ANALYTICAL APPROACHES TO GRAVITATIONAL WAVE SIGNALS

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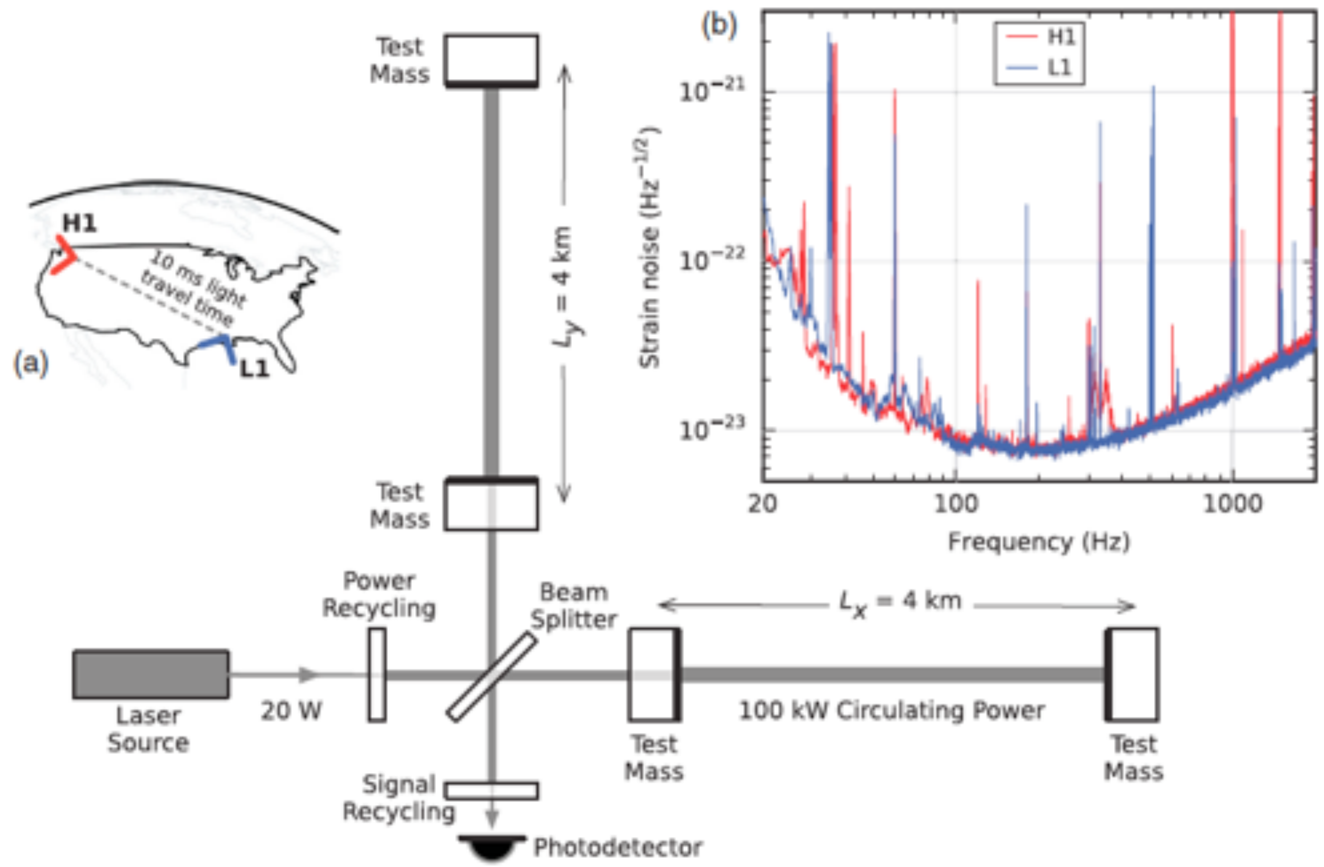
Summer School on Cosmology 2022

