Twisted Holography lecture 2

Last time:

$N=2$ SCFT $\Rightarrow$ 2d chiral algebra

Conformal anomaly $\Leftrightarrow$ BRST anomaly

What are some examples?

$N=4$ SYM

Hypermultiplet in the adjoint rep. of $u(N)$. When we twist, hypermultiplet $\Rightarrow$ symplectic bosons β_1, β_2.

Adjoint matter gives a matrix of symplectic bosons:

$$X_1, X_2 \in gl(N)$$

$$X_{1\,j}^{\,i}(0)\, X_{2\,l}^{\,k}(z) = \frac{1}{z}\delta_j^k \delta_l^i$$

bc ghosts are also matrices

$$b, c \in gl(N)$$

$$b_j^i(0)\, c_l^k(z) = \frac{1}{z}\delta_j^k \delta_l^i$$

X_1, X_2 are two of the 6 scalars of $N=4$ SYM

BRST charge is

$$Q = b^i_j c^j_k c^k_i + c^i_j X^j_{\alpha k} X^k_{\beta i} \varepsilon^{\alpha\beta}$$

There is no BRST anomaly:

Matter representation is $gl(N) \oplus gl(N)$

So $$\mathrm{Tr}_{\text{matter}}(t_a t_b) = 2\,\mathrm{Tr}_{\text{adjoint}}(t_a t_b)$$

An $N=2$ SCFT studied by Seiberg-Witten

Take gauge group $SU(2)$ with 4 fundamental hypermultiplets.

Each fundamental hyper $\Rightarrow$ 2 pairs of symplectic bosons, so 16 matter fields in the chiral algebra:

$I_{r\alpha}$ $\quad r = 1 \dots 8$

$\alpha = 1, 2$ a basis for the fundamental of $SU(2)$

OPE is

$$I_{r\alpha}(0)\, I_{s\beta}(z) = \delta_{rs}\, \varepsilon_{\alpha\beta}\, \frac{1}{z}$$

Note the $U(4)$ flavour symmetry is enhanced to $SO(8)$

Gauge fields give b, c ghosts, adjoint valued

$b_{\alpha\beta}$, $c_{\alpha\beta}$ (symmetric in α, β)

$$Q_{BRST} = \frac{1}{2} bcc + c_{\alpha\beta} I_r^{\alpha} I_r^{\beta}$$

Why is there no BRST anomaly? Take $A \in sl(2)$. We can conjugate A to the form $A = \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$ $\lambda \in \mathbb{C}$

In the fundamental, A has eigenvalues $\lambda, -\lambda$. Therefore

$$\mathrm{Tr}_{\text{matter}}(A^2) = 16\lambda^2$$

The adjoint has basis $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A = \lambda h$

$[A, e] = 2\lambda e$ $[A, f] = -2\lambda f$ $[A, h] = 0$

So, $\mathrm{Tr}_{\text{adjoint}}(A^2) = 8\lambda^2$

and $\mathrm{Tr}_{\text{matter}}(A^2) = 2\,\mathrm{Tr}_{\text{adjoint}}(A^2)$

More generally, there is an infinite sequence of $N=2$ SCFTs with $SO(8)$ flavour symmetry:

Gauge group $Sp(2N)$

($N=1$ this is $SU(2)$)

Matter is:

$$I \in \mathbb{C}^8 \otimes \mathbb{C}^{2N} \qquad I = I_{r\alpha} \qquad r = 1..8, \quad \alpha = 1 .. 2N$$

$$X_1, X_2 \in \Lambda^2_0 \mathbb{C}^{2N} \qquad X_i = X_{i\alpha\beta} = -X_{i\beta\alpha}$$

$$X_{i\alpha\beta}\, \omega^{\alpha\beta} = 0$$

Where $\omega_{\alpha\beta}$ is the symplectic form on $\mathbb{C}^{2N}$

non-zero entries $\quad \omega_{\alpha\,\alpha+N} = 1, \quad \alpha = 1..N$

$$\omega_{\alpha+N,\alpha} = -1 \qquad \alpha = 1 .. N$$

OPEs are

$$I_{r\alpha}(0) I_{s\beta}(z) = \delta_{rs}\, \omega_{\alpha\beta} \frac{1}{z}$$

$$X_{i\alpha\beta}(0)\, X_{j\gamma\delta}(z) = \frac{1}{z} \varepsilon_{ij} (\omega_{\alpha\gamma}\omega_{\beta\delta} - \omega_{\alpha\delta}\omega_{\beta\gamma})$$

Gauge group contributes

$$c_{\alpha\beta} = c_{\beta\alpha}, \quad b_{\alpha\beta} = b_{\beta\alpha}$$

$$b_{\alpha\beta}(0)\, c_{\gamma\delta}(z) = \frac{1}{z}(\omega_{\alpha\gamma}\omega_{\beta\delta} + \omega_{\alpha\delta}\omega_{\beta\gamma})$$

To write things more succinctly: raise/lower indices using $\omega_{\alpha\beta}$

Then, if we write

$$\Omega = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \qquad 2n \times 2n \text{ matrix}$$

bc satisfy

$$b^T \Omega + \Omega b = 0$$
$$c^T \Omega + \Omega c = 0$$

whereas X_1, X_2 satisfy $\quad X_i^T \Omega - \Omega X_i = 0$

In sum b, c matrices

X_1, X_2 matrices

I_r 8 vectors

We would like to study these systems at large N. To do this, we need to describe single string (aka single trace) operators at large, in BRST cohomology

All states in BRST cohomology are invariant under the action of the gauge symmetry, either $U(N)$ or $Sp(N)$

Consider first the case of $\mathcal{N}=4$ SYM.

