Gaussian-ology

Start with $p_G(x;1) \equiv \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Normalisation $\int_{-\infty}^{\infty} dx\, p_G(x;1) = 1$

$$\Big[N^2 \equiv \Big(\int_{-\infty}^{\infty} dx\, p_G(x;1)\Big)^2 = \int \frac{dx\,dy}{2\pi} e^{-(x^2+y^2)/2} = \int_0^{\infty} dr\, r \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-r^2/2} = \int_0^{\infty} dz\, e^{-z} \; [z = r^2/2] = 1 \Big]$$

Mean: $\langle x \rangle = \int_{-\infty}^{\infty} dx \cdot x\, p_G(x;1) = 0$

Variance:

$\langle x^2 \rangle - \langle x \rangle^2 = \int_{-\infty}^{\infty} dx\, x^2 p_G(x;1) = 1$

Odd moments: $\langle x^{2n+1} \rangle = 0$

Even moments: $\langle x^{2n} \rangle = (2n-1)!!$

How to prove the above?

$\longrightarrow$ Fourier representation

$$\mathcal{F}[p_G](k) \equiv \int_{-\infty}^{\infty} dx\, e^{-ikx} p_G(x;1)$$

$$= \langle e^{-ikx} \rangle = e^{-k^2/2}$$

So $\langle x^n \rangle = \int_{-\infty}^{\infty} dx \cdot x^n p_G$

$$= \left(i \frac{\partial}{\partial k}\right)^n \mathcal{F}[p_G](k) \Big|_{k=0}$$

$\longrightarrow$ $n=0$: $\langle 1 \rangle = 1$ ✓

$n=1$: $\langle x \rangle = -ik e^{-k^2/2} \Big|_{k=0} = 0$

$n=2$: $\langle x^2 \rangle = -i^2 \frac{\partial}{\partial k}\left(k e^{-k^2/2}\right) \Big|_{k=0} = 1$

& so on

Another way of writing this:

$$\left(-\frac{\partial}{\partial x}\right)^n P_G(x;1) \equiv H_n(x) P_G(x;1)$$
$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} (-ik)^n e^{-k^2/2}$$

&

$$(-ik)^n e^{-k^2/2} = \int_{-\infty}^{\infty} dx\, e^{-ikx} P_G(x;1) H_n(x)$$

$\left[\text{Recall } \int_{-\infty}^{\infty} dx\, e^{-ikx} = 2\pi \delta_D(k)\right]$

Note that $\langle H_n(x)\rangle = 0$ for all $n \geq 1$.

Introducing variance $\sigma^2 \neq 1$
$\longrightarrow$ dimensional analysis

$$P_G(x;1) \longrightarrow P_G(x;\sigma^2) = \frac{1}{\sigma} P_G(x/\sigma;1)$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

$$\langle 1 \rangle = 1$$
$$\langle x \rangle = 0, \quad \langle x^2 \rangle = \sigma^2$$
$$\langle x^{2n+1} \rangle = 0 \quad ; \quad \langle x^{2n} \rangle = \sigma^{2n} (2n-1)!!$$
$$\mathcal{F}[P_G](k) = \langle e^{-ikx} \rangle = e^{-k^2\sigma^2/2}$$
$$(-ik\sigma)^n e^{-k^2\sigma^2/2} = \int_{-\infty}^{\infty} dx\, e^{-ikx} P_G(x;\sigma^2) H_n(x/\sigma)$$
$$P_G(x;\sigma^2) H_n(x/\sigma) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} (-ik\sigma)^n e^{-k^2\sigma^2/2}$$

Introducing mean $\langle x \rangle = \mu \neq 0$

$x \longrightarrow x - \mu$ in P_G

Fourier Transforms pick up phase $e^{-ik\mu}$

Generating functions

[i.e., function whose Taylor coefficients are the quantity of interest]

$$M(k) = \mathcal{F}[p_G](\mp ik) = \langle e^{kx} \rangle = e^{\mu k} e^{k^2\sigma^2/2}$$

is the moment generating function

[Since $\langle x^n \rangle = \left(i\frac{\partial}{\partial k}\right)^n e^{-k^2/2}\Big|_{k=0}$.
(e.g., with $\mu = 0$)

$\longrightarrow$ with $k \to \mp ik$

$\longrightarrow$ $e^{k^2\sigma^2/2} = \sum_{n=0}^{\infty} \frac{1}{n!} k^n \langle x^n \rangle$]

Cumulant generating function

$$C(k) \equiv \ln M(k)$$

So in this case

$$C(k) = \mu k + \frac{1}{2} k^2 \sigma^2$$

$\longrightarrow$ Defining property of Gaussian distribution: all cumulants of order $n \geq 3$ are zero.