ICTP, Trieste, Italy

13 July 2022

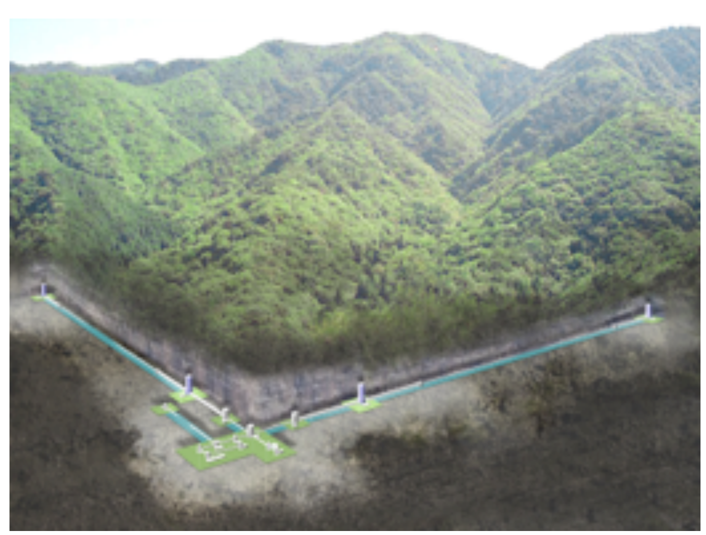
STARTING FROM 14 SEPT 2015: GRAVITATIONAL WAVE (GW) DETECTIONS BY TWO LIGO (+ VIRGO+KAGRA+…) GW DETECTORS



