ICTP 2022 — Inflation

Homework set 3

Problem 3.1: Consider a universe filled with a massive inflaton that decays into radiation with decay rate Γ_ϕ. Denote by ρ_ϕ and ρ_γ the energy densities of the inflaton and of the radiation, respectively, and by $H \equiv \frac{\dot{a}}{a}$ the Hubble rate (a is the scale factor of the Universe, and dot denotes time differentiation).

The system is governed by the equations

$$\begin{aligned} &\dot{\rho}_\phi + \left(3H + \Gamma_\phi\right)\rho_\phi = 0 \\ &\dot{\rho}_\gamma + 4H\rho_\gamma = \Gamma_\phi \rho_\phi \\ &\rho_\phi + \rho_\gamma = 3M_p^2 H^2 \end{aligned} \tag{1}$$

(i) Verify that, for $\Gamma_\phi = 0$, the solutions $\rho_\phi\left(a\right)$ and $\rho_\gamma\left(a\right)$ agree with what we found in homework 1, namely $\rho_\phi \propto a^{-3}$ and $\rho_\gamma \propto a^{-4}$.

(ii) We are interested in the evolution of ρ_γ at very early times. At these times $\rho_\gamma \ll \rho_\phi$, and $\Gamma_\phi \ll H$. Use these approximations in the first and third equations of the above system, and find the approximate solution for $\rho_\phi\left(a\right)$ and $H\left(a\right)$. Take the initial condition $\rho_\phi = \bar{\rho}$ at $a = 1$.

Show that the second equation in the above system can be rewritten as

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{\Gamma_\phi}{aH}\,\rho_\phi \tag{2}$$

Insert the solutions $\rho_\phi\left(a\right)$ and $H\left(a\right)$ into this equation. Solve this equation to obtain the early time solution for $\rho_\gamma\left(a\right)$. Take the initial condition $\rho_\gamma = 0$ (hint: it might be useful to consider the differential equation for the combination $a^4\rho_\gamma$).

(iii) Find the maximum value of $\rho_\gamma\left(a\right)$, and denote it by $\rho_{\rm max}$.

(iv) A common approximation done in studying this problem is to assume that the inflaton instantaneously decays at $\Gamma_\phi = H$. Use the third equation above to find the energy density of radiation at the decay obtained with this assumption. Denote it as $\rho_{\rm inst}$.

(v) Discuss the ratio $\frac{\rho_{\rm max}}{\rho_{\rm inst}}$, showing that it is parametrically given by the ratio between the initial Hubble rate and the inflaton decay rate. Notice that this quantity can be several orders of magnitude greater than one.

Problem 3.2: From the previous problem, we have seen that ρ_γ reaches a peak soon after the end of inflation, and then it decreases as $a^{-3/2}$, while the inflaton is still dominating. Consider a particle X, produced by light particles γ from the thermal bath with a $\gamma + \gamma \to X + X$ process, having cross section proportional to a fixed poweer of the temperature, $\sigma \propto T^n$. Its physical number density is governed by

$$\frac{dN_X}{dt} + 3HN_X = \sigma\, n_\gamma^2 \tag{3}$$

where we recall that $n_\gamma \propto T^3$ (notice that we are neglecting the decrease of n_γ due to the production of X, and the inverse process $X + X \to \gamma + \gamma$; these assumptions are appropriate if the cross section is sufficiently small). Compute the abundance $Y_X \equiv \frac{n_X}{n_\gamma}$ and study wether its growth is dominated by earlier times (when ρ_γ is close to its peak) or by late times (when the decay of the inflaton completes). In the former case, the approximation of instantaneous inflaton decay is a bad approximation for the purpose of computing the final abundance of X. In the latter case, the approximation is good. To see which times dominate, show that the abundance evolves according to

$$Y_X = t^{-5/4} \int_{t_{\rm in}}^{t} \frac{dt'}{t'} C\,(t')^\alpha \tag{4}$$

where C is constant and α is related to n. Find this relation. The production is dominated by the earliest times $t_{\rm in}$ if $\alpha < 0$.