$\longrightarrow$ So any p.d.f. with non-vanishing higher order cumulant(s) is non-Gaussian.

Multivariate Gaussian

Generalisation to $N > 1$ variables.

Now variance σ^2 is replaced with covariance matrix

$$C_{ij} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$$

So diagonal elements are individual variances $\sigma_j^2 = C_{jj}$.

while off-diagonals are cross-correlations.

Full expression:

$$\boxed{P_{(N)}(x_1, \ldots x_N) = \frac{1}{(2\pi)^{N/2} |\det C|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T C^{-1} (x-\mu)}}$$

$\mu = (\mu_1, \ldots \mu_N)$ is mean vector.

Making sense of this

If C is diagonal $= \mathrm{diag}(\sigma_1^2, \ldots, \sigma_N^2)$

then $P_{(N)}(x_1, \ldots x_N) = \prod_{j=1}^{N} P_G(x_j - \mu_j ; \sigma_j^2)$

$\longrightarrow$ obviously normalised correctly

If C is not diagonal, find rotation R such that

$$\Lambda = R C R^T = \mathrm{diag}(\lambda_1^2, \ldots \lambda_N^2) \qquad [R^T R = \mathbb{1}_N]$$

Define $y = R \cdot (x - \mu)$, so $x - \mu = R^T y$

Then

$$P_{(N)}(x_1, \dots x_N) = \frac{1}{(2\pi)^{N/2}\lambda_1 \dots \lambda_N} e^{-\frac{1}{2} y^T \Lambda^{-1} y}$$

[because
$C = R \Lambda R^T$
$\Rightarrow C^{-1} = R \Lambda^{-1} R^T$]

&

$$\left|\frac{\partial y}{\partial x}\right| = |\det R| = 1$$

$$\Rightarrow \quad d^N x \, P_{(N)}(x_1, \dots x_N) = \prod_{j=1}^{N} dy_j \, P_G(y_j ; \lambda_j^2)$$

$\longrightarrow$ again, correctly normalised.

Marginalisation

Say we want to ignore x_N.
What is the resulting distribution
$P_{(N-1)}(x_1, \dots x_{N-1})$?

Again, Fourier transforming helps

Use $\quad P_G(y_j ; \lambda_j^2) = \int_{-\infty}^{\infty} \frac{dq_j}{2\pi} e^{iq_j y_j} e^{-q_j^2 \lambda_j^2 /2}$

$$= d^N x \, P_{(N)}(x_1 \dots x_N) = d^N y \int \frac{d^N q}{(2\pi)^N} e^{iq^T y} e^{-q^T \Lambda q /2}$$

But $\quad y = R \cdot (x - \mu) \;\&\; \Lambda = R C R^T$

So define $\quad k = R^T q \;\Rightarrow\; d^N q \to d^N k$

Then

$$\boxed{P_{(N)}(x_1, \dots x_N) = \int \frac{d^N k}{(2\pi)^N} e^{ik^T (x-\mu)} e^{-k^T C k/2}}$$

Now marginalising is easy.

$$P_{(N-1)}(x_1, \ldots x_{N-1})$$
$$\equiv \int_{-\infty}^{\infty} dx_N \; P_{(N)}(x_1, \ldots x_N)$$
$$= \int dx_N \int \frac{d^N k}{(2\pi)^N} e^{i k^T (x-\mu)} e^{-k^T C k/2}$$
$$= \int \frac{d^N k}{(2\pi)^N} e^{-k^T C k/2} \; e^{i k^T_{(N-1)}(x_{(N-1)} - \mu_{(N-1)})} \underbrace{\int dx_N \, e^{i k_N (x_N - \mu_N)}}_{(2\pi)\delta_D(k_N)}$$

$$= \underbrace{\int \frac{d^{N-1} k}{(2\pi)^{N-1}} e^{i k^T_{(N-1)}(x_{(N-1)} - \mu_{(N-1)})} \; e^{-k^T_{(N-1)} C_{(N-1)} k_{(N-1)}/2}}_{\boxed{\text{N-1 dimensional Gaussian}}}$$

where $C_{(N-1)ij} = C_{ij} \quad , 1 \leq i, j \leq N-1$

i.e., simply ignore the row & column in C that corresponds to the variable being integrated over.

