The Abdus Salam
International Centre
for Theoretical Physics

A Virtual
International Graduate Course on
Tight Closure and its Applications

Supported by ICTP

16 May-25 July 2022

The aim of the course is to provide training in the foundations of commutative algebra and algebraic geometry in prime characteristic and to present some of the exciting recent developments. The local theory of prime characteristic singularities is a beautiful and historically important subject. Singularities which are defined in terms of the behaviour of the Frobenius endomorphism have been labeled “F-singularities”. The course gives an introduction on the most prominent F-singularity classes that emerged from Hochster–Huneke’s tight closure theory. Since its introduction in the late 1980s, it has had a dramatic effect on the field of commutative algebra. Tight closure gives unified proofs and strong generalisations of many major theorems in commutative algebra, as well stimulated recent proofs of longstanding conjectures.

Scientific Committee

Name	**Affiliation**
Maria Evelina Rossi	University of Genoa, Genoa, Italy
Karen Smith	University of Michigan, Ann Arbor, MI, USA
Irena Swanson	Purdue University, West Lafayette, IN, USA
Ngo Viet Trung	Institute of Mathematics, Hanoi, Vietnam
Bernd Ulrich	Purdue University, West Lafayette, IN, USA
Keiichi Watanabe	Nihon University, Tokyo, Japan
Jugal Verma	Indian Institute of Technology Bombay, Mumbai, India

Organising Committee

Name	**Affiliation**
Claudio Arezzo	International Centre forTheoretical Physics
Maria Evelina Rossi	University of Genoa, Genoa, Italy
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Speakers and Titles of lectures

Mondays and Wednesdays

1.00-3.00 pm GMT

Lectures. There will be 3 lectures and a tutorial on each topic each week. The speakers will provide detailed typed notes of their lectures on the day they finish their lectures. These notes will contain exercises which the participants should try to solve and submit to the tutorial instructor before 12 pm GMT on Fridays.

Tutorials. The tutorial on each topic will be held a week after the lectures are delivered. The tutorial instructor will present solutions to problems assigned by the Lecturer that are received from the participants and discuss some material covered during the lectures. Grades will be assigned to the solutions. Four best performers will be awarded cash prizes and certificates.

Title	**Speaker** **Tutor**	**Affiliation**
1. Tight closure	Neil Epstein Kriti Goel	George Mason University Fairfax VA, USA University of Utah, USA
2. Test ideals	Kevin Tucker Kenta Sato	University of Illinois, Chicago, IL, USA Kyushu University, Japan
3. Direct summands	Linquan Ma Cheng Meng	Purdue University, West Lafayette, IN, USA Purdue University, USA
4. F-rational rings and rational singularities	Florian Enescu Kyle Maddox	Georgia State University, Atlanta, GA, USA University of Kansas, USA

5. Hilbert-Kunz multiplicities	Vijaylaxmi Trivedi Mandira Mondal	Tata Institute of Fundamental Research, Mumbai, India Chennai Mathematical Institute, India
6. Briancon-Skoda Theorems	Ian Aberbach Suprajo Das	University of Missouri, Columbia, MO, USA Chennai Mathematical Institute, India
7. Big Cohen-Macaulay algebras	Thomas Polstra Mitra Koley	University of Virginia, Charlottesville, VA, USA ISI, Kolkata
The absolute integral closure	Gennady Lyubeznik	University of Minnesota, Minneapolis, MN, USA
8. Symbolic powers of ideals	Eloisa Grifo Vaibhav Pandey	University of Nebraska, Lincoln-Nebraska, USA University of Utah, USA
9. The localization problem	Wenliang Zhang Sudeshna Roy	University of Illinois, Chicago, IL, USA Tata Institute of Fundamental Research, Mumbai, India
10. Uniform Artin-Rees results	Irena Swanson Kriti Goel	Purdue University, West Lafayette, IN, USA University of Utah, USA

Title, Abstracts and Pre-requisites

Speaker: Craig Huneke, University of Virginia, USA

Title: Where does tight closure come from?

Abstract: The talk will share some details about the first couple of years of the development of tight closure, and give one example of a "tight closure" argument. Time will be given for any questions as well concerning the development of the theory.

(1) **Speaker:** Neil Epstein, George Mason University, Fairfax, VA

Title: An overview of tight closure of ideals

Abstract: This series of lectures serves as an introduction to tight closure of ideals and submodules. Examples and proofs will be provided to ground the students ' understanding of the subject. We place the subject into the general context of closure operations while indicating what is special about the tight closure operation in particular. This provides context for some of the later lecture series on subjects such as Briancon-Skoda theorems, localization problems, test ideals, and direct summands.

(2) Speaker: Kevin Tucker, University of Illinois, Chicago, IL

Title: Test ideals

Abstract: Test ideals were first introduced by Mel Hochster and Craig Huneke in their celebrated theory of tight closure in the 80's and 90's, and since their invention have been closely tied to the theory of Frobenius splittings. Subsequently, test ideals have also found application far beyond their original scope to questions arising in complex analytic geometry, as well as recently even in mixed characteristic. In these lectures, we will overview some of these constructions, with a view towards active research questions. In particular, we will highlight topics such as the connection to the multiplier ideal via reduction to positive characteristic, threshold values and jumping numbers, and applications.

Prerequisites: Sections 1.1, 1.2, 1.5, 1.6, 2.1, 2.2, 3.2, 3.3, 3.5 and the Appendix on Dimension Theory from Bruns-Herzog.

