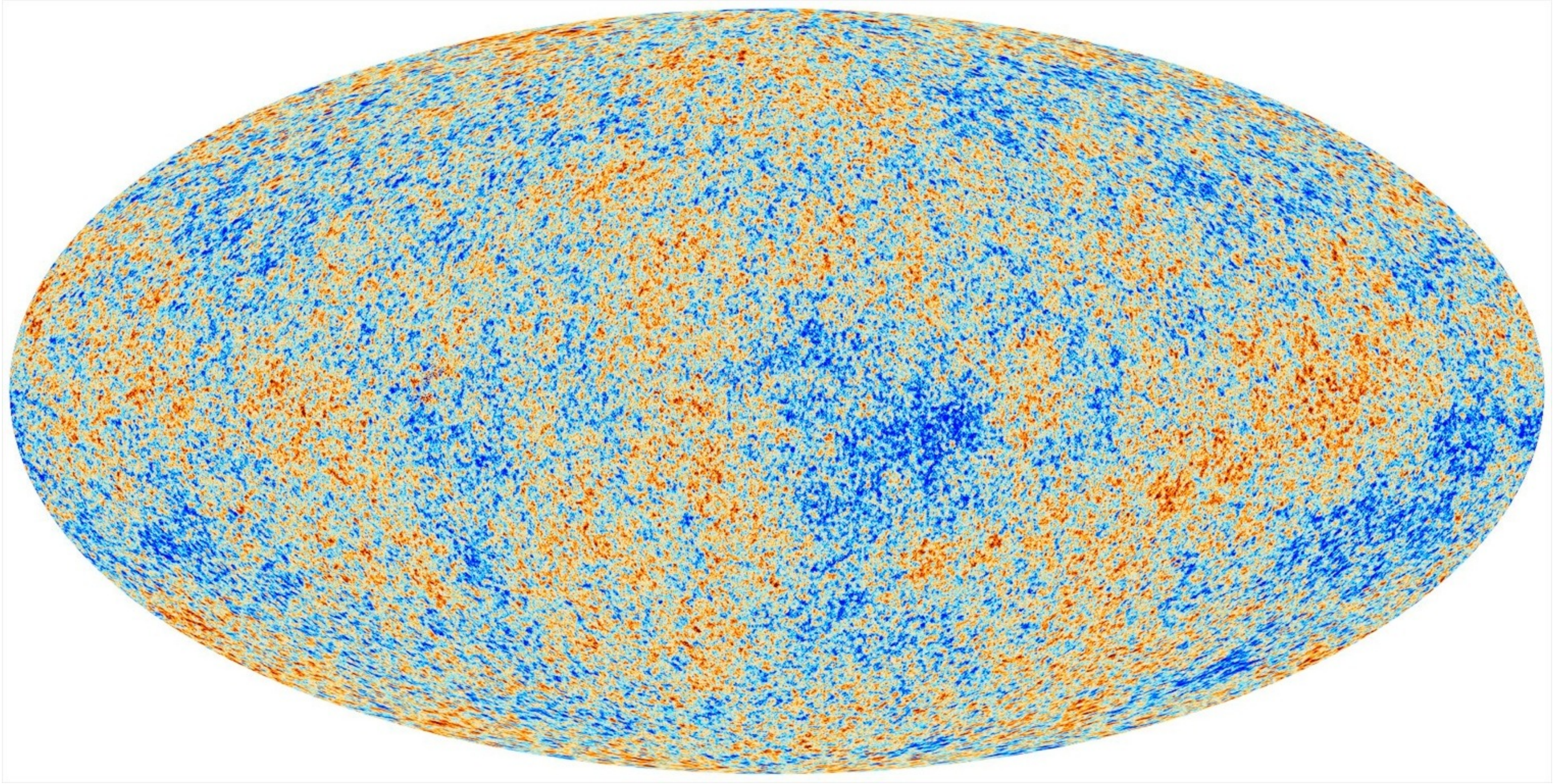
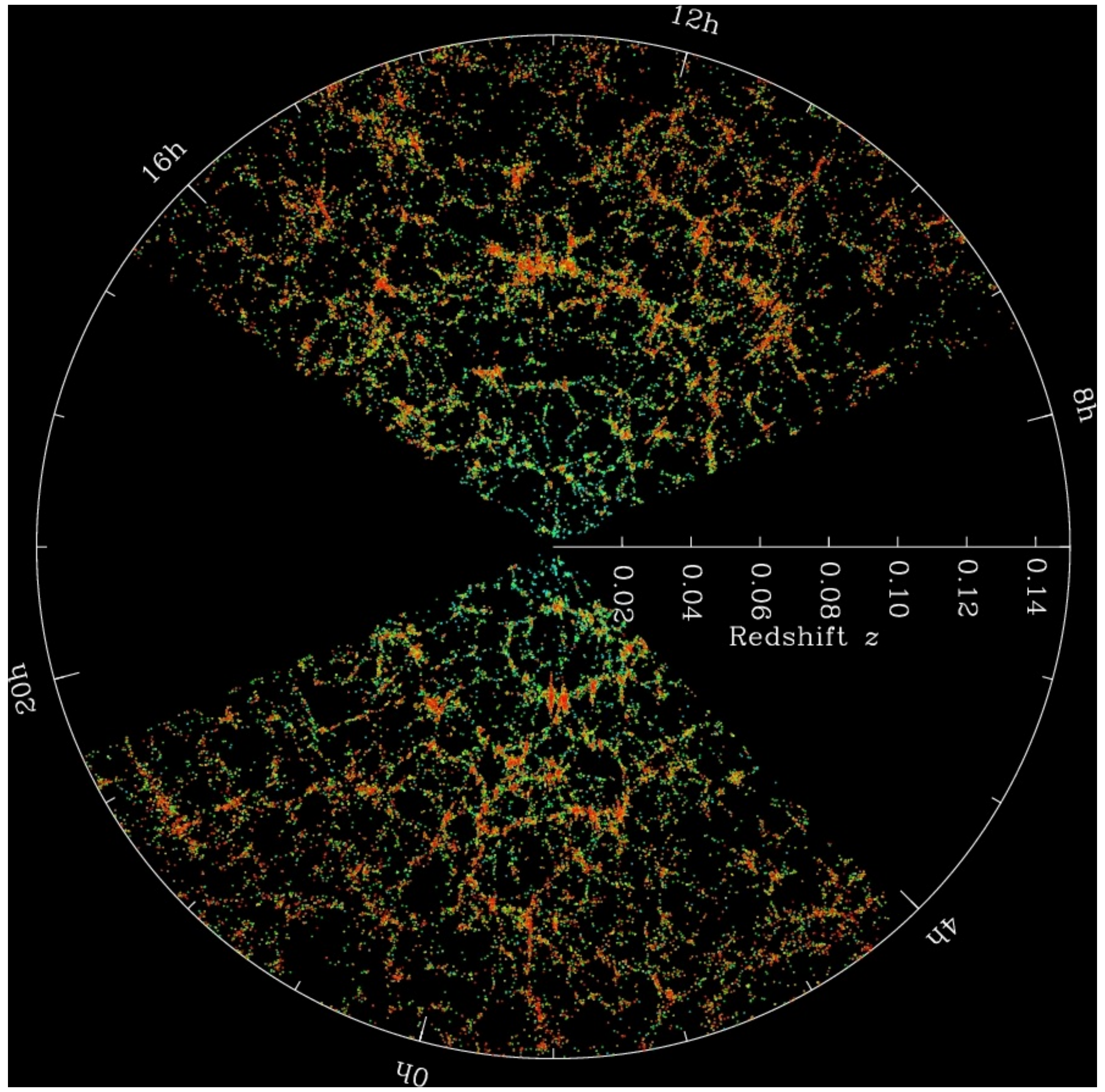
Large-Scale Structure

