Twisted Holography

Lectures by K. Costello

Based on work by K.C., Si Li, Davide Gaiotto, Natalie Paquette, Yehao Zhou, Jihwan Oh, Chris Beem, Leonardo Rastelli, Junya Yagi, ...

AdS/CFT is a conjectured duality between

- $\mathcal{N}=4$ super Yang-Mills
- Type IIB string theory on $AdS_5 \times S^5$

It is very hard to prove, or even state precisely, because it relates strongly coupled gauge theory to quantum gravity.

- Defining strongly coupled gauge theory is one of the most difficult problems in mathematical physics
- Gravity is non-renormalizable - ill defined at the quantum level
- String theory on $AdS_5 \times S^5$ is very hard to define

Lots of very difficult work involved in checking the holographic dictionary

Twisted holography selects a supersymmetric sector of both sides, where everything is much easier!

- Twisted gauge theory: has no coupling constant, easy to compute things explicitly
- $\Longleftrightarrow$ Twisted supergravity: can be defined at the quantum level.

This allows exact calculations to be performed on both sides.

Note: AdS_3 holography is another case when exact computations can be performed - see Gaberdiel's talks

Twisting

Consider e.g. a 4d QFT with $N=2$ supersymmetry.

Let $S_\pm$ be the spin reps of $Spin(4)$

$\dim_{\mathbb{C}} S_\pm = \mathbb{C}^2$ $\qquad SU(2)_+ \times SU(2)_-$

$Q_\alpha \in S_+, \quad \bar{Q}_{\dot\alpha} \in S_-$

Super-charges of $N=2$ theory are

- Q_α^i
- $\bar{Q}_{\dot\alpha i}$

$i = 1, 2$

$\mathbb{C}^4 = S_+ \otimes S_-$

satisfying

$$[Q_\alpha^i, \bar{Q}_{\dot\alpha j}] = \delta^i_j \frac{\partial}{\partial x_{\alpha\dot\alpha}}$$

Traditional Perspective on Twisting:

Theory has an $SU(2)$ R-symmetry transforming the index i

- $Spin(4) = SU(2)_+ \times SU(2)_-$

Change the spin of fields by making

- $SU(2)_+$ act via the diagonal map
- $SU(2)_+ \to SU(2)_+ \times SU(2)_R$

So i index becomes α index

Then Q^i_α becomes Q^β_α $\bar{Q}_{\dot\alpha i}$

and

$$Q = \delta^\beta_\alpha Q^\beta_\alpha \qquad \rightsquigarrow \bar{Q}_{\dot\alpha\alpha}$$

is a scalar supercharge

- $Q^2 = 0$ (because Q^2 is both a scalar and a vector)

Def^n The Hilbert space of the twisted theory is the Q-cohomology of the original Hilbert space:

$$H_{twisted} = \frac{\operatorname{Ker} Q}{\operatorname{Im} Q}$$

Same for local operators etc.

Note Step one (changing the spin of the fields) does nothing on flat space!

Step two (Q cohomology) radically simplifies the theory.

This is the Donaldson-Witten twist of $N=2$ theories - it is topological.

After twisting, supercharges $\bar{Q}_{\dot\alpha i}$ become $\bar{Q}_{\dot\alpha\alpha}$ and

$$[Q, \bar{Q}_{\dot\alpha\alpha}] = \frac{\partial}{\partial x_{\dot\alpha\alpha}} \qquad Q = \delta_\alpha^{\beta} Q^\alpha_\beta$$

- On the Q cohomology of the Hilbert space, $\frac{\partial}{\partial x_{\dot\alpha\alpha}}$ acts by zero.

$Q\psi = 0$

$\frac{\partial}{\partial x_{\dot\alpha\alpha}}\psi = Q\bar{Q}_{\dot\alpha\alpha}\psi$
$= 0$ in coho.

No energy/momentum - hallmark of a topological theory

- If $\mathcal{O}$ is a Q closed local operator, then

$$\frac{\partial}{\partial x_{\dot\alpha\alpha}}\mathcal{O} = Q\bar{Q}_{\dot\alpha\alpha}\mathcal{O}$$

So, correlation functions of Q closed operators are independent of position:

$$\frac{\partial}{\partial x_{\dot\alpha_1\alpha_1}}\langle \mathcal{O}_1(x_{\dot\alpha_1\alpha_1}) \cdots \mathcal{O}_n(x_{\dot\alpha_n\alpha_n})\rangle$$

$$= \langle Q\bar{Q}_{\dot\alpha_1\alpha}\mathcal{O}_1 \cdots \mathcal{O}_n\rangle$$

$$= -\langle \bar{Q}_{\dot\alpha_1\alpha_1}\mathcal{O}_1\, Q\mathcal{O}_2 \cdots \mathcal{O}_n\rangle - \cdots$$

$$- \langle \bar{Q}_{\dot\alpha_1\alpha}\mathcal{O}_1 \cdots Q\mathcal{O}_n\rangle$$

$$= 0.$$

Twists do not have to be topological!

