ICTP 2022 — Inflation

In the next pages you will find the text and the solutions of some homeworks, connected to the lectures on “Inflation” given at the 2022 ICTP Summer School. Some of them have the purpose of proving some statements that were made without proof during the lectures. Some other of them are estimates, made to clarify some concept through quick computations (so sometimes I simplified things, to be able to get more quickly to the answer).

Marco Peloso

Homework set 1

Problem 1.1: For a perfect fluid with energy density ρ and pressure p, in a flat, isotropic, and homogeneous Universe with scale factor a, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_p^2} \quad , \quad 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{p}{M_p^2} \tag{1}$$

where dot denotes time differentiation, and M_p is the reduced Planck mass.

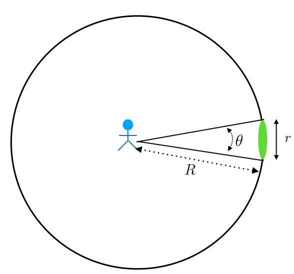
(i) Manipulate these equations, so to obtain the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0 \tag{2}$$

(ii) Denote by $w \equiv \frac{p}{\rho}$ the constant equation of state of the fluid. From the continuity equation, write and solve a differential equation for $\rho\,(a)$. Insert the solution in the first equation of (1) and solve for $a(t)$.

(iii) Specify the solution you found for the cases of $w = 1/3$ (radiation), $w = 0$ (matter), and $w = -1$ (vacuum energy).

Problem 1.2: The figure below depicts us, surrounded by the last scattering surface (the region from which the CMB radiation was emitted, when the scale factor was about 1,100 times smaller than today). For simplicity, assume that it is our horizon, and denote its radius as $d_H \simeq H^{-1}$. The green area represents a causally connected region at that time according to standard cosmology, as observed today. Assuming matter domination from CMB emission to today, and recalling the evolution of the scale factor and of the horizon discussed at lecture, evaluate the ratio r/R and the angle θ under which we observe that region.



Homework set 2

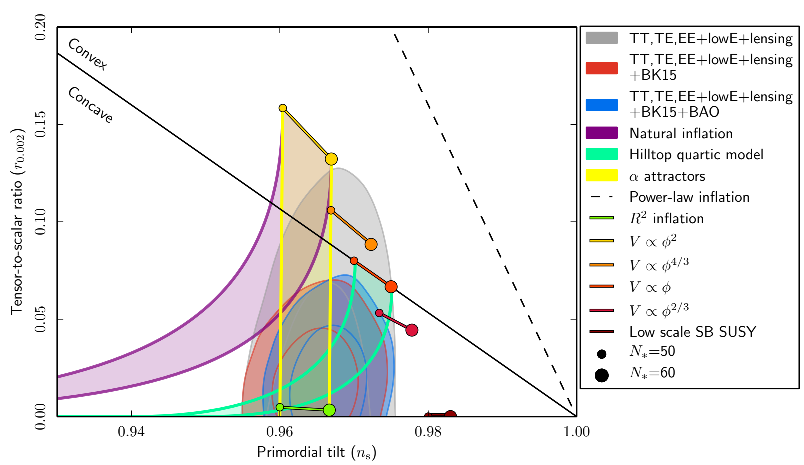
Problem 2.1: In slow roll approximation,

$$3H\dot{\phi} + V' \simeq 0 \quad , \quad H^2 \simeq \frac{V}{3M_p^2} \tag{3}$$

where V is the inflaton potential, a the scale factor, $H \equiv \frac{\dot{a}}{a}$, and where dot and prime denote, respectively, differentiation with respect to time t and to the inflaton field ϕ. (i) From the definition of H, and of the number of e-folds $e^N = \frac{a_{\rm end}}{a}$ (where "end" denotes the end of inflation), show that $\frac{d\phi}{dN} \simeq \frac{M_p^2 V'}{V}$. (ii) Integrate this relation for the inflaton potential $V = \frac{1}{2}m^2\phi^2$, express the slow roll parameters ϵ and η as a function of N, and show that the spectral tilt and the rensor–scalar ratio are given by (for $N \gg 1$)

$$n_s \simeq 1 - 6\epsilon + 2\eta \simeq 1 - \frac{2}{N} \quad , \quad r \simeq 16\epsilon \simeq \frac{8}{N} \tag{4}$$