3. Growth of Structure

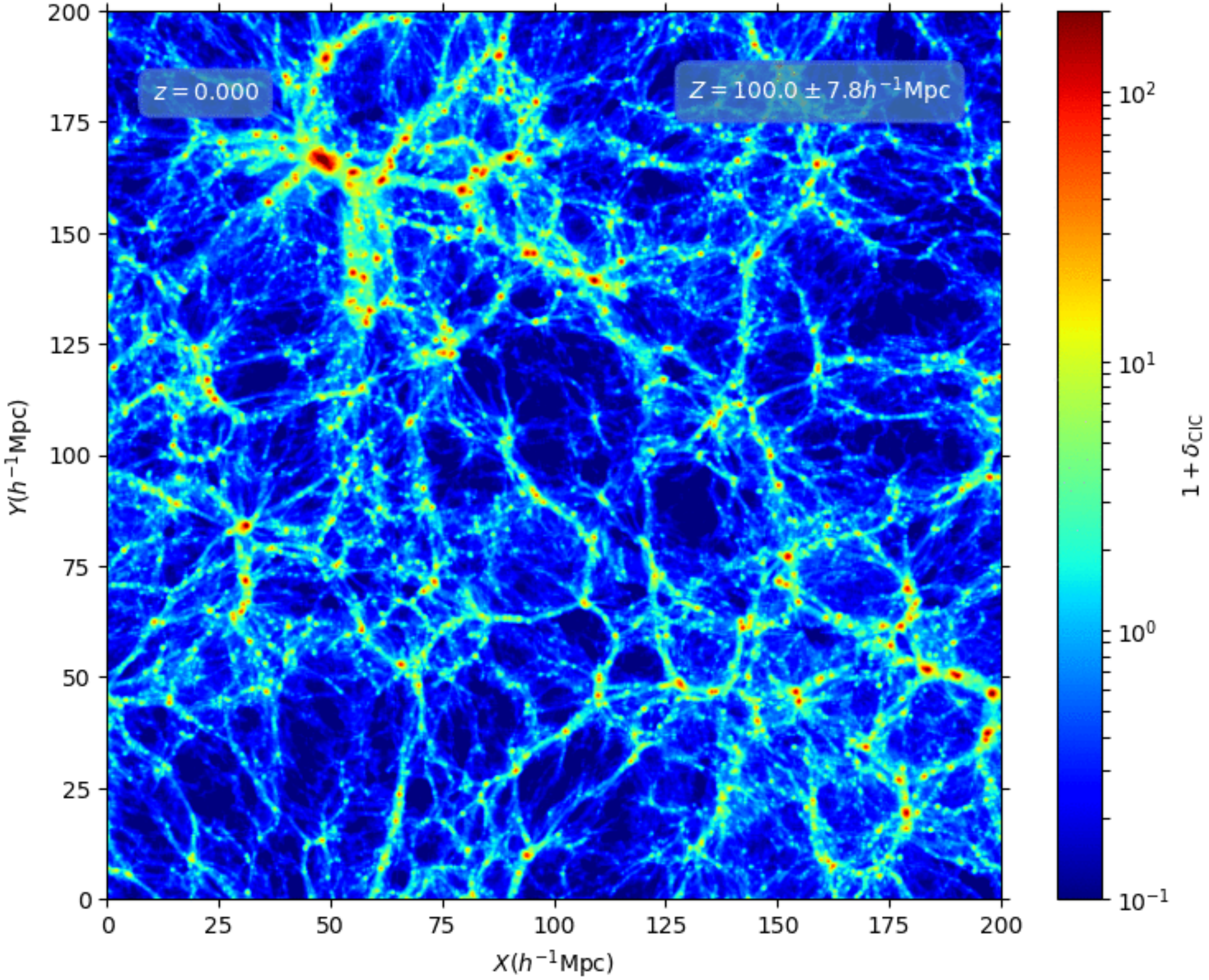




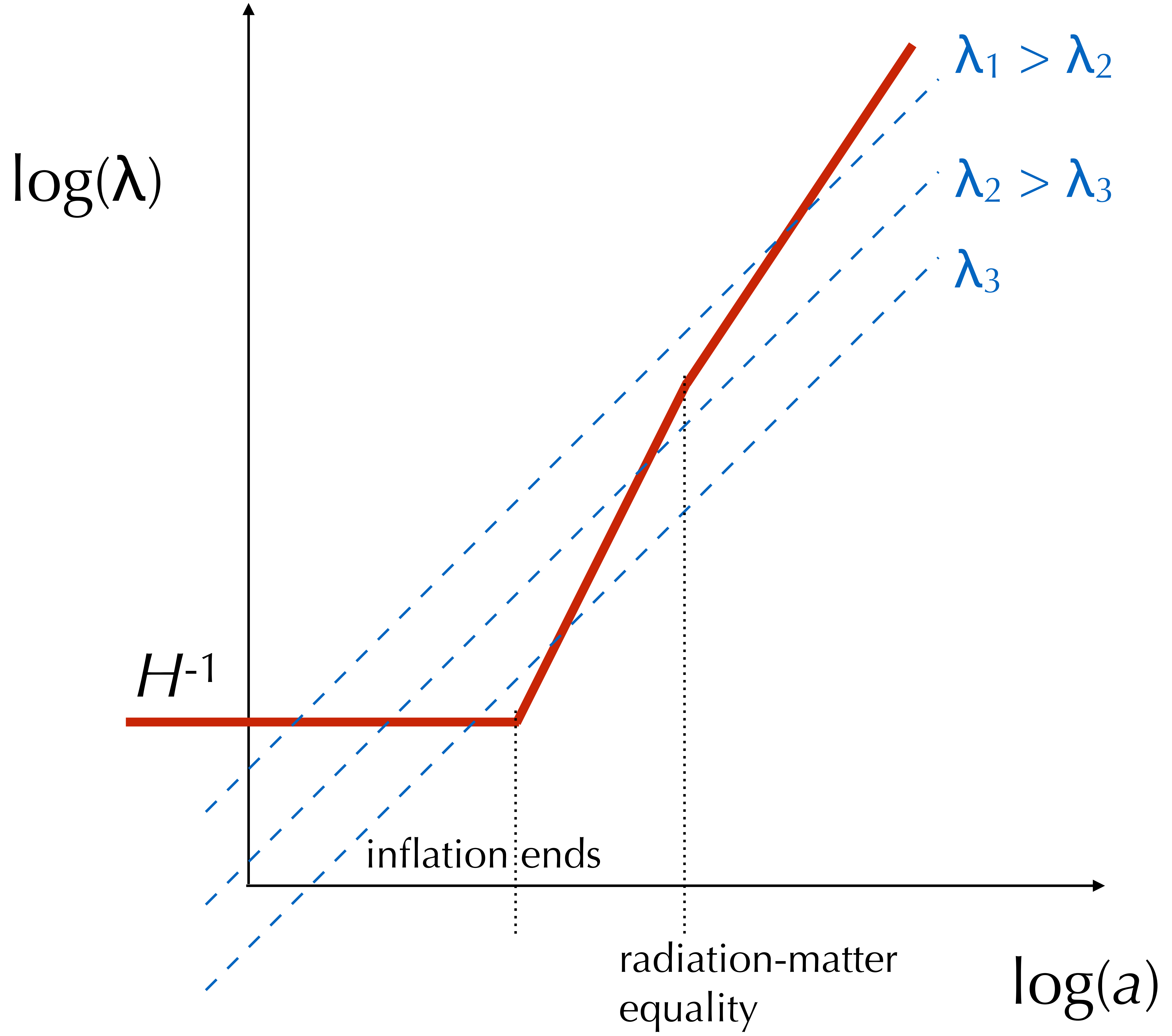
Aseem Paranjape

ICTP Summer School on Cosmology, July 2022

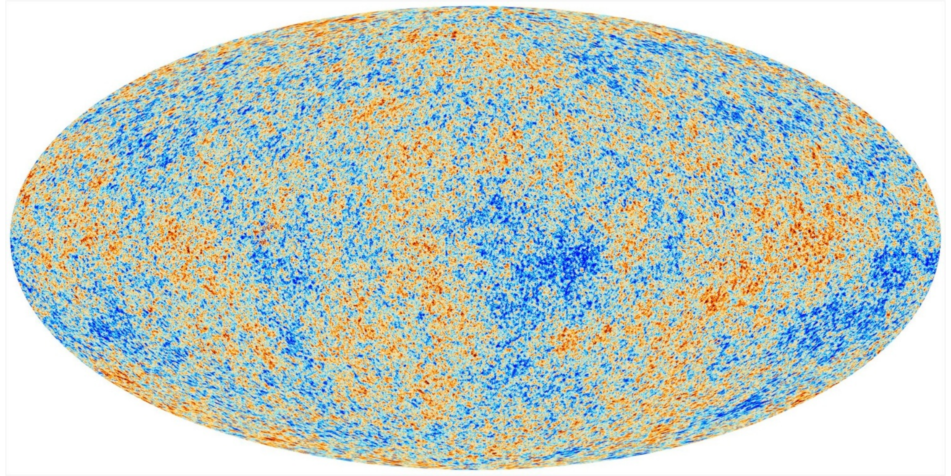
Growth of Structure



Length Scales



Linear evolution



Assume early Universe (say, prior to CMB last scattering) well described using small perturbations on FLRW geometry ($c = 1$). Assume $\Omega_k = 0$.

Metric: $\mathrm{d}s^2 = a(\eta)^2\left[-\left(1+2\varphi(\eta,\mathbf{x})\right)\mathrm{d}\eta^2 + \left(1-2\psi(\eta,\mathbf{x})\right)\mathrm{d}\mathbf{x}^2\right]$ *[See Dodelson, `Modern Cosmology']*

Conformal Hubble parameter: $\mathscr{H} \equiv \mathrm{d}\ln a/\mathrm{d}\eta = aH$

Recall Friedmann equation: $\mathscr{H}^2 = a^2 H_0^2\left[a^{-3}\Omega_{\mathrm{m}0} + a^{-4}\Omega_{\mathrm{R}0} + \Omega_{\Lambda 0}\right]$

Photon distribution: Bose-Einstein with $T = \bar{T}(\eta)\ \left(1+\Theta(\eta,\mathbf{x},\hat{p})\right)$, $\bar{\rho}_\gamma \propto \bar{T}^4$

Dark matter density: $\rho_{\mathrm{dm}} = \bar{\rho}_{\mathrm{dm}}(\eta)\left(1+\delta_{\mathrm{dm}}(\eta,\mathbf{x})\right)$; Baryon density: $\rho_{\mathrm{b}} = \bar{\rho}_{\mathrm{b}}(\eta)\left(1+\delta_{\mathrm{b}}(\eta,\mathbf{x})\right)$

Dark matter peculiar velocity: $\mathbf{v}_{\mathrm{dm}}(\eta,\mathbf{x})$; Baryon peculiar velocity: $\mathbf{v}_{\mathrm{b}}(\eta,\mathbf{x})$

Fourier convention: $f(\eta,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{i\,\mathbf{k}\cdot\mathbf{x}} f_{\mathbf{k}}(\eta) \equiv \int_{\mathbf{k}} \mathrm{e}^{i\,\mathbf{k}\cdot\mathbf{x}} f_{\mathbf{k}}(\eta)$

Linear evolution

Linearise Boltzmann equations and Einstein equations. Work in Fourier space. Define $\dot{f} \equiv \mathrm{d}f/\mathrm{d}\eta$ and $\mu \equiv \hat{p} \cdot \hat{k}$.

Define monopole, dipole moments of photon temperature fluctuation:

$$\Theta_{0\mathbf{k}}(\eta) \equiv \frac{1}{4\pi}\int \mathrm{d}\Omega\, \Theta_{\mathbf{k}}(\eta, \hat{p}) \; ; \quad \Theta_{1\mathbf{k}}(\eta) \equiv \frac{i}{4\pi}\int \mathrm{d}\Omega\, \Theta_{\mathbf{k}}(\eta, \hat{p})\, (\hat{p} \cdot \hat{k}) \; ; \quad \Theta_{\ell\mathbf{k}}(\eta) \equiv \frac{i^{\ell}}{4\pi}\int \mathrm{d}\Omega\, \Theta_{\mathbf{k}}(\eta, \hat{p})\, \mathscr{P}_{\ell}(\hat{p} \cdot \hat{k})$$

Assume irrotational flow (valid since no linear sources):

$\mathbf{v}_{\mathrm{dm},\mathbf{k}} \equiv \hat{k}\, v_{\mathrm{dm},\mathbf{k}}$; $\mathbf{v}_{\mathrm{b},\mathbf{k}} \equiv \hat{k}\, v_{\mathrm{b},\mathbf{k}}$ *[so if $\mathbf{v} = \nabla\omega$, then $v_{\mathbf{k}} = ik\,\omega_{\mathbf{k}}$]*

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Equations (neglecting neutrinos):