$$\therefore \; P_{(N-1)}(x_1, \ldots x_{N-1}) = \frac{1}{(2\pi)^{\frac{N-1}{2}} |\det C_{(N-1)}|^{1/2}} e^{-\frac{1}{2}\underbrace{(x-\mu)^T}_{\text{N-1 dimensional}} C_{(N-1)} \underbrace{(x-\mu)}_{\text{N-1 dimensional}}}$$

Conditional distributions:

Consider bivariate Gaussian

$$p(x_1, x_2) = \int \frac{d^2k}{(2\pi)^2} e^{i k^T (x-\mu)} e^{-k^T C k/2}$$

$$C = \begin{pmatrix} \sigma_1^2 & C_{12} \\ C_{12} & \sigma_2^2 \end{pmatrix}$$

Bayes' thm says

$$p(x_2 | x_1) = \frac{p(x_2, x_1)}{p(x_1)}$$

~~Here~~ $p(x_1) = \cancel{P_G(x_1;\sigma}\; P_G(x_1 - \mu_1 ; \sigma_1^2)$

Note that $\ln p(x_2|x_1)$ is the difference of a quadratic form in (x_2, x_1) and a quadratic form in x_1.

$\Rightarrow$ $\ln p(x_2|x_1)$ is a quadratic form in x_2, parametrised by x_1 (which also appears quadratically)

$\Rightarrow$ $p(x_2|x_1)$ is Gaussian!

$\Rightarrow$ We only need to compute mean & variance

$$\langle x_2 | x_1 \rangle = \int dx_2 \cdot x_2 \, p(x_2|x_1)$$

$$= \int dx_2 \cdot \frac{1}{P_G(x_1)} \int \frac{d^2k}{(2\pi)^2} e^{-k^T C k/2} e^{ik^T(x-\mu)} x_2$$

$$= \frac{1}{P_G(x_1)} \int dx_2 \int \frac{d^2k}{(2\pi)^2} e^{-k^T C k/2} \left(\mu_2 - i\frac{\partial}{\partial k_2}\right) e^{ik^T(x-\mu)}$$

$$= \frac{1}{P_G(x_1)} \int dx_2 \int \frac{d^2k}{(2\pi)^2} e^{ik^T(x-\mu)} \left(\mu_2 + i\frac{\partial}{\partial k_2}\right) e^{-k^T C k/2}$$

$$= \frac{1}{P_G(x_1)} \int \frac{d^2k}{(2\pi)^2} \cdot (2\pi)\delta_D(k_2)\, e^{ik_1(\cancel{x}_1-\mu_1)}$$

$$\times e^{-k^T C k/2} (\mu_2 - ik_2 C_{22} - ik_1 C_{12})$$

$$= \frac{1}{P_G(x_1)} \int \frac{dk_1}{(2\pi)} e^{ik_1(x_1-\mu_1)} e^{-k_1^2\sigma_1^2/2} (\mu_2 - ik_1 C_{12})$$

$$\cancel{= P_G(x_1}$$

$$= \frac{1}{P_G(x_1-\mu_1;\sigma_1^2)} \cdot P_G(x_1-\mu_1;\sigma_1^2)\left[\mu_2 + H_1\left(\frac{x_1-\mu_1}{\sigma_1}\right)\cdot\frac{C_{12}}{\sigma_1}\right]$$

$$\Rightarrow \boxed{\begin{aligned} &\langle x_2|x_1\rangle \\ &= \mu_2 + \frac{C_{12}}{\sigma_1^2}(x_1-\mu_1) \\ &= \langle x_2\rangle + (x_1 - \langle x_1\rangle)\frac{Cov(x_1,x_2)}{Var(x_1)} \end{aligned}}$$

Similar calculation gives

$$\boxed{\begin{aligned} Var(x_2|x_1) &= \langle x_2^2|x_1\rangle - \langle x_2|x_1\rangle^2 \\ &= Var(x_2) - \frac{(Cov(x_1,x_2))^2}{Var(x_1)} \\ &= \sigma_2^2 - C_{12}^2/\sigma_1^2 \end{aligned}}$$

[Verify this]