(iii) Evaluate these quantities for $N = 60$, and compare them with the values in the following Figure (taken from the 2018 Planck paper on inflation)



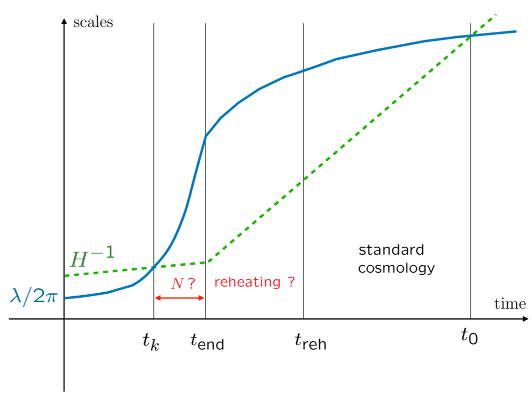
Problem 2.2: A mode of comoving wavenumber k crossed the horizon during inflation when its physical wavelength satisfied $\frac{\lambda}{2\pi} = \frac{a_k}{k} = H_k^{-1}$. The time t_k corresponds to the number of e-folds N inside inflation. Assume that we can effectively describe reheating as being dominated by a source of constant (and unknown) equation of state w. Setting the present value of the scale factor to one, $a_0 = 1$, show that the relation

$$\mathrm{e}^N \equiv \frac{a_{\rm end}}{a_k} = \frac{H_k}{k}\,\frac{a_{\rm end}}{a_{\rm reh}}\,\frac{a_{\rm reh}}{a_0}\,, \tag{5}$$

evaluated for the Planck pivot scale $k = 0.05\,\mathrm{Mpc}^{-1}$, can be rewritten as

$$N \simeq 55.6 + 2\ln\frac{V_k^{1/4}}{10^{16}\,\mathrm{GeV}} + \ln\frac{10^{16}\,\mathrm{GeV}}{\rho_{\rm end}^{1/4}} - \frac{1-3w}{12\,(1+w)}\ln\frac{\rho_{\rm end}}{\rho_{\rm reh}}\,. \tag{6}$$

In these expressions, V is the inflaton potential and ρ is the energy density; the suffixes "end" and "reh" denote, respectively, the end of inflation and reheating. Hint: use conservation of entropy to evaluate the last factor of (5).



Homework set 3

Problem 3.1: Consider a universe filled with a massive inflaton that decays into radiation with decay rate Γ_ϕ. Denote by ρ_ϕ and ρ_γ the energy densities of the inflaton and of the radiation, respectively, and by $H \equiv \frac{\dot{a}}{a}$ the Hubble rate (a is the scale factor of the Universe, and dot denotes time differentiation).

The system is governed by the equations

$$\begin{aligned}
&\dot{\rho}_\phi + (3H + \Gamma_\phi)\,\rho_\phi = 0 \\
&\dot{\rho}_\gamma + 4H\rho_\gamma = \Gamma_\phi \rho_\phi \\
&\rho_\phi + \rho_\gamma = 3M_p^2 H^2
\end{aligned} \tag{7}$$

(i) Verify that, for $\Gamma_\phi = 0$, the solutions $\rho_\phi\,(a)$ and $\rho_\gamma\,(a)$ agree with what we found in homework 1, namely $\rho_\phi \propto a^{-3}$ and $\rho_\gamma \propto a^{-4}$.

(ii) We are interested in the evolution of ρ_γ at very early times. At these times $\rho_\gamma \ll \rho_\phi$, and $\Gamma_\phi \ll H$. Use these approximations in the first and third equations of the above system, and find the approximate solution for $\rho_\phi\,(a)$ and $H\,(a)$. Take the initial condition $\rho_\phi = \bar{\rho}$ at $a = 1$.