Photon Boltzmann equation

DM continuity ; Baryon continuity
DM Euler ; Baryon Euler

Einstein equations

Linear evolution

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Equations (neglecting neutrinos):

$$\dot{\Theta}_{\mathbf{k}} + ik\mu\,\Theta_{\mathbf{k}} = \dot{\psi}_{\mathbf{k}} - ik\mu\,\varphi_{\mathbf{k}} - \dot{\tau}\left[\Theta_{0\mathbf{k}} - \Theta_{\mathbf{k}} + \mu v_{\mathrm{b}\mathbf{k}}\right]\ ;$$ *[τ is electron scattering optical depth: $\dot{\tau} = -\,n_{\mathrm{e}}\sigma_{\mathrm{T}}a$]*

$$\dot{\delta}_{\mathrm{dm},\mathbf{k}} = -\,ikv_{\mathrm{dm},\mathbf{k}} + 3\dot{\psi}_{\mathbf{k}} \quad ; \quad \dot{\delta}_{\mathrm{b},\mathbf{k}} = -\,ikv_{\mathrm{b},\mathbf{k}} + 3\dot{\psi}_{\mathbf{k}}$$

$$\dot{v}_{\mathrm{dm},\mathbf{k}} + \mathscr{H}v_{\mathrm{dm},\mathbf{k}} = -\,ik\,\varphi_{\mathbf{k}}\ ; \quad \dot{v}_{\mathrm{b},\mathbf{k}} + \mathscr{H}v_{\mathrm{b},\mathbf{k}} = -\,ik\,\varphi_{\mathbf{k}} + \dot{\tau}R\left[3i\Theta_{1\mathbf{k}} + v_{\mathrm{b},\mathbf{k}}\right]\ ;$$ *[$R \equiv (4\bar{\rho}_{\gamma})/(3\bar{\rho}_{\mathrm{b}})$]*

$$k^2(\varphi_{\mathbf{k}} - \psi_{\mathbf{k}}) = -\,32\pi Ga^2\,\bar{\rho}_{\gamma}\,\Theta_{2\mathbf{k}}$$

$$3\mathscr{H}\dot{\psi}_{\mathbf{k}} + k^2\psi_{\mathbf{k}} - 3\mathscr{H}^2\varphi_{\mathbf{k}} = -\,4\pi Ga^2\left[\bar{\rho}_{\mathrm{dm}}\delta_{\mathrm{dm},\mathbf{k}} + \bar{\rho}_{\mathrm{b}}\delta_{\mathrm{b},\mathbf{k}} + 4\bar{\rho}_{\gamma}\Theta_{0\mathbf{k}}\right]$$

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Baryon-photon coupling

Acoustic oscillations (CMB, BAO)

Equations (neglecting neutrinos):

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[$R \equiv (4\bar{\rho}_{\gamma})/(3\bar{\rho}_{\mathrm{b}})$]

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Gravitational coupling
Growth of structure

Equations (neglecting neutrinos):

$$\dot{\Theta}_{\mathbf{k}} + ik\mu\, \Theta_{\mathbf{k}} = \underline{\dot{\psi}_{\mathbf{k}} - ik\mu\, \varphi_{\mathbf{k}}} - \dot{\tau}\left[\Theta_{0\mathbf{k}} - \Theta_{\mathbf{k}} + \mu v_{\mathrm{b}\mathbf{k}}\right]\ ;$$ *[τ is electron scattering optical depth: $\dot{\tau} = -n_{\mathrm{e}}\sigma_{\mathrm{T}}a$]*

$$\dot{\delta}_{\mathrm{dm},\mathbf{k}} = -ikv_{\mathrm{dm},\mathbf{k}} + \underline{3\dot{\psi}_{\mathbf{k}}} \quad ; \quad \dot{\delta}_{\mathrm{b},\mathbf{k}} = -ikv_{\mathrm{b},\mathbf{k}} + \underline{3\dot{\psi}_{\mathbf{k}}}$$

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Expansion of Universe

Growth of structure tempered

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Length scales

k versus $\{\mathscr{H}, \partial/\partial\eta\}$

Equations (neglecting neutrinos):

$$\dot{\Theta}_{\mathbf{k}} + \underline{ik\mu\,\Theta_{\mathbf{k}}} = \dot{\psi}_{\mathbf{k}} - \underline{ik\mu\,\varphi_{\mathbf{k}}} - \dot{\tau}\left[\Theta_{0\mathbf{k}} - \Theta_{\mathbf{k}} + \mu v_{\mathrm{b}\mathbf{k}}\right]\ ;$$ *[τ is electron scattering optical depth: $\dot{\tau} = -n_{\mathrm{e}}\sigma_{\mathrm{T}}a$]*

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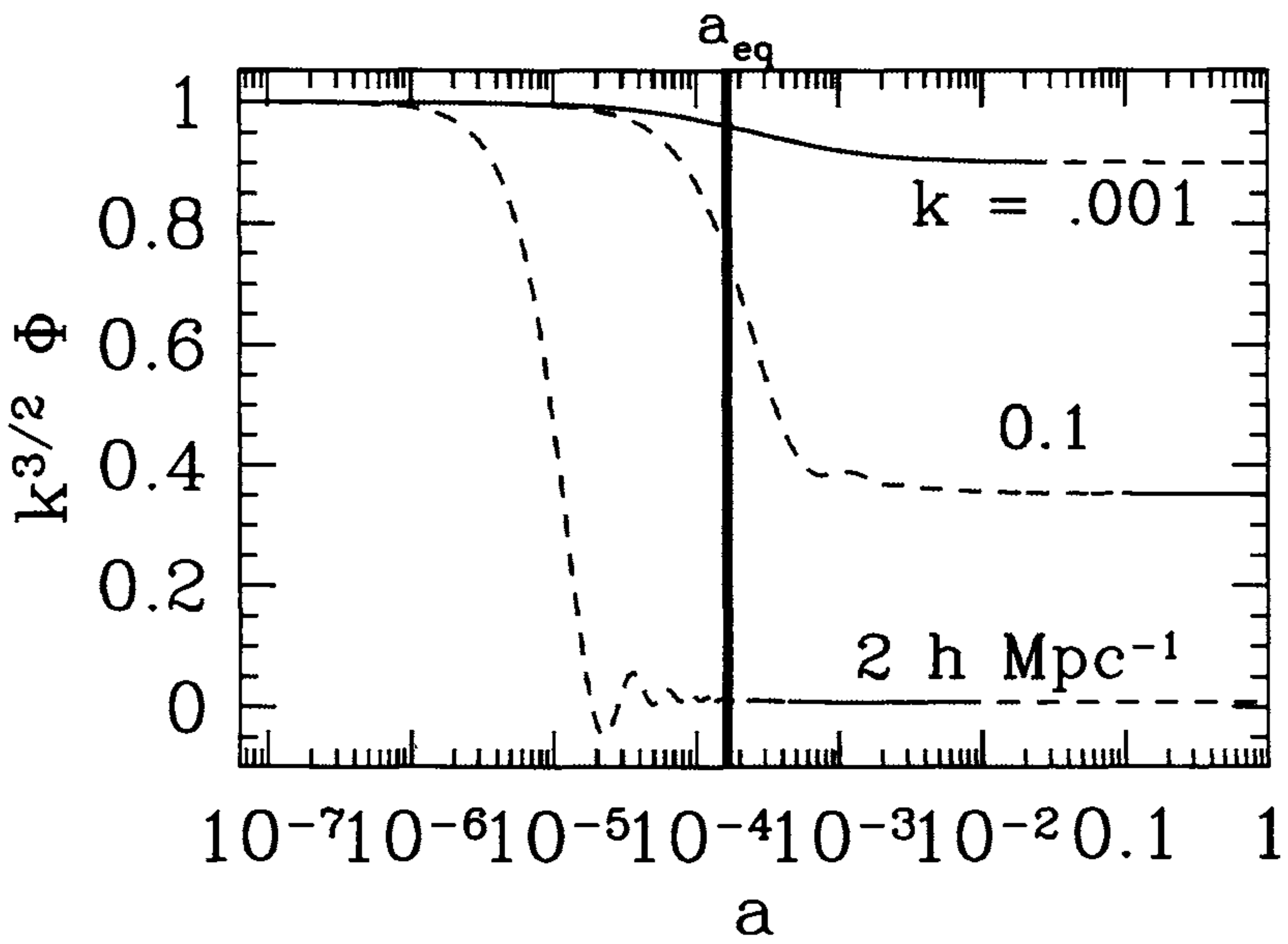
Linear evolution



Relative amplitude:
$\sim k^{-2}\ln(k/k_{\rm eq})$

At late times, all relevant modes sub-Hubble, so:

$\delta_{\mathbf{k}} \sim k^2 \varphi_{\mathbf{k}}$

$P_\delta \sim k^4 P_\varphi \sim k\left(k^3 P_\varphi\right)$

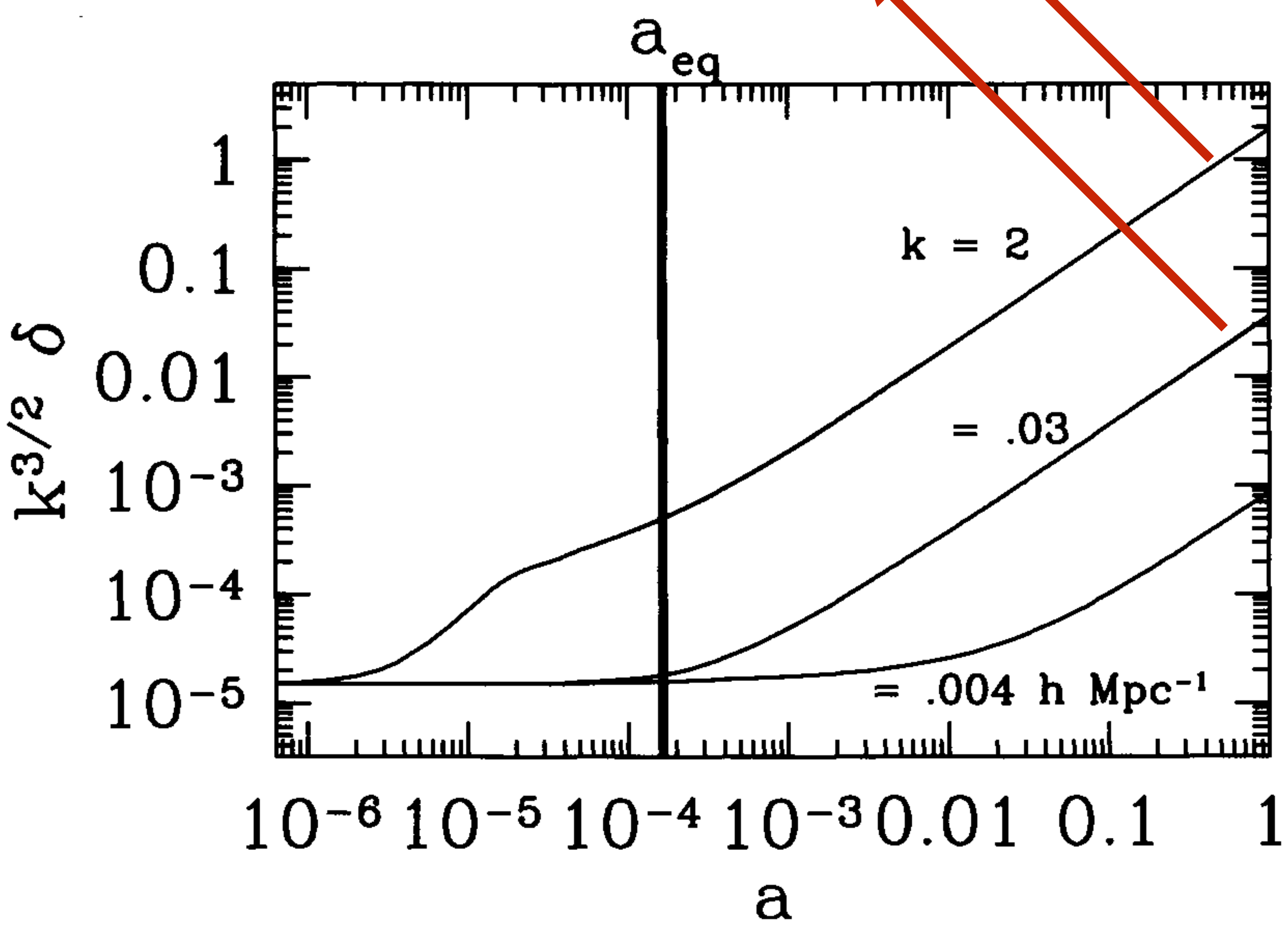
so relative behaviour of P_δ:

$\sim k$ at small k

$\sim k^{-3}\left(\ln(k/k_{\rm eq})\right)^2$ at large k

Linear evolution

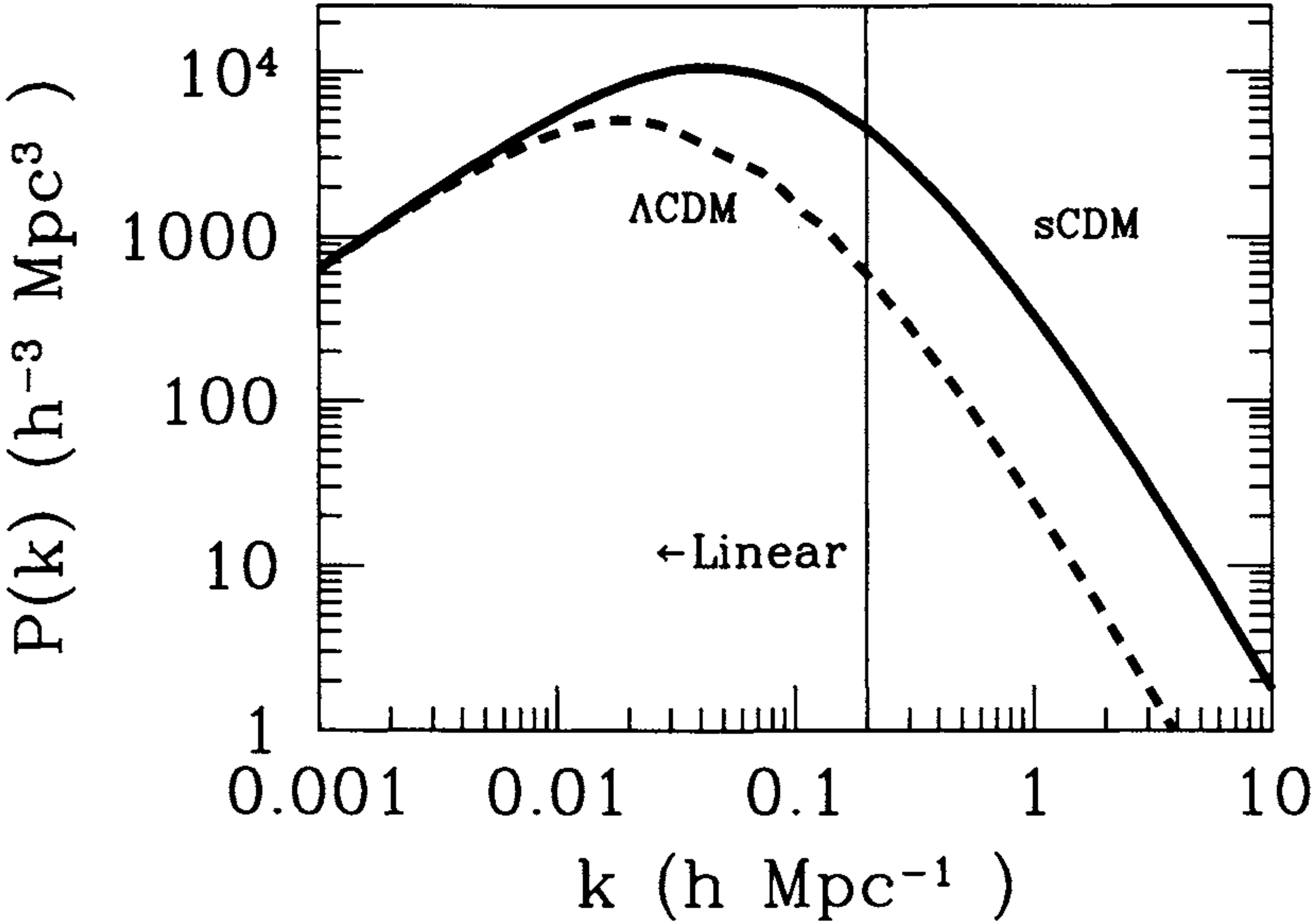
$$\delta(\vec{k}, a) = \frac{3}{5}\frac{k^2}{\Omega_m H_0^2}\Phi_{\rm p}(\vec{k})T(k)D_1(a) \qquad (a > a_{\rm late}).$$



$$\frac{\mathrm{d}^2\delta}{\mathrm{d}a^2} + \left(\frac{\mathrm{d}\ln(H)}{\mathrm{d}a} + \frac{3}{a}\right)\frac{\mathrm{d}\delta}{\mathrm{d}a} - \frac{3\Omega_{\rm m0}}{2a^5}\frac{H_0^2}{H^2}\delta = 0 \quad \left(\Longrightarrow \frac{\mathrm{d}^2\delta}{\mathrm{d}a^2} + \frac{3}{2a}\frac{\mathrm{d}\delta}{\mathrm{d}a} - \frac{3}{2a^2}\delta = 0 \text{ during matter domination}\right)$$

Linear power spectrum

$$\langle\, \delta_{\mathbf{k}}\, \delta^*_{\mathbf{k}'}\, \rangle = (2\pi)^3\, \delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}')\, P(k)$$



Non-linear evolution

At late times (well after photon decoupling), ignore radiation, assume baryons follow dark matter and neglect shear.

Then, setting $\varphi = \psi$,

$$\nabla^2\psi - 3\mathscr{H}\dot{\psi} - 3\mathscr{H}^2\psi = 4\pi G a^2 \bar{\rho}_{\mathrm{m}}\,\delta$$

$$\dot{\delta} + \nabla \cdot \left[(1+\delta)\mathbf{v}\right] = 3\dot{\psi}$$

$$\dot{\mathbf{v}} + \mathscr{H}\,\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\,\nabla\psi\,; \quad [v^2 \equiv \mathbf{v}\cdot\mathbf{v}]$$

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$$\dot{\mathbf{v}} + \mathscr{H}\,\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\,\nabla\psi\,; \quad [v^2 \equiv \mathbf{v}\cdot\mathbf{v}]$$

Defining $\theta \equiv \nabla\cdot\mathbf{v}$, assuming irrotational flow (i.e., $\nabla\times\mathbf{v} = 0$) and $k/\mathscr{H} \gg 1$:

$$\dot{\delta}_{\mathbf{k}} + \theta_{\mathbf{k}} = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} (2\pi)^3\delta_{\rm D}(\mathbf{k}-\mathbf{k}_{12})\,\theta_{\mathbf{k}_1}\,\delta_{\mathbf{k}_2}\,\alpha(\mathbf{k}_1,\mathbf{k}_2)$$
$$\dot{\theta}_{\mathbf{k}} + \mathscr{H}\theta_{\mathbf{k}} + \frac{3}{2}\Omega_{\rm m}\mathscr{H}^2\delta_{\mathbf{k}} = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} (2\pi)^3\delta_{\rm D}(\mathbf{k}-\mathbf{k}_{12})\,\theta_{\mathbf{k}_1}\,\theta_{\mathbf{k}_2}\,\beta(\mathbf{k}_1,\mathbf{k}_2)$$

where $\mathbf{k}_{12} \equiv \mathbf{k}_1 + \mathbf{k}_2$, $\quad \alpha(\mathbf{k}_1,\mathbf{k}_2) = \dfrac{\mathbf{k}_1\cdot\mathbf{k}_{12}}{k_1^2}$, $\quad \beta(\mathbf{k}_1,\mathbf{k}_2) = \dfrac{k_{12}^2(\mathbf{k}_1\cdot\mathbf{k}_2)}{2\,k_1^2\,k_2^2}$

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Then, setting $\varphi = \psi$,

$$\nabla^2\psi - 3\mathscr{H}\dot{\psi} - 3\mathscr{H}^2\psi = 4\pi G a^2 \bar{\rho}_{\rm m}\,\delta + \mathcal{O}\left((\nabla\psi)^2, \mathscr{H}^2\psi^2\right)$$
$$\dot{\delta} + \nabla\cdot\left[(1+\delta)\mathbf{v}\right] = 3\dot{\psi} + \mathcal{O}\left(\mathbf{v}\cdot\nabla\psi, \delta\dot{\psi}\right)$$
$$\dot{\mathbf{v}} + \mathscr{H}\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla\psi + \mathcal{O}\left(\mathscr{H}\psi\mathbf{v}, \mathscr{H}v^2\mathbf{v}, \psi\nabla\psi, v^2\nabla\psi\right)$$

Defining $\theta \equiv \nabla\cdot\mathbf{v}$, assuming irrotational flow (i.e., $\nabla\times\mathbf{v} = 0$) and $k/\mathscr{H} \gg 1$:

$$\dot{\delta}_{\mathbf{k}} + \theta_{\mathbf{k}} = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}(2\pi)^3\delta_{\rm D}(\mathbf{k}-\mathbf{k}_{12})\,\theta_{\mathbf{k}_1}\,\delta_{\mathbf{k}_2}\,\alpha(\mathbf{k}_1,\mathbf{k}_2)$$
$$\dot{\theta}_{\mathbf{k}} + \mathscr{H}\theta_{\mathbf{k}} + \frac{3}{2}\Omega_{\rm m}\mathscr{H}^2\delta_{\mathbf{k}} = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}(2\pi)^3\delta_{\rm D}(\mathbf{k}-\mathbf{k}_{12})\,\theta_{\mathbf{k}_1}\,\theta_{\mathbf{k}_2}\,\beta(\mathbf{k}_1,\mathbf{k}_2)$$

where $\mathbf{k}_{12} \equiv \mathbf{k}_1 + \mathbf{k}_2$, $\quad \alpha(\mathbf{k}_1,\mathbf{k}_2) = \dfrac{\mathbf{k}_1\cdot\mathbf{k}_{12}}{k_1^2}$, $\quad \beta(\mathbf{k}_1,\mathbf{k}_2) = \dfrac{k_{12}^2(\mathbf{k}_1\cdot\mathbf{k}_2)}{2\,k_1^2\,k_2^2}$