Bardeen et al. (1986) [BBKS] Appendix D
generalises this to multivariate conditional Gaussians

If $p(\vec{Y_A}, \vec{Y_B})$ is Gaussian with $\langle \vec{Y_A} \rangle = 0 = \langle \vec{Y_B} \rangle$
then
$p(\vec{Y_B} | \vec{Y_A})$ is multivar. Gaussian
with

$$\langle \vec{Y_B} | \vec{Y_A} \rangle = \langle Y_B \otimes Y_A \rangle \langle Y_A \otimes Y_A \rangle^{-1} \vec{Y_A}^T$$

& $\langle \Delta Y_B \otimes \Delta Y_B | \vec{Y_A} \rangle$

$$= \langle Y_B \otimes Y_B \rangle - \langle Y_B \otimes Y_A \rangle \langle Y_A \otimes Y_A \rangle^{-1} \langle Y_A \otimes Y_B \rangle$$

where $\Delta Y_B \equiv Y_B - \langle Y_B | Y_A \rangle$.

Random Fields & Correlations

Inflation tells us that "initial" density field had tiny fluctuations whose Fourier modes obeyed (nearly) Gaussian Statistics.

$\delta_{\vec{k}} \sim$ Gaussian means that its real & imaginary parts are independent Gaussian variables.

Start with cube of side L $[V = L^3]$

$$\delta_{\vec{k}} = \int_V d^3x \, e^{-i\vec{k}\cdot\vec{x}} \delta(\vec{x})$$

$$\delta(\vec{x}) = \frac{1}{V} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} \delta_{\vec{k}}$$

(Later we'll take the continuum limit
$$\frac{1}{V}\sum_{\vec{k}} \to \int \frac{d^3k}{(2\pi)^3} \text{)}$$

Note that $\delta_{\vec{k}} = a_{\vec{k}} + i b_{\vec{k}}$ is a weighted sum of Gaussians

$\Rightarrow$ $\delta(\vec{x})$ is also a weighted sum of Gaussians

$\Rightarrow$ $\delta(\vec{x})$ is itself Gaussian distributed

Translational invariance (statistical homogeneity)

$\Rightarrow$ $\langle \delta(\vec{x})\, \delta(\vec{x}+\vec{r}) \rangle$ independent of $\vec{x}$

$\Rightarrow$ $\sum_{\vec{k}} \sum_{\vec{k}'} e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot(\vec{x}+\vec{r})} \langle \delta_{\vec{k}} \delta^*_{\vec{k}'} \rangle$

independent of $\vec{x}$

$\Rightarrow$ $\langle \delta_{\vec{k}} \delta^*_{\vec{k}'} \rangle \sim \delta^K_{\vec{k},\vec{k}'}$

$\Rightarrow$ modes with different $\vec{k}$ are uncorrelated or independent.

$\rightarrow$ Important exception:

$$\delta_{-\vec{k}} = \delta^*_{\vec{k}}$$

because we want $\delta(\vec{x})$ to be real.

But this simply means that all d.o.f. are contained in the right half cube "$\vec{k} > 0$" [Actually, e.g., $k_z > 0$]

There's no contradiction since each $\delta_{\vec{k}}$ has 2 d.o.f ($a_{\vec{k}}$ & $b_{\vec{k}}$), so

total number of d.o.f = total number of modes.

How to count d.o.f in Gaussian field?

Fix $\vec{k}$ (">0"), $\delta_{\vec{k}} = a_{\vec{k}} + i b_{\vec{k}}$

Then

$$da_{\vec{k}}\, db_{\vec{k}}\, g_{\vec{k}}(a_{\vec{k}}, b_{\vec{k}}) = \text{Gaussian}$$
$$= \frac{1}{2\pi\mu_{\vec{k}}^2} e^{-\frac{1}{2\mu_{\vec{k}}^2}(a_{\vec{k}}^2 + b_{\vec{k}}^2)}\, da_{\vec{k}}\, db_{\vec{k}}$$

P.d.f. of modes is

$$\mathcal{P}[\{\delta_{\vec{k}}\}] = \prod_{\text{"}\vec{k}>0\text{"}} g_{\vec{k}}(\delta_{\vec{k}})$$
$$\propto e^{-\sum_{\text{"}\vec{k}>0\text{"}} (a_{\vec{k}}^2 + b_{\vec{k}}^2)/2\mu_{\vec{k}}^2}$$