Show that the second equation in the above system can be rewritten as

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{\Gamma_\phi}{aH}\,\rho_\phi \tag{8}$$

Insert the solutions $\rho_\phi\,(a)$ and $H\,(a)$ into this equation. Solve this equation to obtain the early time solution for $\rho_\gamma\,(a)$. Take the initial condition $\rho_\gamma = 0$ (hint: it might be useful to consider the differential equation for the combination $a^4\rho_\gamma$).

(iii) Find the maximum value of $\rho_\gamma\,(a)$, and denote it by $\rho_{\rm max}$.

(iv) A common approximation done in studying this problem is to assume that the inflaton instantaneously decays at $\Gamma_\phi = H$. Use the third equation above to find the energy density of radiation at the decay obtained with this assumption. Denote it as $\rho_{\rm inst}$.

(v) Discuss the ratio $\frac{\rho_{\rm max}}{\rho_{\rm inst}}$, showing that it is parametrically given by the ratio between the initial Hubble rate and the inflaton decay rate. Notice that this quantity can be several orders of magnitude greater than one.

Problem 3.2: From the previous problem, we have seen that ρ_γ reaches a peak soon after the end of inflation, and then it decreases as $a^{-3/2}$, while the inflaton is still dominating. Consider a particle X, produced by light particles γ from the thermal bath with a $\gamma + \gamma \to X + X$ process, having cross section proportional to a fixed power of the temperature, $\sigma \propto T^n$. Its physical number density is governed by

$$\frac{dN_X}{dt} + 3HN_X = \sigma\, n_\gamma^2 \tag{9}$$

where we recall that $n_\gamma \propto T^3$ (notice that we are neglecting the decrease of n_γ due to the production of X, and the inverse process $X + X \to \gamma + \gamma$; these assumptions are appropriate if the cross section is sufficiently small). Compute the abundance $Y_X \equiv \frac{n_X}{n_\gamma}$ and study wether its growth is dominated by earlier times (when ρ_γ is close to its peak) or by late times (when the decay of the inflaton completes). In the former case, the approximation of instantaneous inflaton decay is a bad approximation for the purpose of computing the final abundance of X. In the latter case, the approximation is good. To see which times dominate, show that the abundance evolves according to

$$Y_X = t^{-5/4} \int_{t_{\rm in}}^{t} \frac{dt'}{t'} C\,(t')^\alpha \tag{10}$$

where C is constant and α is related to n. Find this relation. The production is dominated by the earliest times $t_{\rm in}$ if $\alpha < 0$.

Solutions

Problem 1.1

(i) We take the time derivative of the first of (1)

$$2\frac{\dot{a}\ddot{a}}{a^2} - 2\frac{\dot{a}^3}{a^3} = \frac{\dot{\rho}}{3M_p^2} \tag{11}$$

which we rewrite as

$$\dot{\rho} = 6M_p^2\frac{\dot{a}}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right] \tag{12}$$

Take the second equation in (1) minus three times the first equation in (1)

$$2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 = -\frac{p}{M_p^2} - \frac{\rho}{M_p^2} \tag{13}$$

From the last two equations we just wrote, we have

$$\dot{\rho} = 3M_p^2\frac{\dot{a}}{a} \times \left[-\frac{p}{M_p^2} - \frac{\rho}{M_p^2}\right] \tag{14}$$

which immediately gives the continuity equation indicated in the text.

(ii) We have

$$\frac{d\rho}{dt} + 3\frac{1}{a}\frac{da}{dt}\left(1+w\right)\rho = 0 \tag{15}$$

and we consider the differential

$$d\rho + 3\left(1+w\right)\frac{da}{a}\,\rho = 0 \tag{16}$$

We separate variables

$$\frac{d\rho}{\rho} = -3\left(1+w\right)\frac{da}{a} \tag{17}$$

and integrate

$$\int_{\rho_{\text{in}}}^{\rho}\frac{d\rho}{\rho} = -3\left(1+w\right)\int_1^a\frac{da}{a} \tag{18}$$