<u>Recovering linear sub-Hubble evolution:</u>

Ignore quadratic terms (r.h.s.), differentiate 1st, use 1st and second to eliminate $\theta_{\mathbf{k}}$ and $\dot{\theta}_{\mathbf{k}}$:

$$\ddot{\delta}_{\mathbf{k}} + \mathscr{H}\dot{\delta}_{\mathbf{k}} - \frac{3}{2}\Omega_{\rm m}(\eta)\mathscr{H}^2\,\delta_{\mathbf{k}} = 0$$

Solutions are

$\delta = D_- \propto \mathscr{H}/a = H$ (decaying mode)
and
$\delta = D_+ \propto H\displaystyle\int^a {\rm d}a'/\left(a'H(a')\right)^3$ (growing mode)

Note: during matter domination, $D_+ \propto a$

Non-linear evolution

At late times (well after photon decoupling), ignore radiation, assume baryons follow dark matter and neglect shear.
Then, setting $\varphi = \psi$,

$$\nabla^2\psi - 3\mathscr{H}\dot{\psi} - 3\mathscr{H}^2\psi = 4\pi G a^2 \bar{\rho}_{\mathrm{m}}\,\delta + \mathcal{O}\left((\nabla\psi)^2, \mathscr{H}^2\psi^2\right)$$
$$\dot{\delta} + \nabla\cdot\left[(1+\delta)\mathbf{v}\right] = 3\dot{\psi} + \mathcal{O}\left(\mathbf{v}\cdot\nabla\psi, \delta\dot{\psi}\right)$$
$$\dot{\mathbf{v}} + \mathscr{H}\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla\psi + \mathcal{O}\left(\mathscr{H}\psi\mathbf{v}, \mathscr{H}v^2\mathbf{v}, \psi\nabla\psi, v^2\nabla\psi\right)$$

Defining $\theta \equiv \nabla\cdot\mathbf{v}$, assuming irrotational flow (i.e., $\nabla\times\mathbf{v} = 0$) and $k/\mathscr{H} \gg 1$:

$$\dot{\delta}_{\mathbf{k}} + \theta_{\mathbf{k}} = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}(2\pi)^3\delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_{12})\,\theta_{\mathbf{k}_1}\,\delta_{\mathbf{k}_2}\,\alpha(\mathbf{k}_1,\mathbf{k}_2)$$
$$\dot{\theta}_{\mathbf{k}} + \mathscr{H}\theta_{\mathbf{k}} + \frac{3}{2}\Omega_{\mathrm{m}}\mathscr{H}^2\delta_{\mathbf{k}} = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}(2\pi)^3\delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_{12})\,\theta_{\mathbf{k}_1}\,\theta_{\mathbf{k}_2}\,\beta(\mathbf{k}_1,\mathbf{k}_2)$$

where $\mathbf{k}_{12} \equiv \mathbf{k}_1 + \mathbf{k}_2$, $\quad \alpha(\mathbf{k}_1,\mathbf{k}_2) = \dfrac{\mathbf{k}_1\cdot\mathbf{k}_{12}}{k_1^2}$, $\quad \beta(\mathbf{k}_1,\mathbf{k}_2) = \dfrac{k_{12}^2(\mathbf{k}_1\cdot\mathbf{k}_2)}{2\,k_1^2\,k_2^2}$

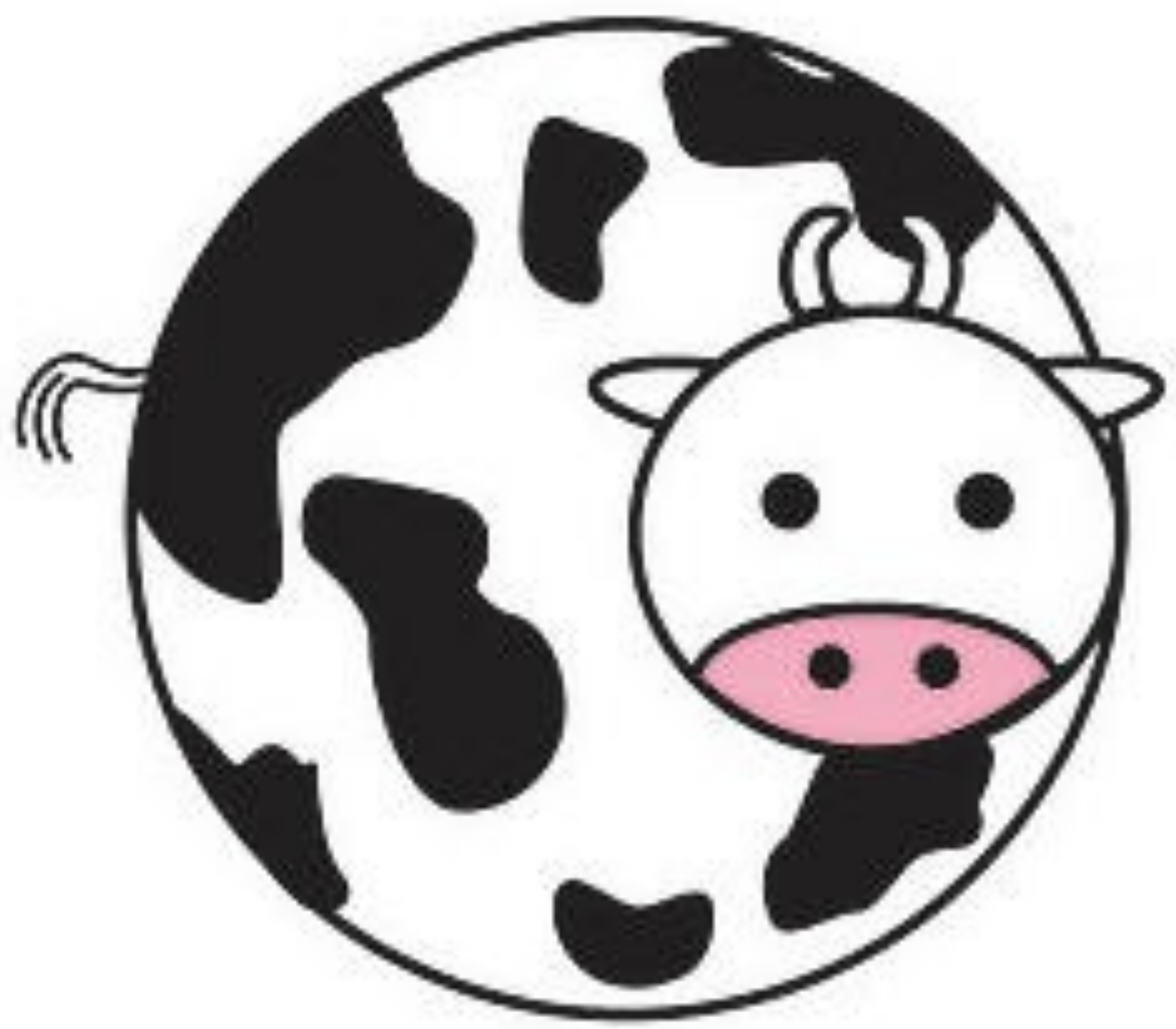
Comments:

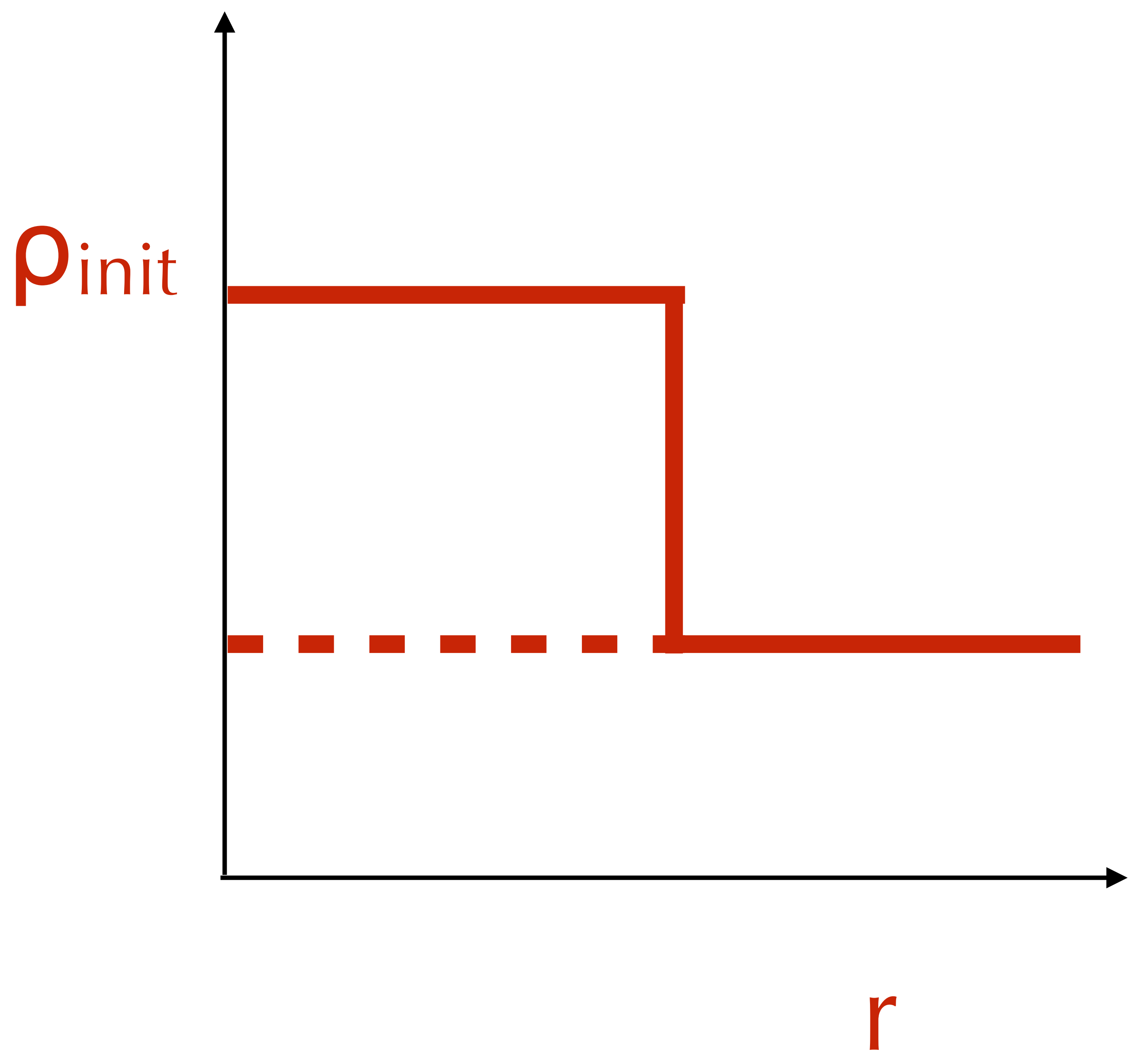
- Generic solutions will clearly be non-linear in the ICs (e.g., PT expansion).
  Hence, e.g., non-zero 3-point function:
  late time field is non-Gaussian even if ICs are perfectly Gaussian.