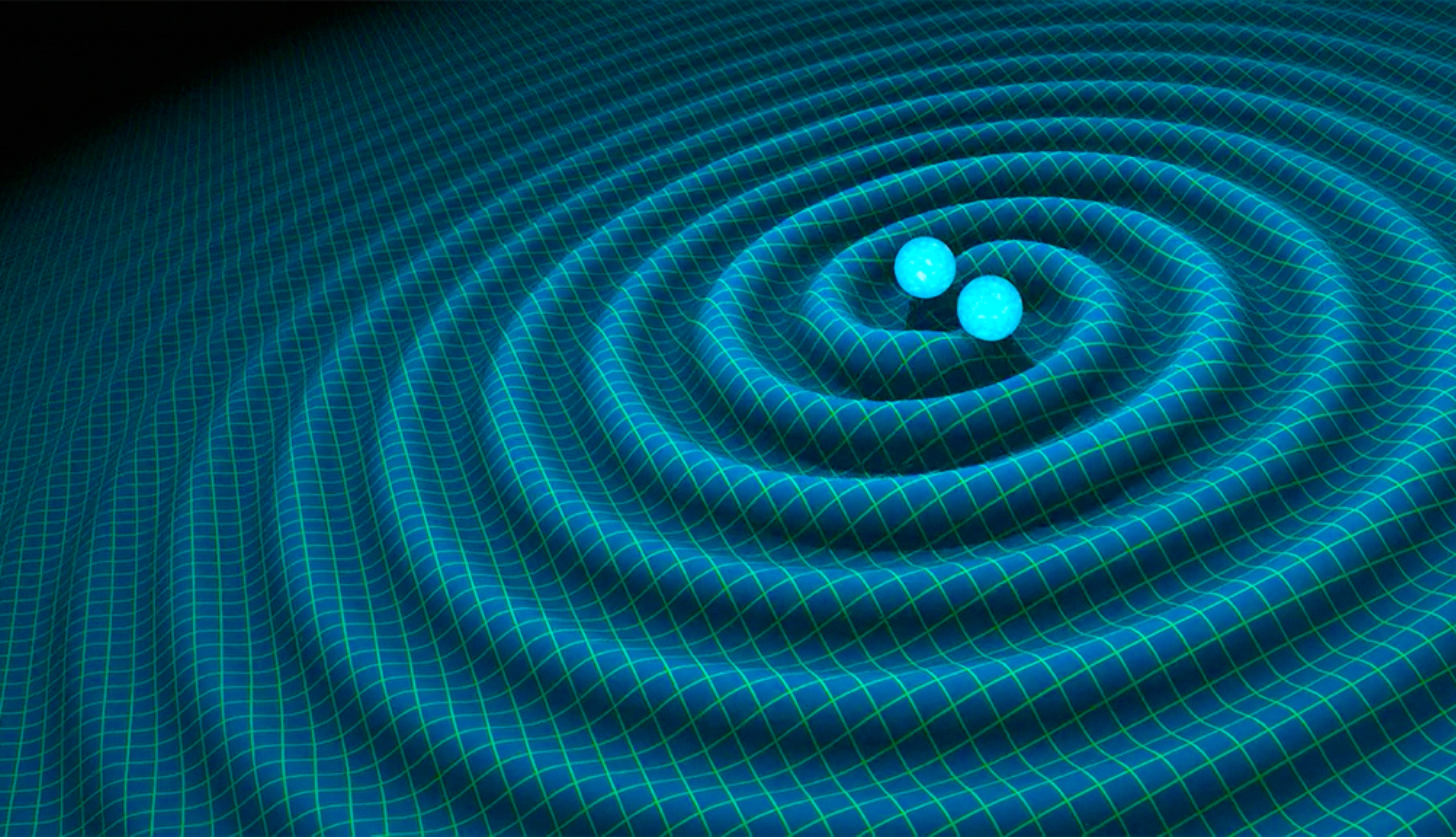
LIGO Hanford

LIGO Livingston

KAGRA



Virgo (IT)



$$m_1 = 36^{+5}_{-4} M_\odot$$