E.g. $\mathcal{N}=1$ theory in dimension 4 has a holomorphic twist (first studied by Johansson, Nekrasov, mid 90s)

Supercharges:

$$Q^{\alpha}, \quad \bar{Q}^{\dot\alpha}$$

The twist breaks $Spin(4)$ to $SU(2)_-$ rotates $\bar{Q}^{\dot\alpha}$, fixes Q^{α}

Let

- $Q = Q^1$ (explicitly break $SU(2)_+$)

Let

$$\left.\begin{aligned} \bar{z}_{\dot\alpha} &= x_{\dot\alpha 1} \\ z_{\dot\alpha} &= x_{\dot\alpha 2} \end{aligned}\right\} \text{hol. coordinates on } \mathbb{R}^4 = \mathbb{C}^2$$

Then,

$$\boxed{[Q^1, \bar{Q}^{\dot\alpha}] = \frac{\partial}{\partial \bar{z}_{\dot\alpha}}}$$

- Correlation functions of Q closed operators are now holomorphic:

$$\frac{\partial}{\partial \bar{z}_{\dot\alpha_1}} \langle \mathcal{O}_1(z_{\dot\alpha_1}, \bar{z}_{\dot\alpha_1}) \cdots \mathcal{O}_n(z_{\dot\alpha_n}, \bar{z}_{\dot\alpha_n}) \rangle = 0$$

↖ CR equation

Superconformal twists and the Ω background

To analyze twisted $\mathcal{N}=4$ SYM holographically, we will perform a kind of twist which localize the theory to a 2d chiral theory.

There are two approaches to this:

1) Superconformal Twist (Beem, Rastelli, ...)
2) Ω background (Nekrasov, Yagi, ...)

I will sketch the method, giving the answer in detail but not the derivation.

Consider an $\mathcal{N}=2$ SCFT in dimension 4.

8 supercharges

$$Q^{\alpha i},\ \bar{Q}^{\dot\alpha}_i \qquad i=1,2$$

which satisfy
$$[Q^{\alpha i}, \bar{Q}^{\dot\alpha}_j] = \delta^i_j \frac{\partial}{\partial x_{\alpha\dot\alpha}}$$

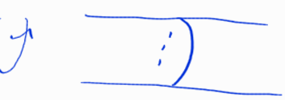
Set

$$Q = Q^{11} + \bar{Q}^{\dot 2}_2 + \varepsilon\left(Q^{22} + \bar{Q}^{\dot 1}_1\right)$$

Then
$$Q^2 = \varepsilon\underbrace{\left(\frac{\partial}{\partial x_{1\dot 1}} + \frac{\partial}{\partial x_{2\dot 2}}\right)}_{\text{This is a spacelike vector}} \qquad \underset{x_{1\dot1},\, x_{2\dot2}}{I} \times \underset{\substack{x_{1\dot2}\\ x_{2\dot1}}}{\mathbb{R}^2}$$

Make this direction periodic, and call it θ:

$$Q^2 = \varepsilon \frac{\partial}{\partial \theta}$$

θ  $\times\ \mathbb{R}^2$

Note also $\dfrac{\partial}{\partial x_{1\dot 2}}$ is Q exact.

$N=1$ chiral multiplet

$\Rightarrow$ φ bosonic

$$\frac{\partial^k}{\partial z_1} \frac{\partial^l}{\partial z_2} \varphi \rightsquigarrow \prod \frac{1}{(1-q^k)(1-p^l)}$$

ψ fermionic

$$\frac{\partial^k}{\partial z_1} \frac{\partial^l}{\partial z_2} \psi$$

$x_{1\dot{2}} = z$ is a holomorphic coordinate on $\mathbb{R}^2 = \mathbb{C}$

Theorem (J. Oh, J. Yagi, ..)

- The SCFT extends to the cigar

 $\times\ \mathbb{C}$

- Consider S^1 invariant operators at the tip of the cigar. Here $Q^2 = \frac{\partial}{\partial\theta}$ = Rotation on Cigar

$$Q^2 \mathcal{O} = 0$$

by S^1 invariance

Q-cohomology is a chiral algebra: $\rightsquigarrow$ of S^1 inv. operators at tip

$$\frac{\partial}{\partial \bar z}\mathcal{O} = Q\bar{Q}^{\dot{2}}_{1}\mathcal{O}$$

(coords: $x_{1\dot{2}} = \bar z$ $\quad x_{2\dot{1}} = z$)

This equation implies correlators are meromorphic functions of $z \in \mathbb{C}$:

$$\frac{\partial}{\partial \bar z_i}\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n)\rangle = 0$$

$\mathcal{O}_i$ Q-closed and rotation invariant.