But $\delta_{\vec{k}}^* = \delta_{-\vec{k}} \Rightarrow a_{-\vec{k}} = a_{\vec{k}},\ b_{-\vec{k}} = -b_{\vec{k}}$

i.e $a_{-\vec{k}}^2 = a_{\vec{k}}^2$ & $b_{-\vec{k}}^2 = b_{\vec{k}}^2$

$$\Rightarrow \mathcal{P}[\{\delta_{\vec{k}}\}] \propto \exp\left(-\sum_{\text{all } \vec{k}} \frac{1}{2} \frac{a_{\vec{k}}^2 + b_{\vec{k}}^2}{2\mu_{\vec{k}}^2}\right)$$

or

$$\boxed{\mathcal{P}[\{\delta_{\vec{k}}\}] \propto \exp\left(-\sum_{\text{all } \vec{k}} \frac{|\delta_{\vec{k}}|^2}{2P(\vec{k})}\right)}$$

$$\boxed{P(\vec{k}) = 2\mu_{\vec{k}}^2 = \langle |\delta_{\vec{k}}|^2 \rangle}$$

(statistical isotropy)

Rotational invariance demands

$$P(\vec{k}) = P(k)$$

$P(k)$ is called the power spectrum.

Alternative counting

Recast $a_{\vec{k}}, b_{\vec{k}}$ as amplitude $r_{\vec{k}}$ & phase $\phi_{\vec{k}}$

$$\therefore \quad \delta_{\vec{k}} = r_{\vec{k}}\, e^{i\phi_{\vec{k}}}$$

$$\longrightarrow \quad g_{\vec{k}}(r_{\vec{k}}, \phi_{\vec{k}})\, dr_{\vec{k}}\, d\phi_{\vec{k}}$$

$$= \frac{1}{2\pi \mu_k^2} e^{-r_{\vec{k}}^2/2\mu_k^2}\, r_{\vec{k}}\, dr_{\vec{k}}\, d\phi_{\vec{k}}$$

$$= \frac{d\phi_{\vec{k}}}{(2\pi)} \cdot \frac{d(r_{\vec{k}}^2)}{P(k)} e^{-r_{\vec{k}}^2/P(k)}$$

I.e., $\phi_{\vec{k}}$ is uniformly distributed in $[0, 2\pi)$ and $r_{\vec{k}}^2 = |\delta_{\vec{k}}|^2$ is exponentially distributed with $\langle r_{\vec{k}}^2 \rangle = P(k)$.

Uncorrelated modes

$$\Rightarrow \quad P(\delta_{\vec{k}}, \delta_{\vec{k}'}) \sim g_{\vec{k}}(\phi_{\vec{k}}, r_{\vec{k}})\, g_{\vec{k}'}(\phi_{\vec{k}'}, r_{\vec{k}'})$$

$\Rightarrow$ Amplitudes as well as phases of different modes are uncorrelated.

$\longrightarrow$ So non-Gaussianity can show up as non-uniformity of phases, or phase correlations, or amplitude correlations (despite translational invariance)

In real space,

$$|\delta_{\vec{k}}|^2 = \int d^3x\, d^3y\, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\, \delta(\vec{x})\, \delta(\vec{y})$$

$$\therefore P[\{\delta(\vec{x})\}] \propto \exp -\frac{1}{2} \sum_{\vec{k}} \int d^3x\, d^3y\, \delta(\vec{x})\, \delta(\vec{y})\, \frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{P(k)}$$

Define

$$F(\vec{x}-\vec{y}) = \sum_{\vec{k}} \frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{P(k)}$$

$$\therefore P[\{\delta(\vec{x})\}] \propto \exp -\frac{1}{2} \int d^3x\, d^3y\, \delta(\vec{x})\, F(\vec{x}-\vec{y})\, \delta(\vec{y})$$

Note that $F(\vec{x}-\vec{y}) \not\propto \delta_D(\vec{x}-\vec{y})$,

i.e. uncorrelated modes
$\Rightarrow$ real space 2-point function is non-trivial.

$F(\vec{x}-\vec{y})$ is like an inverse covariance matrix.