$$m_2 = 29^{+4}_{-4} M_\odot$$

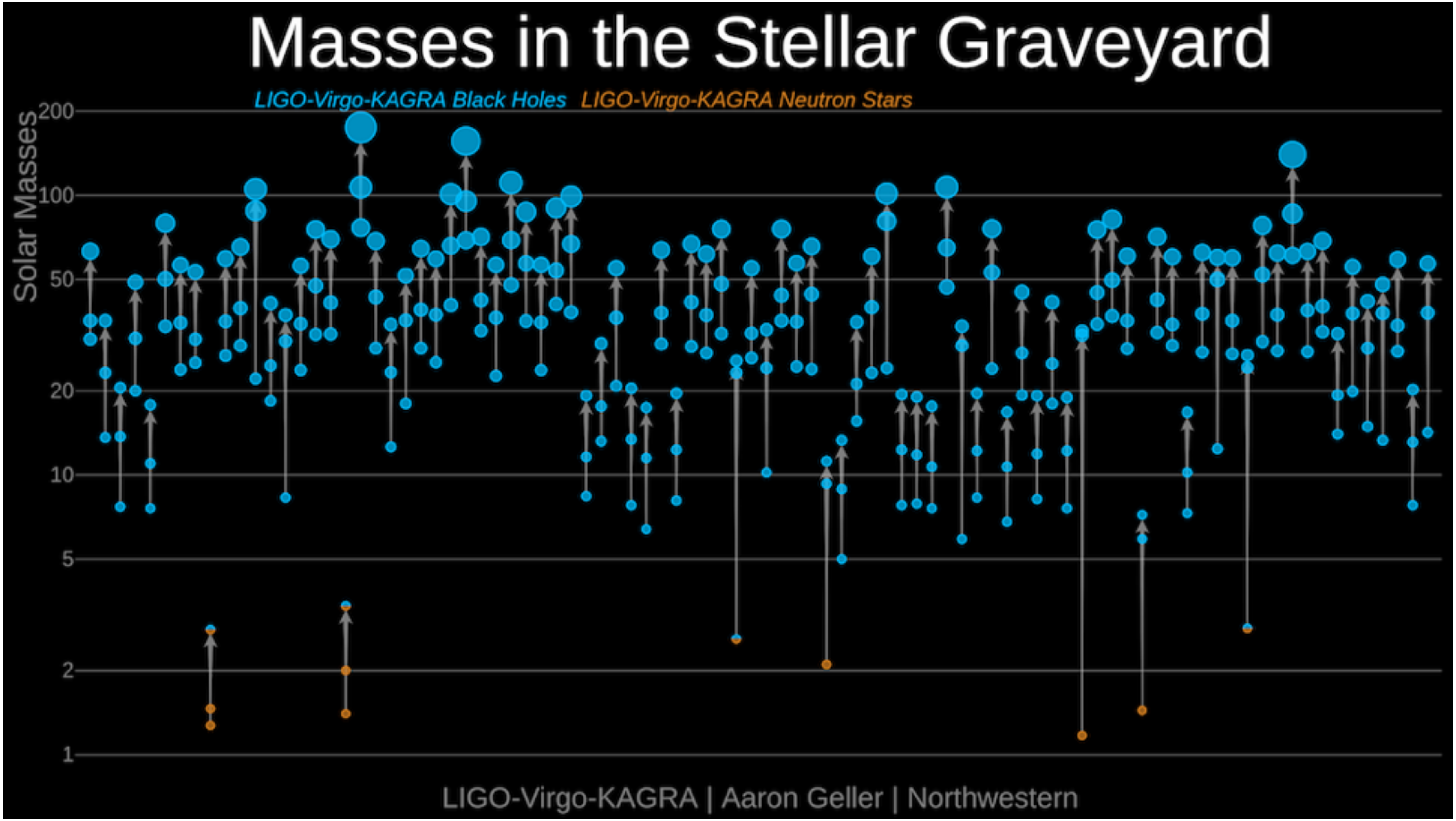
$$\chi_{\rm eff} = -0.06^{+0.17}_{-0.18}$$

