}^\#_\chi$ finite-to-one $\Rightarrow \nu$ MME
- Gurevich 1969, 1970: (Σ, σ) has at most countably many MME's.

<u>THM</u> (Sarig) In the above context, $\exists$ at most countably many MME's.

QUESTION: How to go beyond and prove finiteness and/or uniqueness?

UH diffeos (Bowen): if f is transitive, then (Σ, σ) is transitive

BCS

NUH diffeos (Buzzi-Crovisier-Sarig):

- Rodrigues Hertz-Rodrigues Hertz-Tahzibi-Ures: relate homoclinic classes and SRB measures.

Step 1: Every SRB is supported in a single hom. class.

Step 2: Every hom. class supports at most one SRB.

Hence, if f is topologically transitive then $\exists$ at most one SRB.

Here, a <u>dynamical Sard's lemma</u> is used: "metric" transversality and Lebesgue measure on W^u $\Rightarrow$ actual transversality somewhere. Here, <u>low dimension</u> is essential.

su-rectangles

- BCS: In NUH context, homoclinic classes might not be disjoint but their intersection carries no entropy
- BCS: each homoclinic class is coded by a transitive (Σ, σ).

This is part of Step 2

- BCS: every measure with entropy >0 is supported in a homoclinic class. Here, a new dynamical Sard's lemma is used. Regularity on f and low dimension are essential.
- BCS: there are finitely many hom. classes with "large" entropy.

THM (Buzzi - Crovisier - Sarig) If $f: M^2 \to M^2$ is C^∞, transitive with $h_{top}(f) > 0$, then $\exists!$ MME.

Setup: $\varphi : M^3 \to M^3$ flow s.t. $X = d\varphi$ is $C^{1+\beta}$ and $X \neq 0$ everywhere, $h_{top}(\varphi) > 0$.

- N = global Poincaré section
- $f : N \to N$ Poincaré map
- NUH_x for $\varphi \leftrightarrow NUH_{x'}$ for f: ✓
- NUH^*_x for $\varphi \leftrightarrow NUH^*_{x'}$ for f: what is NUH^*_x?

Boundary effect

- Lima-Sarig: fixing μ on M, $\exists N$ s.t. $NUH^*_{x'}$ "carries" μ.

1-parameter family N_t + double counting + Borel-Cantelli lemma.

THM (Lima–Sarig) In the above context, φ has at most countably many MME.

Setup: $f: M^2 \setminus \mathscr{S} \to M^2$ $C^{1+\beta}$ with singularities and $h_{top}(f) > 0$, e.g. billiard maps

- Adapted measure: μ is adapted if $\log d(x, \mathscr{S}) \in L^1(\mu)$.

The adapted and hyperbolic measures are supported in $NUH^{\#}_{\chi}$.

Problem: are measures with large entropy adapted?

In general: wide open

THM (Baladi–Demers) For many 2-dim dispersing billiards, $\exists!$ MME and it is adapted.

Anisotropic spaces

Setup: $f: M^n \to M^n$ $C^{1+\beta}$ diffeo with $h_{top}(f) > 0$.

THM (Ben Ovadia) In the above context, f has at most countably many hyperbolic MME.

Setup: $f: M \to M$ as in Araujo-Lima-Poletti.

Problem: relate large entropy with $\begin{cases} \text{hyperbolicity} \\ \text{adaptedness} \end{cases}$

THM (Araujo-Lima-Poletti) In the above context, f has at most countably many hyperbolic and adapted MME.

Ergodic properties of equilibrium states:

- In (Σ, σ), if μ = erg. equilibrium state of Hölder continuous potential, then μ = Bernoulli or Bernoulli x rotation.

THM (Sarig) If $f: M^2 \to M^2$ $C^{1+\beta}$ diffeo and μ as above with $h_\mu(f) > 0$ is either Bernoulli or Bernoulli $\times$ rotation.

In particular, it applies to MME

THM (Lima–Sarig) If $\varphi: M^3 \to M^3$ s.t. $X = d\varphi \in C^{1+\beta}$ and $X \neq 0$ everywhere, then μ as above with $h_\mu(\varphi) > 0$ is either Bernoulli or Bernoulli $\times$ rotation. If φ is additionally contact, then μ is Bernoulli.

THM (Ben Ovadia) If $f: M^n \to M^n$ $C^{1+\beta}$ diffeo, then μ as above + hyperbolic is either Bernoulli or Bernoulli $\times$ rotation.

ALP

<u>THM</u> (Araujo-Lima-Poletti) In the context of ALP, then μ as above + hyperbolic + adapted is either Bernoulli or Bernoulli $\times$ rotation.

Open: If φ is contact, is μ Bernoulli?

For rank one, μ is K by Call-Thompson, hence Bernoulli.