- Assumptions of zero shear and vorticity valid until \`shell crossing' (multi-streaming in phase space).
  For genuine CDM, multi-streaming will occur at small enough scales at any time during MD.
  So $\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}$ will access multi-streamed scales.
- Coarse-graining of some kind is thus needed. See literature on renormalised perturbation theory (RPT) or effective field theory (EFT).

Non-linear approximations

Spherical Collapse





$$d^2R / dt^2 = -GM(<R,t) / R^2$$

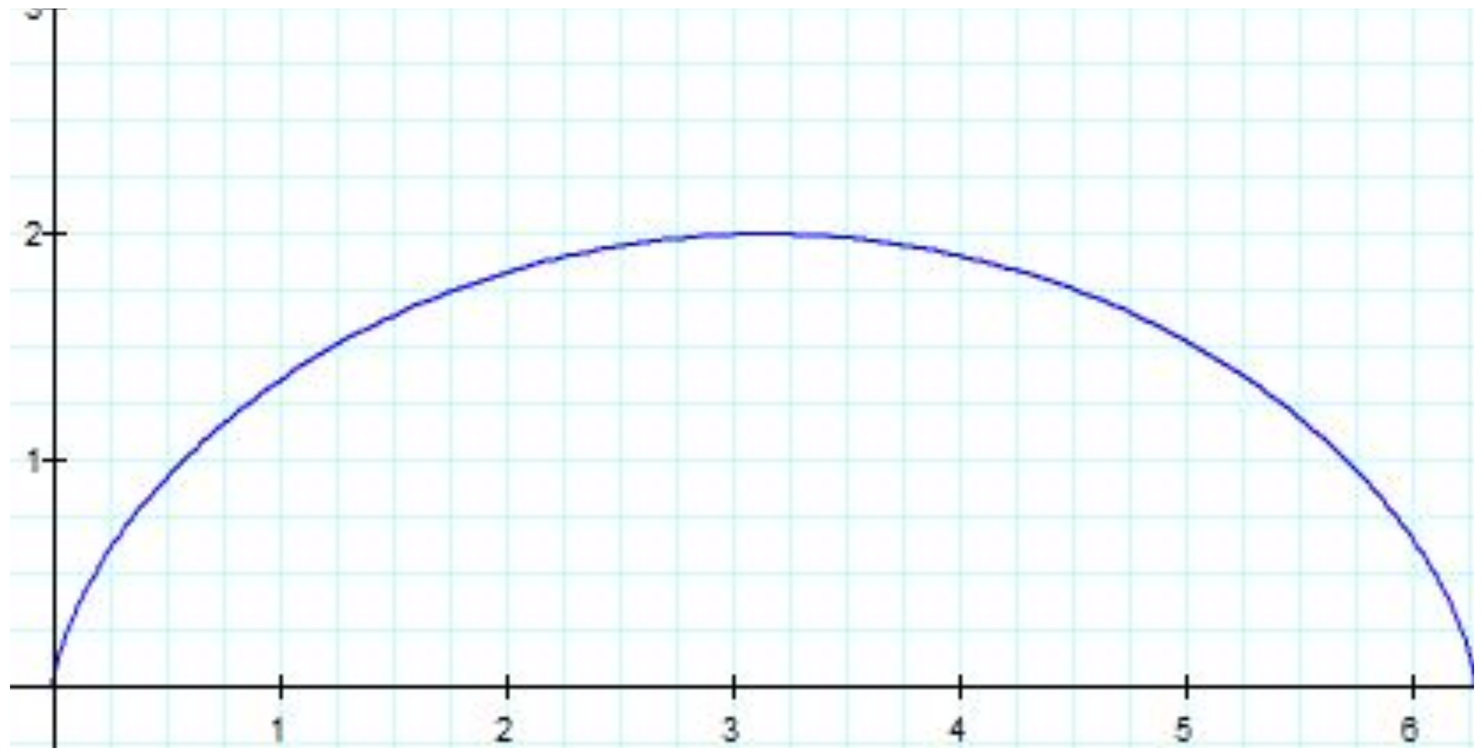
Assume no shell-crossing: $M(<R,t) = M(<R_{init},t_{init}) = \text{constant}$

Spherical Collapse

Nonlinear solution:

$$R \propto 1 - \cos\theta \quad ; \quad t \propto \theta - \sin\theta$$

$$\frac{\rho}{\bar{\rho}} = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3}$$



Linearly extrapolated value:

$$\delta_{\mathrm{L}} = \frac{3}{5}\left(\frac{3}{4}\right)^{2/3}(\theta - \sin\theta)^{2/3}$$

Value at collapse $\boxed{\delta_c = 1.686}$

Virialisation:

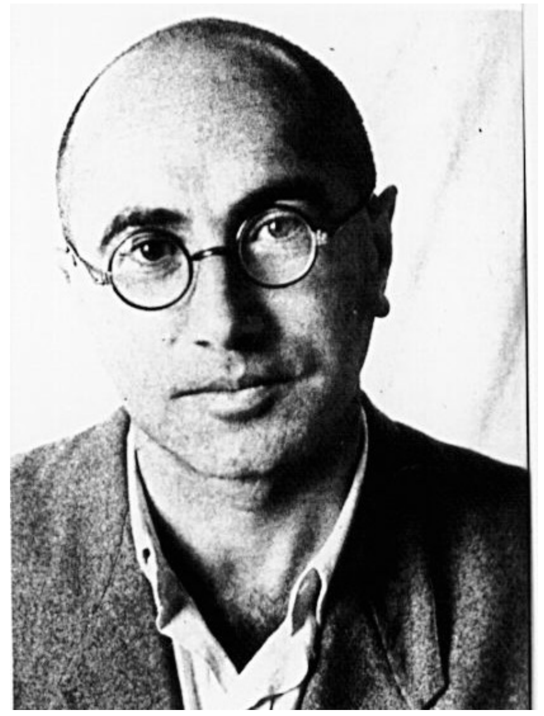
2 (K.E.) + (P.E.) = 0

Also **K.E. + P.E. = constant**

$R(t_{vir}) = R(t_{ta}) / 2$

$\Delta_{vir} = 18\pi^2 \simeq 178$

Zel′dovich Approximation (1970)

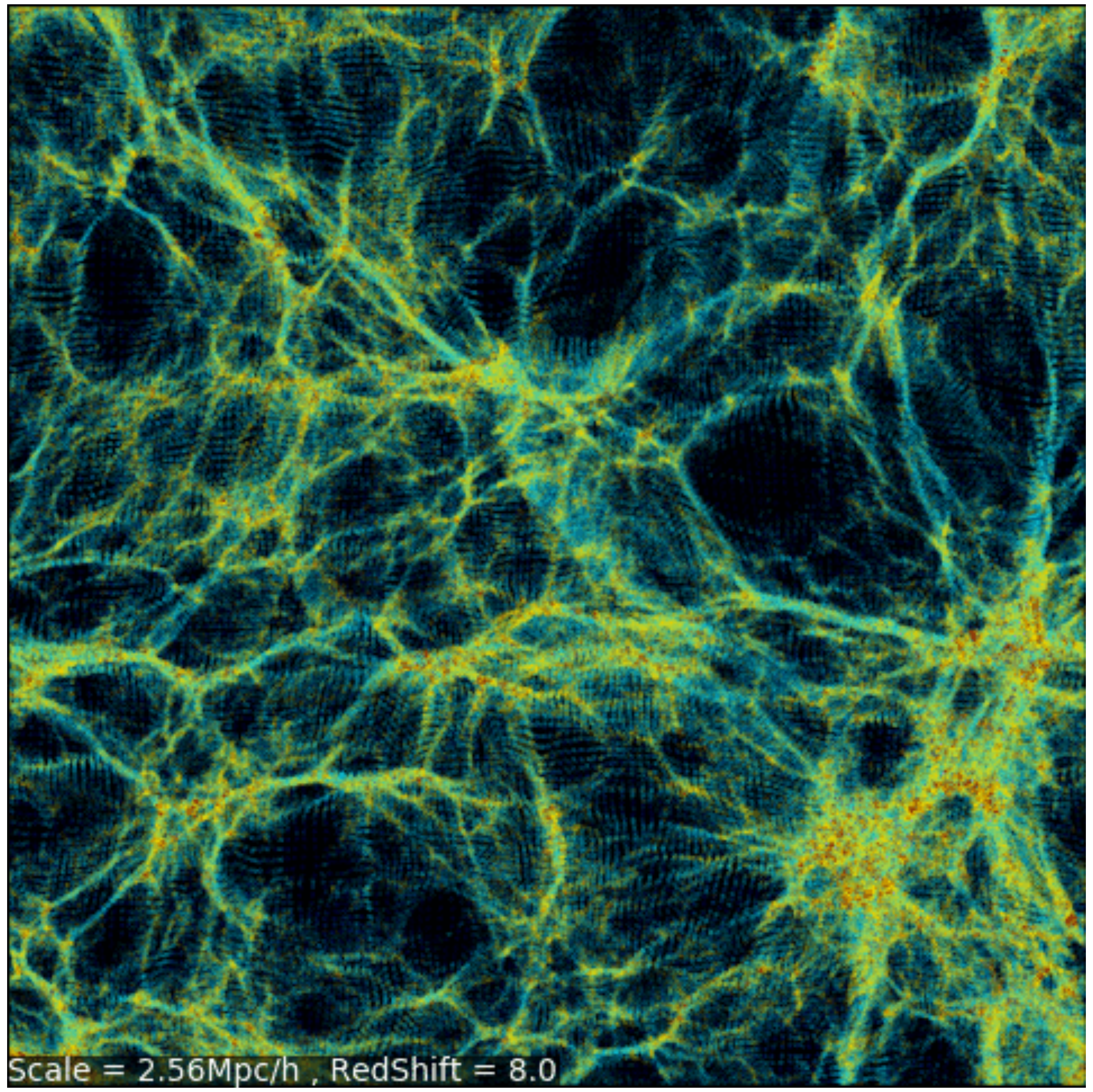


Ya. B. Zel′dovich

“Straight line motion”