$$D_{\rm L} = 410^{+160}_{-180} {\rm Mpc}$$

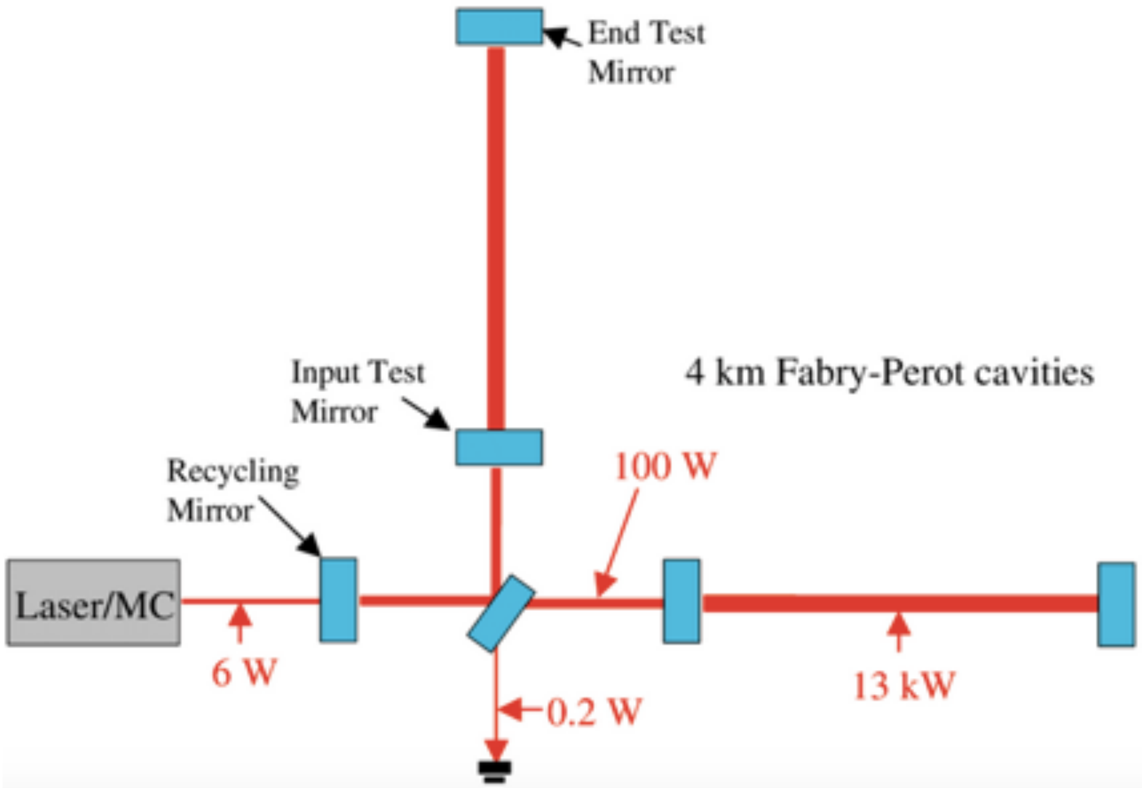
LIGO-Virgo p>0.5 Events

(O1-O2-O3a-O3b; nov 2021)

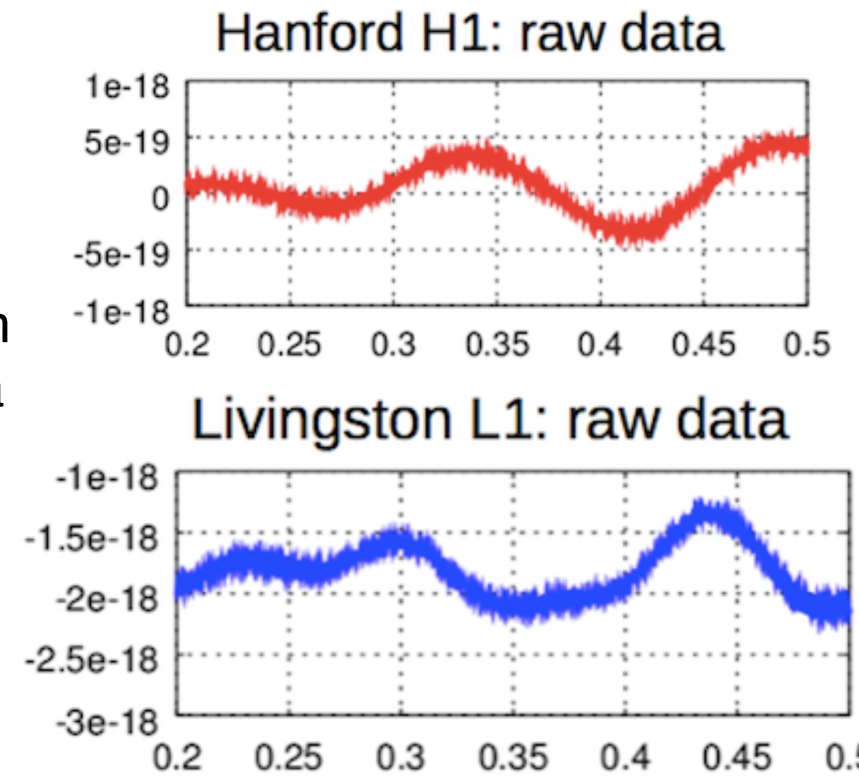
90 events, incl.: 2 NS-NS; 3 NS-BH; 85 BH-BH



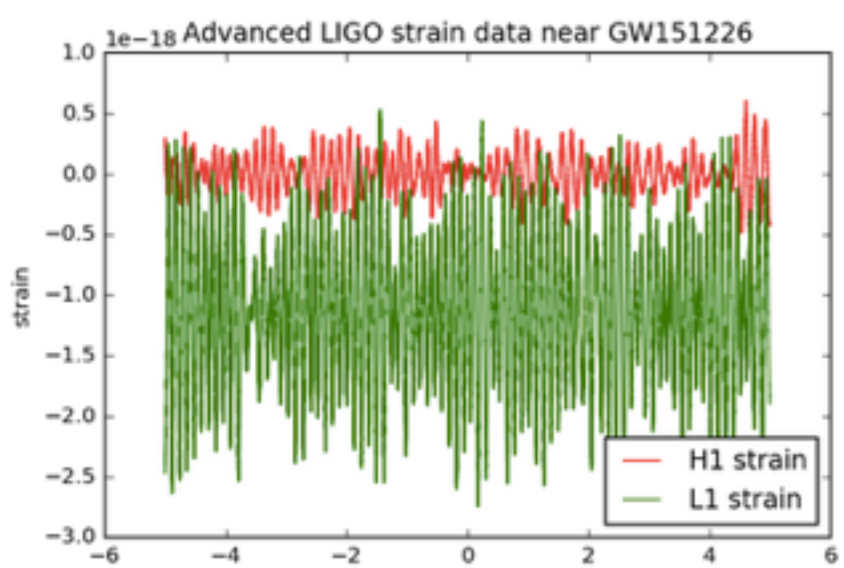
LIGO Raw Data for First Binary Black Hole Events