This is integrated to give

$$\ln \frac{\rho}{\rho_{\rm in}} = -3\,(1+w)\,\ln a \tag{19}$$

and, taking the exponential of this,

$$\rho = \rho_{\rm in}\, a^{-3(1+w)} \tag{20}$$

Next, we inert this in the first of (1)

$$\frac{\dot a}{a} = \frac{\rho^{1/2}}{\sqrt{3}M_p} = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\, a^{-\frac{3(1+w)}{2}} \tag{21}$$

which we again rewrite in separate form

$$\frac{da}{a} \times a^{\frac{3(1+w)}{2}} = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\, dt \tag{22}$$

We need to distinguish two cases

$$\begin{aligned} w = -1 \quad & \to \quad \int_1^a \frac{da}{a} = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p} \int_{t_{\rm in}}^t dt \\ & \to \quad \ln\; a = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\; (t - t_{\rm in}) \\ & \to \quad a = \exp\left[\frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\; (t - t_{\rm in})\right] \end{aligned} \tag{23}$$

and

$$\begin{aligned} w \neq -1 \quad & \to \quad \int_1^a da\, a^{\frac{3(1+w)}{2}-1} = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p} \int_{t_{\rm in}}^t dt \\ & \to \quad \frac{2}{3\,(1+w)}\left[a^{\frac{3(1+w)}{2}} - 1\right] = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\; (t - t_{\rm in}) \end{aligned} \tag{24}$$

Imposing $a = 0$ at $t = 0$ amounts in

$$\frac{2}{3\,(1+w)}\,[0-1] = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\; (0 - t_{\rm in}) \quad \Rightarrow \quad t_{\rm in} = \frac{2M_p}{\sqrt{3}\,(1+w)\,\rho_{\rm in}^{1/2}} \tag{25}$$

namely, the constant factors cancel, and

$$\begin{aligned} w \neq -1 \quad & \rightarrow \quad a^{\frac{3(1+w)}{2}} = \frac{3\,(1+w)}{2}\,\frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\,t \\ & \rightarrow \quad a = \left[\frac{3\,(1+w)}{2}\,\frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\right]^{\frac{2}{3(1+w)}}\,t^{\frac{2}{3(1+w)}} \\ & \rightarrow \quad a = \left(\frac{t}{t_{\rm in}}\right)^{\frac{2}{3(1+w)}} \end{aligned} \tag{26}$$

where we recognized the above expression for $t_{\rm in}$. Notice that indeed $a = 1$ for $t = t_{\rm in}$.

(vii) We have

$$\begin{aligned} & \text{radiation}, \quad w = \frac{1}{3} \;\;, \;\; a = \left(\frac{t}{t_{\rm in}}\right)^{1/2} \\ & \text{non}-\text{relativistic matter}, \quad w = 0 \;\;, \;\; a = \left(\frac{t}{t_{\rm in}}\right)^{2/3} \\ & \text{cosmological constant}, \quad w = -1 \;\;, \;\; a = \exp\left[\frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\,\left(t - t_{\rm in}\right)\right] \end{aligned} \tag{27}$$

Problem 1.2

(This is an estimate, and we do not enter into the definition of the various horizons in FLRW, nor we account for the small corrections due to the present accelerated phase), We denote as $d_{\rm CMB}$ the size of the horizon at CMB emission, when the scale factor was $a_{\rm CMB} = \frac{a_0}{1100}$. In our estimate, we identify the radius of the last scattering surface observed by us with the present horizon. The horizon grows as t, which, in matter domination, scales as $a^{3/2}$. Therefore

$$R = d_{\rm CMB}\,\frac{a_0^{3/2}}{a_{\rm CMB}^{3/2}} \tag{28}$$

The radius r in the figure corresponds to the region that was of size $d_{\rm CMB}$ at CMB emission. The size of any given physica region scales as a, and so

$$r = d_{\rm CMB}\,\frac{a_0}{a_{\rm CMB}} \tag{29}$$

We therefore have the ratio (notice that $d_{\rm CMB}$ simplifies in the ratio)

$$\frac{r}{R} = \sqrt{\frac{a_{\rm CMB}}{a_0}} \simeq 0.03 \tag{30}$$

and the angle

$$\theta = 2 \arcsin \frac{r}{2R} \simeq 1.7^\circ \tag{31}$$

Problem 2.1

(i) We use the definition of H and N to "trade" dt for dN:

$$dN\, {\rm e}^N = -\frac{a_{\rm end}}{a^2}\, da \quad \Rightarrow \quad dN = -\frac{da}{a} = -H\, dt \tag{32}$$