$$\mathbf{v}(t,\mathbf{q}) = f(t)\,\nabla\psi(\mathbf{q})$$

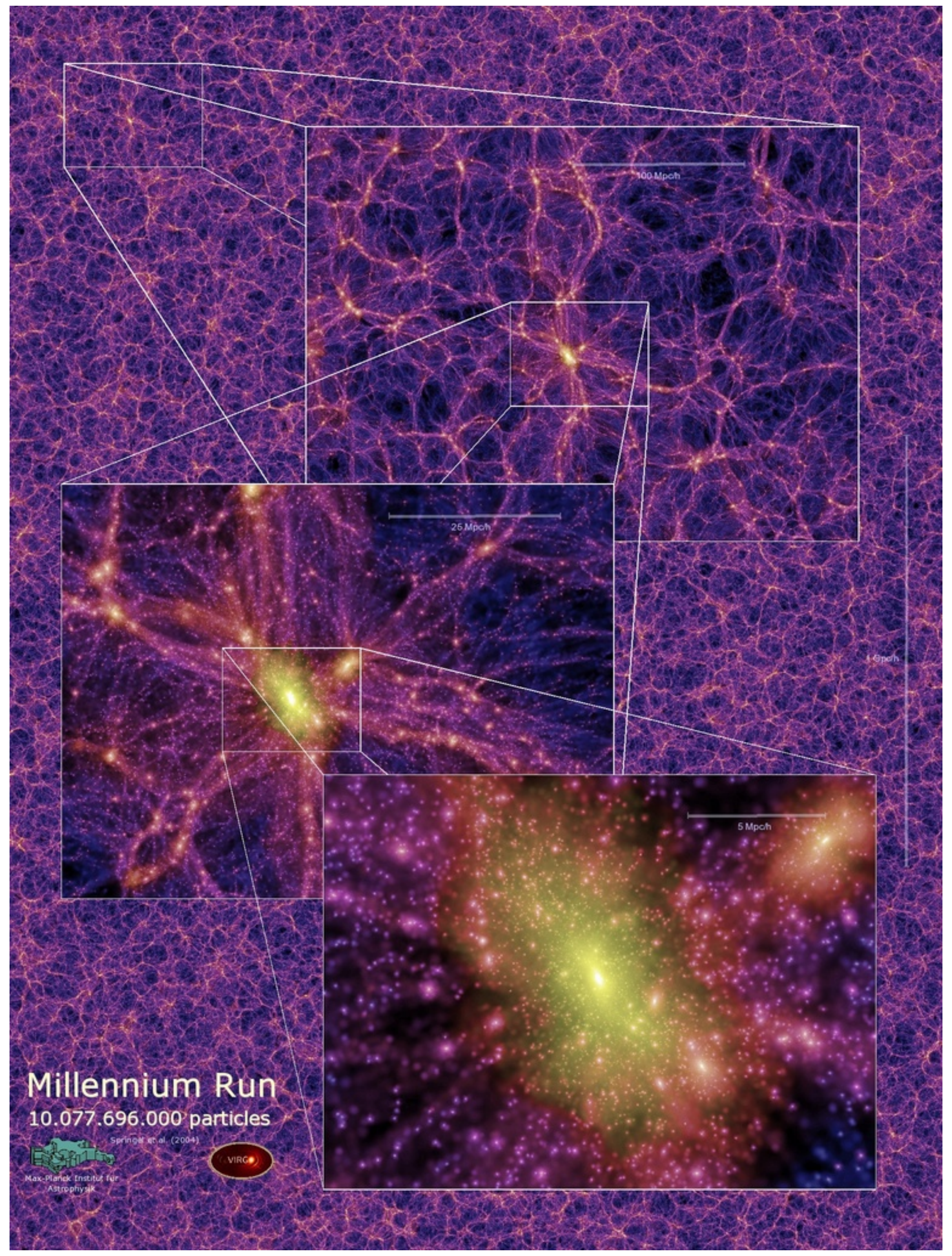
determined by initial conditions



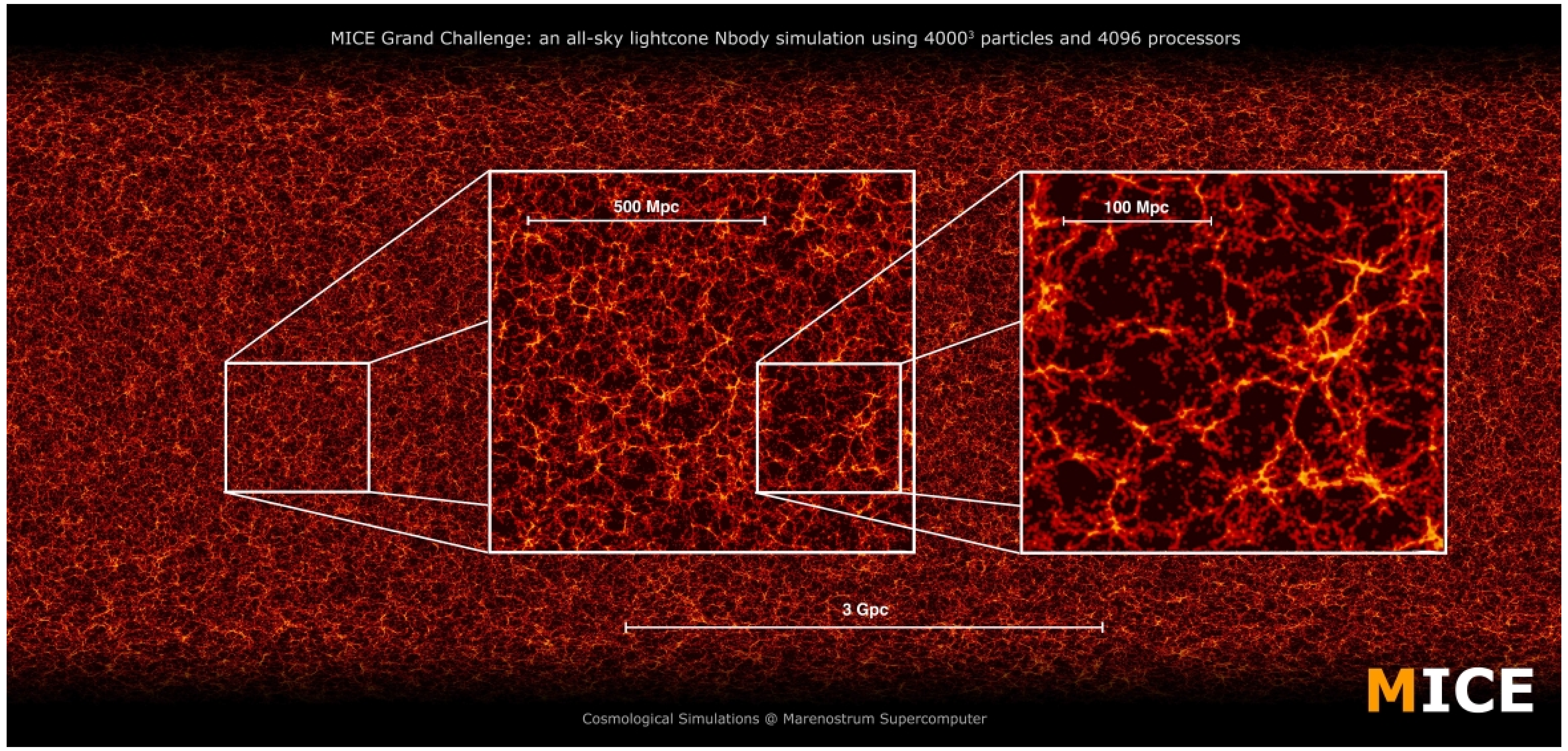
determined by linear perturbation theory
(~ dD_1/dt)

(Movie: courtesy Sujatha Ramakrishnan)

Simulating the Universe



Millennium series

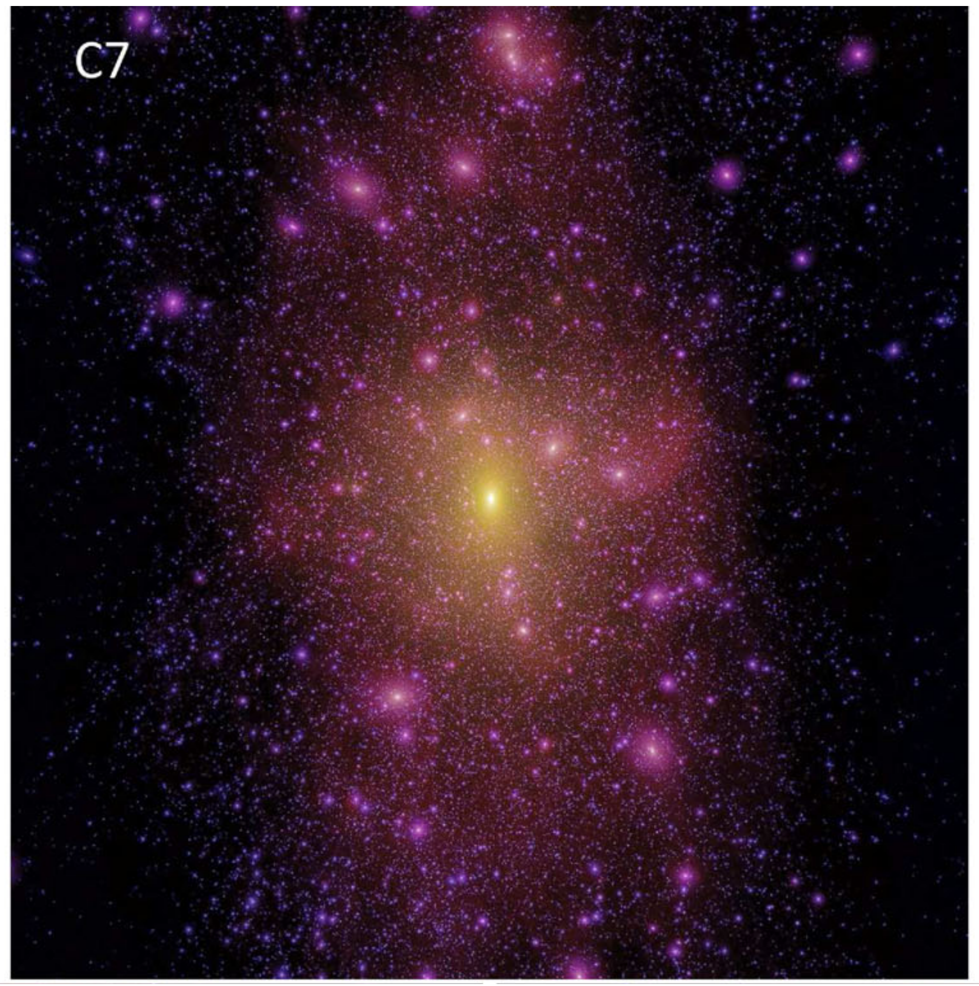


MICE series

Also COYOTE, DEUS, Bolshoi, MultiDARK, Jubilee...

Simulations: CDM

Numerical techniques



Lovell+ (2014)

***Goal*:**
Solve collisionless Boltzmann eqn with cold ICs

***Approach*:**
N-body technique

- Sample phase space distribution function with mass tracers (`particles') and follow their positions and velocities (Newton's law augmented by Poisson equation).
- Avoid small scale 2-body effects through `force softening' (Newton's gravitational law with a core radius).
- Code efficiency + accuracy increases by combining Fourier techniques on particle mesh (PM) for large scale forces with direct calculations for small scale forces.
- Test for convergence of various statistics with $N_{\rm part}$, softening scale, PM grid size, etc.