(3) **Speaker:** Linquan Ma, Purdue University, West Lafayette, IN

Title: Direct summands of regular rings

Abstract: We will use F-singularity theory to give quick proof of a celebrated result of Hochster and Roberts that direct summands (or more generally, pure subrings) of regular rings are Cohen-Macaulay, in the positive characteristic case. We will also discuss some of its generalizations. The key idea behind is a notion called strong F-regularity.

Prerequisites: Chapters 1,2,3 of Bruns-Herzog.

(4) **Speaker:** Florian Enescu, Georgia State University, Atlanta, GA

Title: F-rational rings and rational singularities

Abstract: These lectures will introduce the notion of F-rational rings and present its place in tight closure theory. The major results on F-rationality will be presented, highlighting the Gorenstein case as well as the implications on the local cohomology of a local ring with support in its maximal ideal. Connections to birational geometry, namely rational singularities, will also be discussed.

Prerequisites: Chapters 1,2,3 in Bruns and Herzog (and Matsumura, Chapters 6 and 7). For connections to birational geometry, chapter 3 in Hartshorne is useful.

(5) **Speaker:** Vijaylaxmi Trivedi, Tata Institute of Fundamental Research,

Mumbai, India

Title: Introduction to Hilbert-Kunz multiplicity

Abstract: These lectures will be a self-contained introduction to the theory of Hilbert-Kunz multiplicity eHK. This will involve proving existence of the HK multiplicity, implication of various bounds on the HK multiplicity. We briefly give several applications of this invariant to other seemingly unrelated characteristic p invariants. In the end we list some open problems in this area.

Prerequisites:

(1) Chapter 5 Section 13 and 14 of Commutative ring theory by H. Matsumura
(2) Section 8.2 of Cohen-Macaulay rings by Bruns-Herzog

(6) **Speaker:** Ian Aberbach, University of Missouri, Columbia, MO, USA

Title: Briancon-Skoda Theorems

Abstract: The original Briancon-Skoda theorem was a statement made about the ring of convergent power series over the complex numbers. Hochster saw that the statement was, in fact, a purely algebraic one. Roughly speaking, the theorem says that in a regular local ring, a sufficiently high power of any ideal (with that power being no more than the dimension of the ring) is contained in any reduction of the ideal. In a short time, the theorem was proved for regular local rings by Avinash Sathaye and Lipman and for certain ideals in rational singularities by Lipman and Teissier.

With the introduction of tight closure, Hochster and Huneke showed that a vast generalization of the Briancon-Skoda theorem holds in rings of positive characteristic, with an elementary proof. Moreover, tight closure methods also allowed for the investigation of the coefficients in the theorem, allowing, for instance, early cases of the theorem that over a regular ring, the Cohen-Macaulayness of an associated graded ring of an ideal implies the Cohen-Macaulayness of the Rees ring. We will discuss these results and a number of other generalizations and improvements of the Briancon-Skoda theorem.

Pre-requisites from Bruns-Herzog : 2.2 and 4.5 (in the original 1993 version)
Pre-requisites from Matsumura: Section 14 and section 19

(7) **Speaker:** Thomas Polstra, University of Virginia, Charlottesville, VA

Title: Big Cohen-Macaulay algebras and their applications

Abstract: One of the big successes of prime characteristic commutative algebra is that the absolute integral closure of an excellent local domain or prime characteristic is a weakly functorial big Cohen-Macaulay algebra. We will discuss applications of the existence of weakly functorial Cohen-Macaulay algebras as they relate to the various homological conjectures, colon capturing property of tight closure, and homological properties of pure subrings of regular rings.

(8) **Speaker:** Gennady Lyubeznik, University of Minnesota, Minneapolis, MN, USA

Title: Cohen-Macaulayness of the absolute integral closure in characteristic $p>0$.

Abstract: A celebrated result of Melvin Hochster and Craig Huneke says that the absolute integral closure of a Noetherian excellent domain in characteristic $p>0$ is Cohen-Macaulay. I am going to discuss a shorter proof of this result (under slightly different assumptions) due to Craig Huneke and myself.

(9) **Speaker:** Eloisa Grifo, University of Nebraska, Lincoln, NE, USA

Title: Symbolic powers of ideals

Abstract: Symbolic powers arise naturally from the theory of primary decomposition, and appear ubiquitously throughout commutative algebra. This topic also has connections to geometry: for a radical ideal in a polynomial ring over an algebraically closed field, the nth symbolic power of I describes the polynomials that vanish to order n on the corresponding variety. In this lecture series, we will introduce the basics of the theory of symbolic powers and discuss some of the main open problems in this field, with a focus on the many contributions of Hochster and Huneke's work to the topic.

Pre-requisites: Matsumura's Commutative Ring Theory sections 6 and 16.

(10) **Speaker:** Wenliang Zhang, University of Illinois, Chicago, IL, USA

Title: The localization problem

Abstract: In this lecture series, we will discuss topics centered around the problem whether tight closure commutes with localization. These topics are closely related to singularities in characteristic p, such as 'F-regularity'. We will discuss Hochster-Huneke's original approach using a uniform annihilation of local cohomology modules, as well as more recent developments due to Lyubeznik-Smith and Brenner-Monsky.

(11) **Speaker:** Irena Swanson, Purdue University, West Lafayette, IN, USA

Title: Uniform Artin-Rees results

Abstract: The lectures will start with the classical Artin-Rees Lemma and how it is used. We will then go through the main points of the proofs of various uniform versions and applications. The emphasis will be on characteristic-p results, but we will also look at other contexts. One of the goals is to showcase the power of tight closure methods