This gives

$$\frac{d\phi}{dN} \simeq -\frac{1}{H}\frac{d\phi}{dt} \simeq \frac{V'}{3H^2} \simeq \frac{M_p^2\, V'}{V} \tag{33}$$

(ii) For the inflaton potential $V = \frac{1}{2}m^2\phi^2$, this relation reads

$$\frac{d\phi}{dN} \simeq \frac{2M_p^2}{\phi} \tag{34}$$

which is integrated as

$$\int_{\phi_{\rm end}}^{\phi} d\phi\, \phi \simeq 2M_p^2 \int_0^N dN \tag{35}$$

and solved by

$$\phi^2 - \phi_{\rm end}^2 \simeq 4M_p^2 N \tag{36}$$

We can also evaluate the slow roll parameters

$$\epsilon \equiv \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 = \frac{2\, M_p^2}{\phi^2} \quad , \quad \eta \equiv M_p^2\, \frac{V''}{V} = \frac{2\, M_p^2}{\phi^2} \tag{37}$$

Inflation ends at $\epsilon \simeq 1$ so that $\phi_{\rm end} \simeq \sqrt{2}\, M_p$. We then find

$$\phi^2 \simeq (4N+2)\, M_p^2 \simeq 4\, N\, M_p^2 \;\; , \;\; N \gg 1 \tag{38}$$

The slow roll parameters are therefore

$$\epsilon = \eta \simeq \frac{1}{2N} \tag{39}$$

and

$$n_s \simeq 1 - \frac{2}{N} \;\; , \;\; r \simeq \frac{8}{N} \tag{40}$$

For $N = 60$ we obtain $n_s \simeq 0.967$ and $r \simeq 0.13$ in agreement with the figure in the text.

Problem 2.2

From the definition of a Megaparsec, and from the use of natural units, we have

$$k = 0.05\,\text{Mpc}^{-1} \simeq 3.2 \times 10^{-40}\,\text{GeV} \tag{41}$$

From the Friedmann equation during inflation

$$H_k = \frac{V_k^{1/2}}{\sqrt{3}\,M_p} \simeq 2.4 \times 10^{13}\,\text{GeV} \left(\frac{V_k^{1/4}}{10^{16}\,\text{GeV}} \right)^2 \tag{42}$$

Therefore, from the first factor in (5),

$$\ln \frac{H_k}{k} \simeq 121.7 + 2\, \ln \frac{V_k^{1/4}}{10^{16}\,\text{GeV}} \tag{43}$$

We then use the fact that the energy of a species with equation of state w scales as $\rho \propto a^{-3(1+w)}$, to write

$$\ln \frac{a_{\rm end}}{a_{\rm reh}} = \frac{1}{3\,(1+w)}\, \ln \frac{\rho_{\rm reh}}{\rho_{\rm end}} \tag{44}$$

Conservation of entropy during standard cosmology, is expressed as $g_{*s}\,T^3\,a^3 =$ const, where T is the temperature of the thermal bath and g_{*s} counts the effecctive bosonic relativistic degrees of freedom for entropy. Assuming Standard Model content at reheating, $g_{*s,{\rm reh}} = 106.75$. Treating the neutrinos as massless (as they are effectively massless when they decoupled) today we have $g_{*s,0} \simeq 3.91$. From this, and from the current temperature $T_0 \simeq 2.73\,\text{K} \simeq 2.35 \times 10^{-13}\,\text{GeV}$ we find

$$106.75 \times T_{\rm reh}^3\, a_{\rm reh}^3 \simeq 3.91 \times \left(2.35 \times 10^{-13}\,\text{GeV}\right)^3\, a_0^3 \tag{45}$$