Continuum limit.

$$\frac{1}{V}\sum_{\vec{k}} \rightarrow \int \frac{d^3k}{(2\pi)^3} \quad ; \quad P(k)\big|_{\text{discrete}} \rightarrow V\,P(k)$$

$$\delta^K_{\vec{k},\vec{k}'} \rightarrow \frac{(2\pi)^3}{V}\,\delta_D(\vec{k}-\vec{k}')$$

So $\quad \langle \delta_{\vec{k}}\,\delta^*_{\vec{k}'} \rangle = \delta^K_{\vec{k},\vec{k}'}\,P(k)$

$$\downarrow$$

$$\boxed{\langle \delta_{\vec{k}}\,\delta^*_{\vec{k}'} \rangle = (2\pi)^3\,\delta_D(\vec{k}-\vec{k}')\,P(k)}$$

1-point p.d.f of $\delta(\vec{x})$:

$$p(\delta) = P_G(\delta;\sigma^2)$$

where

$$\sigma^2 \equiv \langle \delta(\vec{x})^2 \rangle = \int \frac{d^3k\,d^3k'}{(2\pi)^6}\,e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\,\langle \delta_{\vec{k}}\,\delta^*_{\vec{k}'} \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3}\,P(k) \equiv \int d\ln k\,\Delta^2(k)$$

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}$$

For CDM-like spectrum, $P(k)$ has too much UV power

$\rightarrow$ δ needs to be smoothed

$$\delta_R(\vec{x}) = \int d^3x'\,\delta(\vec{x}')\,W_R(\vec{x}-\vec{x}') = \int \frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{x}}\,\delta_{\vec{k}}\,W_R(\vec{k})$$

usually spherically symmetric

$$p(\delta_R) = P_G(\delta_R;\sigma^2(R))$$

$$\sigma^2(R) = \int d\ln k\,\Delta^2(k)\,W_R(k)^2$$

[Try proving this using $p(\delta_R) = \int \mathcal{D}\{\delta_k\}\,P(\delta_R,\{\delta_k\}) = \int \mathcal{D}\{\delta_k\}\,\delta_D\!\left(\delta_R - \frac{1}{V}\sum_{k}\delta_k W_R(k) e^{ikx}\right) P(\{\delta_k\})$]

E.g. Common quantity is σ_8 where

$$\sigma_8^2 = \int d\ln k\,\Delta^2_{\text{lin}}(k)\,W^{(\text{TopHat})}_{R=8\,\text{Mpc}/h}(k)^2$$

Correlation function

- Recall $F(\vec{x}-\vec{y})$ was analog of inverse covariance matrix.

- Analog of covariance matrix is the correlation function

$$\boxed{\xi(\vec{r}) \equiv \langle \delta(\vec{x})\, \delta(\vec{x}+\vec{r}) \rangle}$$

Simple calculation shows

$$\xi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, P(k)$$

So that, due to rotational invariance

$$\boxed{\xi(\vec{r}) = \xi(r) = \int d\ln k\ \Delta^2(k) \left(\frac{\sin kr}{kr}\right)}$$

Examples :

- $P(k) \propto k^n$, $-3 \leq n \leq -1$

  $\Delta^2(k) \propto k^{n+3}$

  $\xi(r) \propto r^{-(n+3)} \left| \Gamma(n+2) \sin \frac{n\pi}{2} \right|$

- $P(k) \propto k\, e^{-kR_c}$ [toy CDM ($\sim$ WDM)]

  $\Delta^2(k) \propto k^4 e^{-kR_c}$

  $$\xi(r) \propto \frac{(3R_c^2 - r^2)R_c^4}{(R_c^2 + r^2)^3} = \frac{(3 - (r/R_c)^2)}{(1 + (r/R_c)^2)^3}$$

  As $r \to 0$: $\xi(r) \to$ constant

  As $r \to \infty$: $\xi(r) \to 0^-$, having crossed zero at ~~$r = R_c/\sqrt{3}$~~ $r = \sqrt{3} R_c$.

- Full CDM : $P(k) = A\,(k/k_0)^{\overset{\text{primordial slope, } \approx 1}{n_s}}\, T(k)^2$

  $T(k)$ describes decay of modes that entered the Hubble radius during radiation domination.

  $T(k \to 0) \to 1$

  $$T(k \gg k_{eq}) \to \frac{12}{(k/k_{eq})^2} \ln(k/8k_{eq})$$

  where $k_{eq} = a_{eq} H(a_{eq})$, $\Omega_{m0}\, a_{eq}^{-3} = \Omega_{rad,0}\, a_{eq}^{-4}$

  $\xi(r)$ needs numerical evaluation