Have X_1, X_2 adjoint-valued.

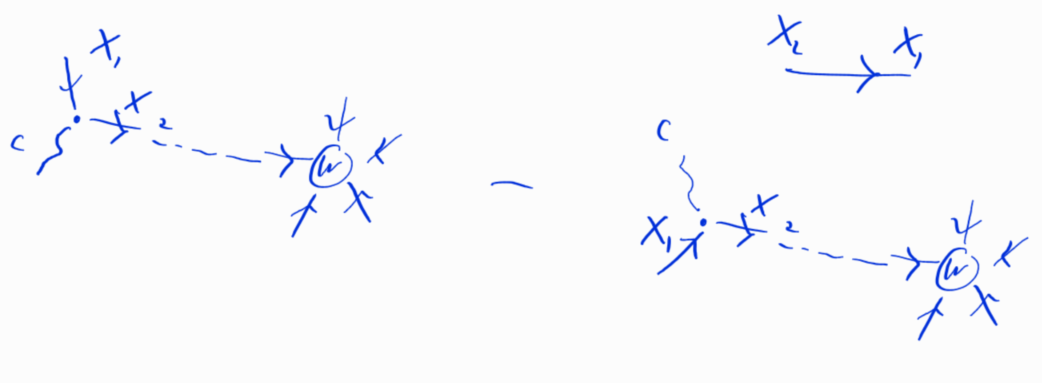
Lemma $\operatorname{Tr}(X_1^k)$ is BRST closed.

Proof:

BRST current is $\operatorname{Tr} bcc + \operatorname{Tr}(c(X_1 X_2 - X_2 X_1))$

To compute $\oint q_{BRST}(z) \operatorname{Tr}(X_1^k)(0)$

we need to look at first order poles in the OPE. Only one Wick contraction is possible:



So, $\mathrm{tr}(c(X_1 X_2 - X_2 X_1))(z)\ \mathrm{tr}(X_1^n)(0)$

$$= \sum_{r+s=n} \mathrm{tr}(X_1^r c X_1 X_1^{s-1}) - \sum_{r+s=n} \mathrm{tr}(X_1^{r-1} X_1 c X_2^s)$$

$$= 0.$$

From this we can build many more states. $SU(2)$ acts as an R symmetry, and X_1, X_2 are a doublet of $N \times N$ matrices. Therefore any state built from $\mathrm{Tr} X_1$ by applying an $SU(2)$ generator is also BRST closed. These are symmetrized traces

$$\mathrm{Tr}(X_1^{(r} X_2^{s)}) = \text{symmetrization of } \mathrm{Tr}(X_1^r X_2^s)$$

<u>Theorem</u> At large N there are four towers of single trace operators

$\mathrm{Tr}(X_1^{(r} X_2^{s)})$ bosonic, spin $\frac{r+s}{2}$

$$\left.\begin{matrix} \mathrm{Tr}(b X_1^{(r} X_2^{s)}) \\ \mathrm{Tr}(\partial c\, X_1^{(r} X_2^{s)}) \end{matrix}\right\} \text{fermionic, spin } \frac{r+s}{2}+1$$

$\varepsilon^{ij}\, \mathrm{Tr}(\partial X_i X_j X_1^{(r} X_2^{s)})$ bosonic, spin $2 + \frac{r+s}{2}$

+ terms with ghosts

All operators are obtained as normally ordered products of derivatives of these operators.

Large N operators of the model with $SO(8)$ flavour symmetry:

Theorem In this case the single-trace operators at large N are:

- Open string states

$$J_{rs}[n,m] = I_r X_1^{(n} X_2^{m)} I_s$$

where r,s runs from 1 to 8

and

$$J_{rs}[n,m] = - J_{sr}[n,m]$$

so $J_{rs}[n,m]$ transform in the adjoint of $SO(8)$

- The two bosonic towers from the $N=4$ case.

In this case all operators are bosonic.

We want to find a string theory / gravitational theory that is holographically dual to these chiral algebras.

This is the topological string.

Conjectural relationship between physical and topological string:

$$\text{IIB on } \mathbb{D}_{\varepsilon} \times \mathbb{D}_{-\varepsilon} \times \mathbb{C}^3$$

Twisting $\rightsquigarrow$ Topological B-model on $\mathbb{C}^3$

Conjectures like this have been made since the early days: e.g. Bershadsky, Cecotti, Ooguri, Vafa in 94.

There is lots of evidence but no first-principles proof. Best work along these lines:

Saberi-Williams, Eager-Hahner

Give a first principles computation that the Q cohomology of free supergravity matches the fields of the top'l string.

Outline of our approach:

IIB on $\underset{-\varepsilon}{\mathbb{D}} \times \underbrace{\underset{\varepsilon}{\mathbb{D}} \times \mathbb{C}} \times \mathbb{C}^2$

with D3 brane

$\Longrightarrow$ Top'l. String on $\mathbb{C}^3$ with D1 brane on $\mathbb{C} \subseteq \mathbb{C}^3$

Theory on D1 brane $=$ the chiral algebra

Backreact D1 branes in top'l string gives top'l string on $SL_2(\mathbb{C}) = AdS_3 \times S^3$

Twisted holography:

Top'l. String on $SL_2(\mathbb{C}) \cong$ Large N chiral algebra

Conjecture Top'l. string on $SL_2(\mathbb{C})$ is a twist of IIB on $AdS_5 \times S^5$