Namely,

$$\frac{a_{\rm reh}}{a_0} \simeq \frac{7.8\times 10^{-14}\,{\rm GeV}}{T_{\rm reh}} \tag{46}$$

For the thermal bath at the end of reheating, $\rho_{\rm reh}^{1/4} = \left(\frac{\pi^2}{30}\,g_*\right)^{1/4} T_{\rm reh} \simeq 2.43\,T_{\rm reh}$. As a consequence

$$\ln\frac{a_{\rm reh}}{a_0} \simeq \ln\frac{1.9\times 10^{-13}\,{\rm GeV}}{\rho_{\rm reh}^{1/4}} \simeq -66.1 + \ln\frac{10^{16}\,{\rm GeV}}{\rho_{\rm reh}^{1/4}} \tag{47}$$

Combining the three factors

$$\begin{aligned}
N &\simeq 55.6 + 2\,\ln\frac{V_k^{1/4}}{10^{16}\,{\rm GeV}} + \frac{1}{3\,(1+w)}\,\ln\frac{\rho_{\rm reh}}{\rho_{\rm end}} + \ln\frac{10^{16}\,{\rm GeV}}{\rho_{\rm reh}^{1/4}} \\
&= 55.6 + 2\,\ln\frac{V_k^{1/4}}{10^{16}\,{\rm GeV}} + \ln\frac{10^{16}\,{\rm GeV}}{\rho_{\rm end}^{1/4}} + \ln\frac{\rho_{\rm end}^{1/4}}{\rho_{\rm reh}^{1/4}} + \frac{1}{3\,(1+w)}\,\ln\frac{\rho_{\rm reh}}{\rho_{\rm end}} \\
&= 55.6 + 2\,\ln\frac{V_k^{1/4}}{10^{16}\,{\rm GeV}} + \ln\frac{10^{16}\,{\rm GeV}}{\rho_{\rm end}^{1/4}} + \left[\frac{1}{4} - \frac{1}{3\,(1+w)}\right]\ln\frac{\rho_{\rm end}}{\rho_{\rm reh}} \\
&= 55.6 + 2\,\ln\frac{V_k^{1/4}}{10^{16}\,{\rm GeV}} + \ln\frac{10^{16}\,{\rm GeV}}{\rho_{\rm end}^{1/4}} - \frac{1-3w}{12\,(1+w)}\,\ln\frac{\rho_{\rm end}}{\rho_{\rm reh}}
\end{aligned} \tag{48}$$

as we wanted to prove.

Problem 3.1

(i) For $\Gamma_\phi = 0$, we have

$$\frac{d\rho_\phi}{dt} + 3\frac{1}{a}\,\frac{da}{dt}\rho_\phi = 0 \tag{49}$$

We consider the differential, and separate variables,

$$\frac{d\rho_\phi}{\rho_\phi} = -3\frac{da}{a} \tag{50}$$

which we integrate

$$\int_{\rho_{\phi,{\rm in}}}^{\rho_\phi}\frac{d\phi}{\phi} = -3\int_{a_{\rm in}}^{a}\frac{da}{a} \tag{51}$$

to find

$$\ln \frac{\rho_\phi}{\rho_{\rm phi,in}} = -3 \ln \frac{a}{a_{\rm in}} \tag{52}$$

which we exponentiate so to write

$$\rho_\phi = \rho_{\phi,\rm in} \left(\frac{a_{\rm in}}{a}\right)^3 \tag{53}$$

The computation is identical for radiation, with 3 replaced by 4

$$\rho_\gamma = \rho_{\gamma,\rm in} \left(\frac{a_{\rm in}}{a}\right)^4 \tag{54}$$

(ii) Assuming $\Gamma_\phi \ll H$ and $\rho_\gamma \ll \rho_\phi$, the first and third equation rewrite

$$\begin{aligned} \dot{\rho_\phi} + 3H\rho_\phi = 0 \\ \rho_\phi = 3H^2 M_p^2 \end{aligned} \tag{55}$$