Counting periodic trajectories:

Notation: $f: M \to M$, $\mathrm{Per}_n(f) = \#$ periodic points of period n.

$\varphi: M \to M$, $\mathrm{Per}_T(\varphi) = \#$ " " $\le T$.

- Gurevich 1969, 1970: In (Σ, σ) with $h_{top}(\sigma) = h > 0$, if $\exists$ MME then $\mathrm{Per}_n(f) \ge \text{const} \times e^{hn}$ for $n \gg 1$.

<u>THM</u> (Sarig) If $f: M^2 \to M^2$ $C^{1+\beta}$ with $h_{top}(f) = h > 0$ and if $\exists$ MME, then $\exists\, p \ge 1$ s.t. $\mathrm{Per}_{np}(f) \ge \text{const} \times e^{hnp}$ for $n \gg 1$.

e.g. when $f \in C^\infty$, by Newhouse.

THM (Buzzi) If additionally f is transitive, then

$\mathrm{Per}_n(f) \geq \text{const} \times e^{hn}$ for $n \gg 1$.

THM (Ben-Ovadia) If $f: M^n \to M^n$ $C^{1+\beta}$ with $h_{top}(f) = h > 0$

has hyperbolic MME, then $\exists\, p \geq 1$ s.t. $\mathrm{Per}_{np}(f) \geq \text{const} \times e^{hnp}$

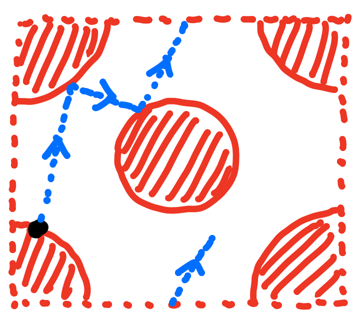
for $n \gg 1$.

THM (Buzzi) If additionally f is transitive, then

$\mathrm{Per}_n(f) \geq \text{const} \times e^{hn}$ for $n \gg 1$.

THM (Lima-Matheus, Baladi-Demers, Buzzi) For many 2-dim

dispersing billiards, $\mathrm{Per}_n(f) \geq \text{const} \times e^{hn}$ for $n \gg 1$.



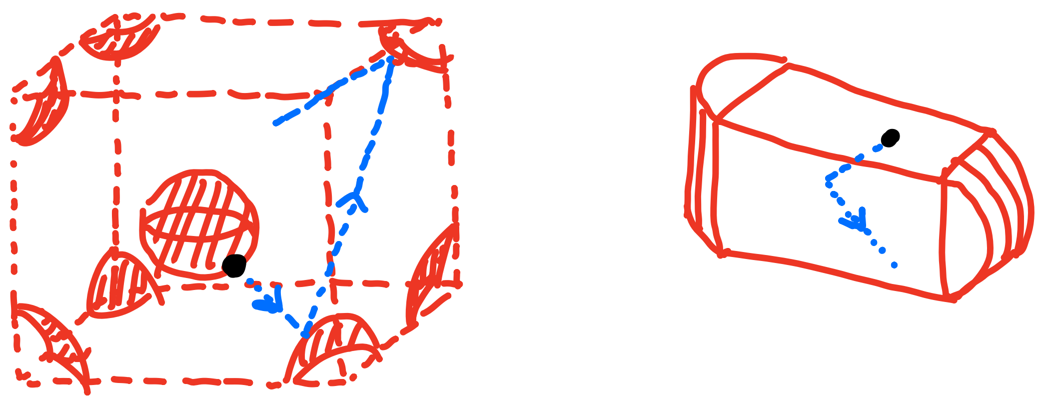
THM (Lima–Sarig) If $\varphi: M^3 \to M^3$ s.t. $X = d\varphi \in C^{1+\beta}$ and $X \neq 0$ everywhere, $h_{top}(\varphi) = h > 0$ and if $\exists$ MME, then $\mathrm{Per}_T(\varphi) \geq \text{const} \times \frac{e^{hT}}{T}$ for $T \gg 1$.

e.g. when $f \in C^\infty$, by Newhouse.

THM (ALP) If $\varphi: M^n \to M^n$ s.t. $X = d\varphi \in C^{1+\beta}$ and $X \neq 0$ everywhere, $h_{top}(\varphi) = h > 0$ and if $\exists$ hyperbolic MME, then $\mathrm{Per}_T(\varphi) \geq \text{const} \times \frac{e^{hT}}{T}$ for $T \gg 1$.

THM (ALP) For Viana maps, $\mathrm{Per}_n(f) \geq \text{const} \times e^{hn}$ for $n \gg 1$.

<u>THM</u> (ALP) For the following billiards



we have $\mathrm{Per}_n(f) \geq \text{const} \times e^{hn}$ for $n \gg 1$, where $h = h_{\mu_{SRB}}(f) > 0$.

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