***Typical application*:**
Periodic cubic box in comoving coordinates.

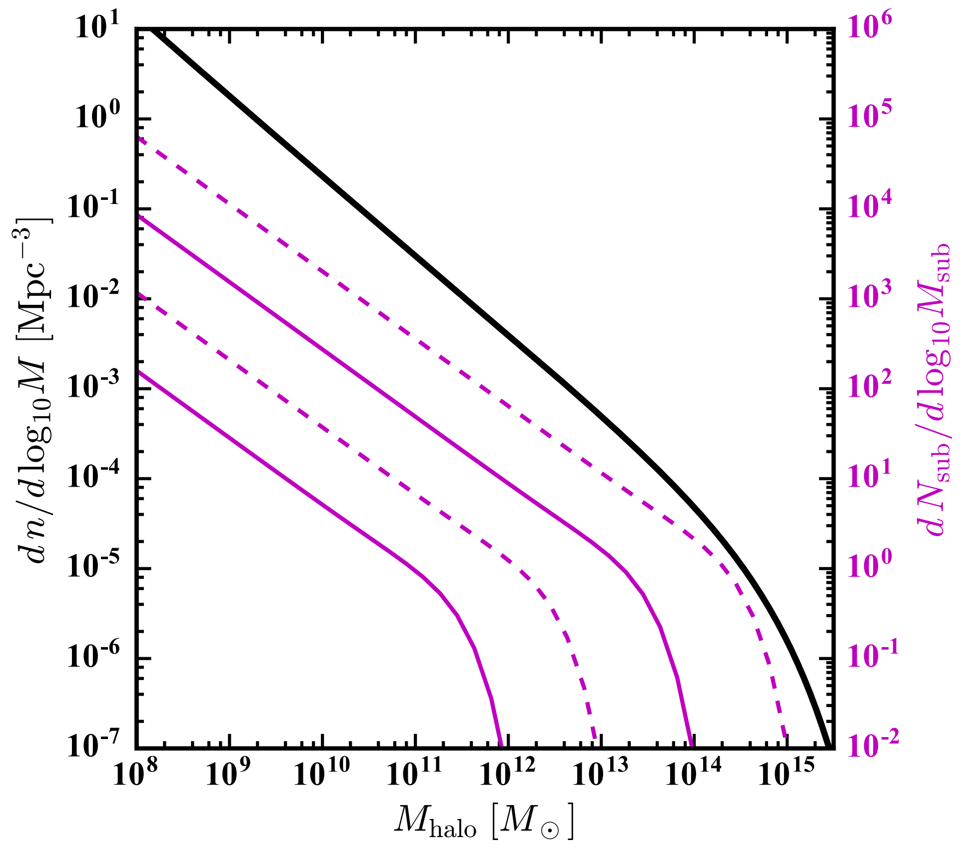
Simulations: CDM

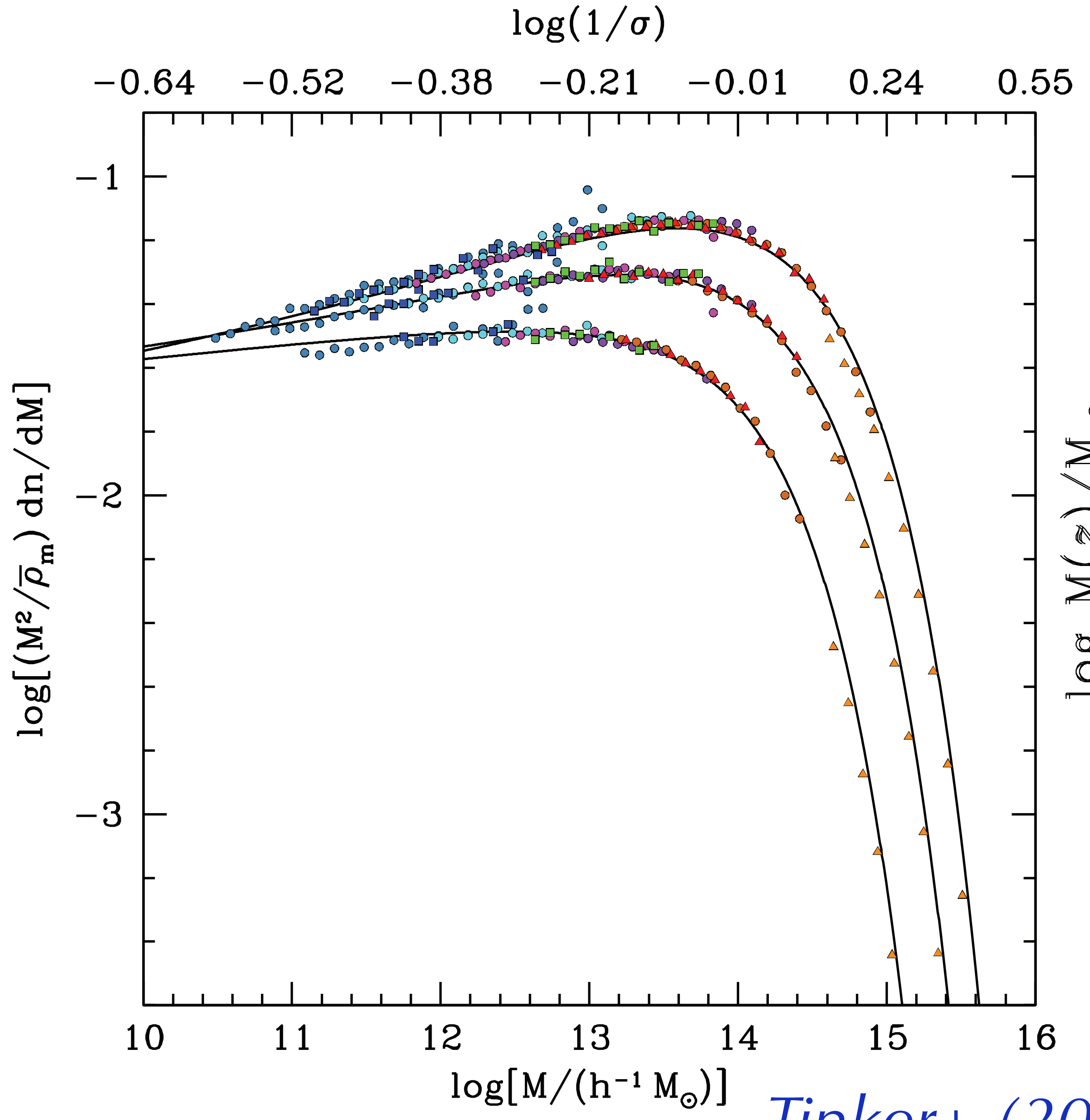
Products

Post-processing:
Identify halos, substructure, merger tree.

Predictions:
Halo + subhalo mass functions, accretion history, clustering (halos/DM).

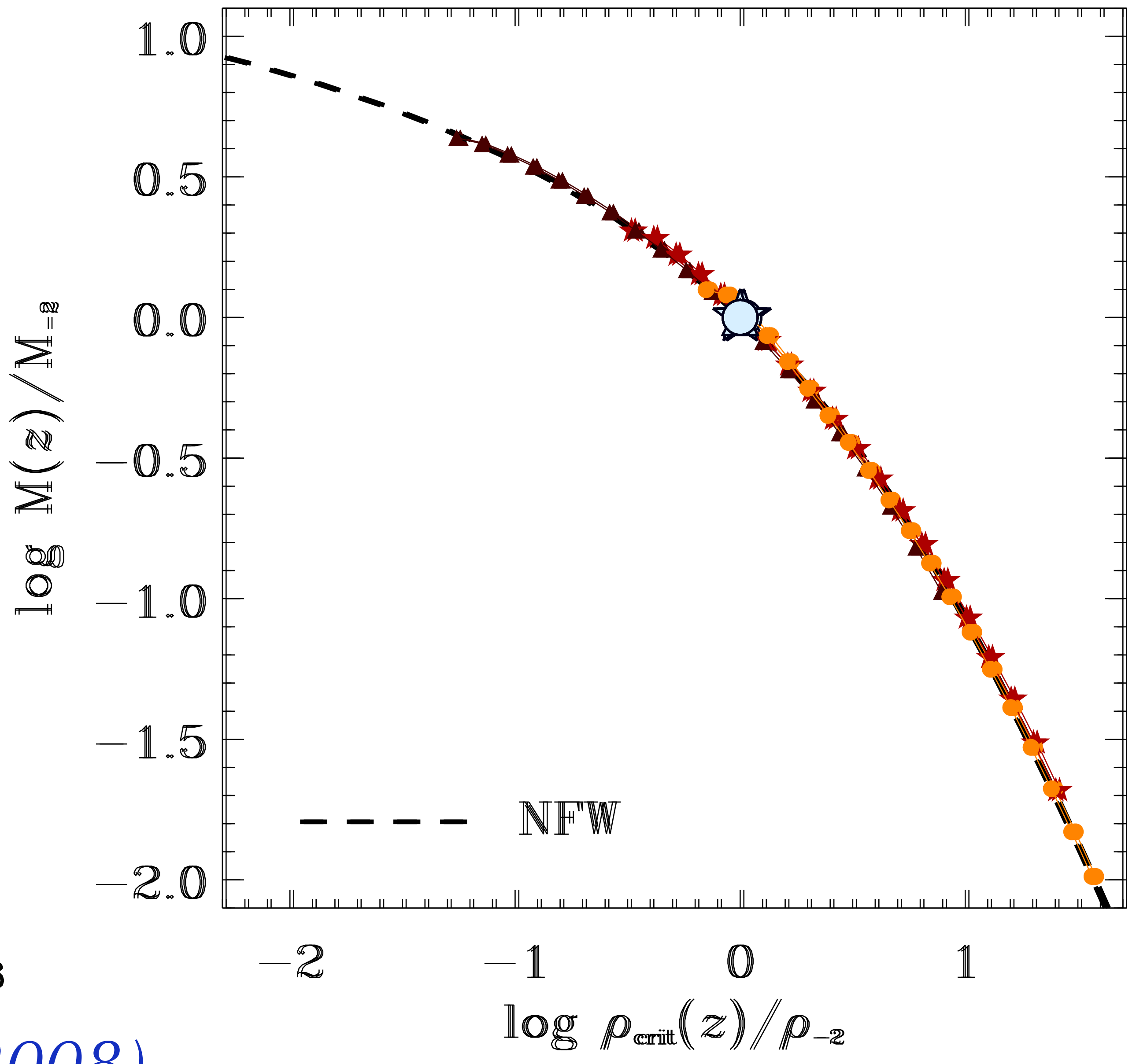
Bullock & Boylan-Kolchin (2017)





Tinker+ (2008)

Ludlow+ (2016)



Fosalba+ (2015)