From the first equation we obtained eq. (53), which we rewrite (using the initial values specified in the text) as

$$\rho_\phi = \bar{\rho}\, a^{-3} \tag{56}$$

The other equation then gives

$$H = \frac{\rho_\phi^{1/2}}{\sqrt{3}M_p} = \frac{\bar{\rho}^{1/2}}{\sqrt{3}M_p} \frac{1}{a^{3/2}} \tag{57}$$

We now need to solve

$$\frac{d\rho_\gamma}{dt} + 4H\,\rho_\gamma = \Gamma_\phi\, \rho_\phi \tag{58}$$

We divide by H to write

$$\frac{d\rho_\gamma}{dt} a \frac{dt}{da} + 4\rho_\gamma = \frac{\Gamma_\phi}{H}\, \rho_\phi \tag{59}$$

or

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{\Gamma_\phi}{aH}\, \rho_\phi \tag{60}$$

or

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{\Gamma_\phi}{a} \frac{\sqrt{3}M_p\, a^{3/2}}{\bar{\rho}^{1/2}} \frac{\bar{\rho}}{a^3} \tag{61}$$

or

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{C}{a^{5/2}} \quad , \quad C \equiv \sqrt{3}\Gamma_\phi M_p \bar{\rho}^{1/2} \tag{62}$$

As suggested in the text, we note that

$$\frac{d\left(a^4\rho_\gamma\right)}{da} = a^4\,\frac{d\rho_\gamma}{da} + 4\,a^3\,\rho_\gamma = a^4\left(\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma\right) \tag{63}$$

Therefore

$$\frac{d\left(a^4\rho_\gamma\right)}{da} = C\,a^{3/2} \tag{64}$$

We integrate it to write

$$\int_0^{a^4\rho_\gamma} d\left(a^4\,\rho_\gamma\right) = C\int_1^a a^{3/2}da \tag{65}$$

which gives

$$a^4\rho_\gamma = C\frac{2}{5}a^{5/2}\Big|_1^a \;\Rightarrow\; \rho_\gamma = \frac{2\,C}{5}\,\frac{a^{5/2}-1}{a^4} \tag{66}$$

(iii) We need to find the maximum of the function

$$f\left(a\right) \equiv \frac{a^{5/2}-1}{a^4} \tag{67}$$

for $a > 1$. We see that this function starts at zero (at the initial value $a = 1$ of the scale factor), then it grows, due to the inflaton decay, and then it decreases, since the dilution from the expansion is stronger than the production from the inflaton. We have

$$\frac{df}{da} = \frac{d}{da}\left(a^{-3/2} - a^{-4}\right) = -\frac{3}{2}a^{-5/2} + 4a^{-5} = 0 \tag{68}$$

giving

$$a^{5/2} = \frac{8}{3} \;\Rightarrow\; a = \left(\frac{8}{3}\right)^{2/5} \tag{69}$$

so that

$$f_{\max} = \left(\frac{8}{3}\right)^{-8/5}\left(\frac{8}{3} - 1\right) \simeq 0.35 \tag{70}$$

Therefore

$$\rho_{\max} \simeq \frac{2}{5}\,\sqrt{3}\Gamma_\phi M_p \bar{\rho}^{1/2}\,0.35 \simeq 0.24\Gamma_\phi M_p \bar{\rho}^{1/2} \tag{71}$$

(iv) In this case we have

$$\Gamma_\phi = H = \frac{\rho_{\rm inst}^{1/2}}{\sqrt{3}M_p} \quad \Rightarrow \quad \rho_{\rm inst} = 3M_p^2\Gamma_\phi^2 \tag{72}$$

(v) The ratio is

$$\frac{\rho_{\rm max}}{\rho_{\rm inst}} \simeq \frac{0.24\bar{\rho}^{1/2}}{3M_p\Gamma_\phi} = \frac{0.24 \times \sqrt{3}M_p H_{\rm initial}}{3M_p\Gamma_\phi} \simeq 0.14\,\frac{H_{\rm initial}}{\Gamma_\phi} \tag{73}$$

Problem 3.2

Let us denote the physical number density of particles in the thermal bath as $n_\gamma \propto T^3$. Let us denote with n_X the physical number density of the particle X, and with $Y_X \equiv \frac{n_X}{n_\gamma}$ the abundance of the particle relative to the thermal bath.