Different Perspective: (Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees)

- Instead of using Q as above use $\mathbb{Q}$ a supercharge in superconformal algebra $psu(2,2|2)$
- $\mathbb{Q}$ cohomology gives the same chiral algebra
- There is more literature on this approach (historically it was first)

So far:

4d $\mathcal{N}=2$ QFT ⇝ 2d chiral algebra

What is the algebra? Start from a Lagrangian theory, we will build it explicitly

$\mathcal{N}=2$ Free Hypermultiplet

$\varphi_i, \bar{\varphi}_i \quad i=1,2$

$\psi_i^\alpha, \bar{\psi}_i^{\dot{\alpha}}$

$$\int \varphi_i \Box \bar{\varphi}_i + \psi_i \not{\partial} \bar{\psi}_i$$

Chiral algebra is symplectic bosons:

β_i: bosonic

OPE like that of free fermions

$\beta_i(z)$ (comes from φ_i), spin $\frac{1}{2}$

OPE $\beta_i(0)\beta_j(z) = \varepsilon_{ij}\frac{1}{z}$

Comes from Lagrangian $\int \beta_i \bar{\partial} \beta_j \varepsilon^{ij}$

Gauge theory $\mathcal{N}=2$ vector multiplet contributes

b_a, c^a ghosts (a the colour index)

- b_a is spin one
- c^a spin zero

both fermionic

OPE is $b_a(0)c^b(z) = \delta_a^{\ b}\frac{1}{z}$

BRST Charge:

- $Q_{BRST} = \oint \frac{1}{2} f_{ab}^{\ \ c} :b_c c^a c^b:$

for pure gauge theory.

Coupled system, gauge theory with matter:

- $\beta_i \in R$ a representation of g which is symplectic.

Action given by a current

$\omega_{ij} = -\omega_{ji}$

g-inv. pairing

$$J_a = \frac{1}{2} M_a^{ij} :\beta_i \beta_j:$$

BRST charge for the coupled system is

$$\frac{1}{2} f_{ab}{}^{c} :b_c c^a c^b: + c^a J_a \qquad Q_{BRST}^2 = 0$$

<u>Theorem</u> The BRST charge is nilpotent if and only if, for $t_a, t_b \in g$

$$2\,Tr_g(t_a t_b) = tr_R(t_a t_b)$$

□.

(Similar, but easier than, the calculation of the critical dimension of string theory)

This condition is the same as the one that guarantees a 4d $N=2$ theory is superconformal.

<u>Goal:</u> Study these chiral algebras holographically, at large N (for gauge group $su(N)$, $sp(N)$, ..)

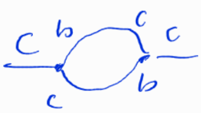
Q^2:

Gauge sector

OPE.

Tree level (one Wick contraction): vanishes by Jacobi.

One loop:



$$bcc(0)\; bcc(z) \sim \frac{1}{z^2} c^a(0) c^b(z)\, 2\,Tr_g(t_a t_b)$$

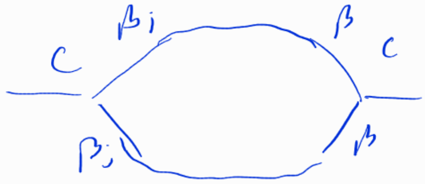
2: b, c run around loop

c is fermionic

$$c^a(0) c^b(0) \overbrace{Tr(t_a t_b)}^{\text{symmetric}} = 0$$

$bcc(0)\, bcc(z) \sim \frac{-1}{z} c^a(0)\, \partial c^b(0)\, 2\,\mathrm{Tr}_g(t_a t_b)$

Matter: Same:



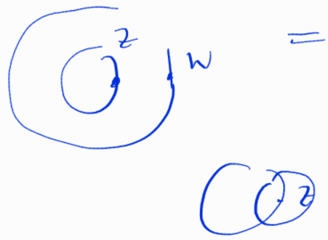
$\rightsquigarrow \frac{1}{z} c^a \partial c^b\, \mathrm{Tr}_R(t_a t_b)$

So,

$Q_{BRST}(0)\, Q_{BRST}(z)$ has no pole at $z=0$

iff $\quad \mathrm{Tr}_R(t_a t_b) = 2\,\mathrm{Tr}_g(t_a t_b)$

$\left[\oint Q_{BRST}(z), \oint Q_{BRST}(w) \right]$



$= \oint_z \oint_{|w|=\varepsilon} Q_{BRST}(z)\, Q_{BRST}(z+w)$

$\neq 0$ unless trace identity holds.

1) $N=2$ theory has a twist

hol/topl. on $\quad \mathbb{R}^2 \times \mathbb{C}$

Yagi et al: $\rightsquigarrow$ Poisson VOA

Remnant of supercont. symmetry.

Ω b.g. $\overset{?}{\longleftrightarrow}$ $\quad S$ twist

differ by

something Q exact

$\underbrace{\mathbb{C}^2/\Gamma \times \underline{\mathbb{C}}}_{D3} \times \underset{\mathbb{R}^2}{\text{flat}}$

D1 brane in B-model: same chiral alg.

Hol. twist:

$\mathbb{R}^4 = \mathbb{C}^2$ coords z_1, z_2 $\bar{z}_1, \bar{z}_2$

$\frac{\partial}{\partial \bar{z}_i}$ is Q exact

Q-coho: $\frac{\partial}{\partial \bar{z}_i} \mathcal{O} = 0$

Top' twist: ALSO have $\frac{\partial}{\partial z_i} \mathcal{O} = 0$