The number density of X is governed by

$$\frac{dn_X}{dt} + 3Hn_X = \sigma\, n_\gamma^2 \tag{74}$$

Typically we are interested in the abundance of a particle X, defined by the ratio between the number density of this particle and that of the thermal bath,

$$Y_X \equiv \frac{n_X}{n_\gamma} \tag{75}$$

Particles are produced by the thermal bath formed immediately, with this high $T_{\rm rh}$. However, these particles are diluted by the radiation that keeps being produced by the decaying inflaton. This dilution effect becomes less and less relevant as ρ_ϕ decreases below ρ_γ, which happens when $H \simeq \Gamma_\phi$.

Let us compute the evolution equation for Y_X.

$$\begin{aligned} \frac{dY_X}{dt} &= \frac{1}{n_\gamma}\frac{dn_X}{dt} - \frac{n_X}{n_\gamma^2}\frac{dn_\gamma}{dt} \\ &= \frac{1}{n_\gamma}\left[-3Hn_X + \sigma n_\gamma^2\right] - \frac{n_X}{n_\gamma^2}\frac{dn_\gamma}{dT}\frac{dT}{dt} \end{aligned} \tag{76}$$

Since $n_\gamma = CT^3$ (where C is a constant), we have $dn_\gamma = 3T^2 C dT = 3T^2 \frac{n_\gamma}{T^3} dT$, namely $\frac{dn_\gamma}{dT} = \frac{3n_\gamma}{T}$. Therefore

$$\frac{dY_X}{dt} = -3HY_X + \sigma\, n_\gamma - \frac{n_X}{n_\gamma^2}\frac{3n_\gamma}{T}\frac{dT}{dt} \tag{77}$$

which we rewrite as

$$\frac{dY_X}{dt} + 3\left[H + \frac{1}{T}\frac{dT}{dt}\right] Y_X = \sigma\, n_\gamma \tag{78}$$

We know from the text that $\rho_\gamma \propto a^{-3/2}$, meaning that $T \propto \rho_\gamma^{1/4} \propto a^{-3/8}$. We also know that the inflaton is dominating and that it is effectiively behaving as matter, so that $a \propto t^{2/3}$, and $T \propto t^{-1/4}$. We then have

$$H + \frac{1}{T}\frac{dT}{dt} = \frac{2}{3t} + \frac{\frac{d}{dt}\left(t^{-1/4}\right)}{t^{-1/4}} = \frac{5}{12\,t} \tag{79}$$

We also have

$$\sigma n_X \propto T^{3+n} \propto t^{-\frac{n+3}{4}} \tag{80}$$

so that

$$\frac{dY_X}{dt} + \frac{5}{4}\frac{Y_X}{t} = C\, t^{-\frac{n+3}{4}} \tag{81}$$

where C is a constant.

It is convenient to define $\tilde{Y}_X \equiv t^{5/4}\, Y_X$, so that

$$\frac{d\tilde{Y}_X}{dt} = \frac{5}{4} t^{1/4} Y_X + t^{5/4}\frac{dY_X}{dt} = t^{5/4}\left(\frac{dY_X}{dt} + \frac{5}{4}\frac{Y_X}{t}\right) \tag{82}$$

so that

$$\frac{d\tilde{Y}_X}{dt} = C\, t^{-\frac{n+3}{4}}\, t^{5/4} = C t^{-\frac{n}{4}+\frac{1}{2}} \tag{83}$$

which we integrate to obtain

$$Y_X = t^{-5/4}\int_{t_{\rm in}}^{t}\frac{dt'}{t'} C\,(t')^{-\frac{n}{4}+\frac{3}{2}} \tag{84}$$

We see that the production is dominated by the earliest times if

$$-\frac{n}{4} + \frac{3}{2} < 0 \quad\Rightarrow\quad n > 6 \tag{85}$$