GW150914 from
LIGO open data

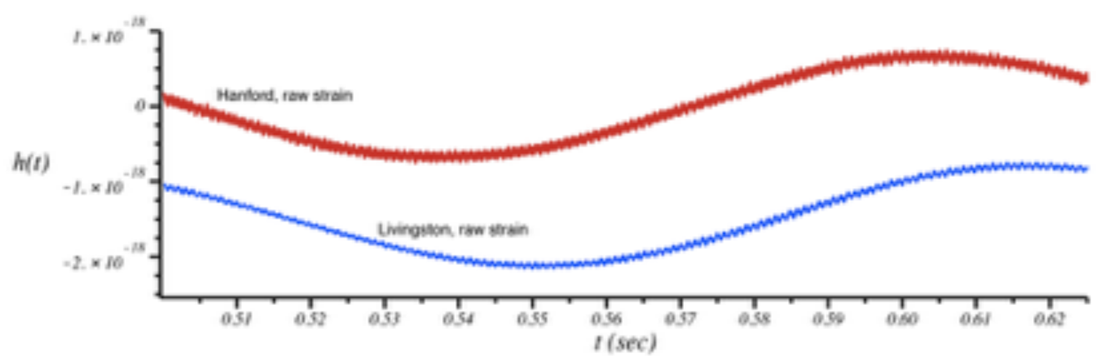


GW151226 from
LIGO open data

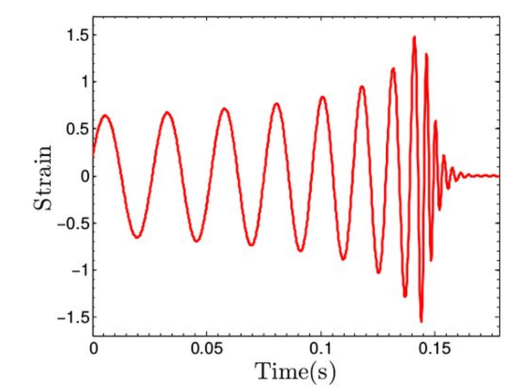


$$\left[\frac{\delta L}{L}\right]_{\rm obs} \sim 10^{-18}$$
$$\gg h_{GW} \lesssim 10^{-21}$$

GW170104 from
LIGO open data



Real
GW
signal

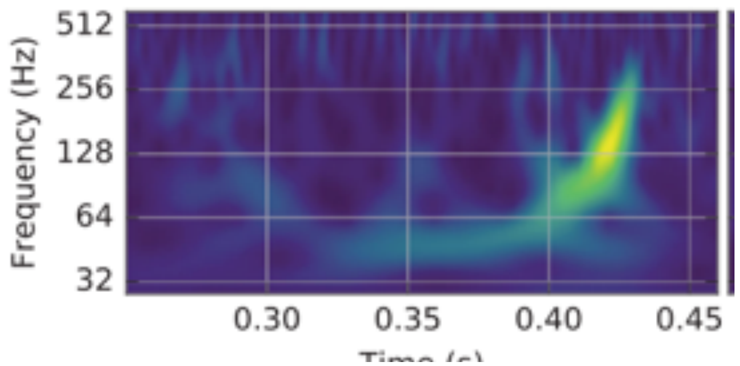


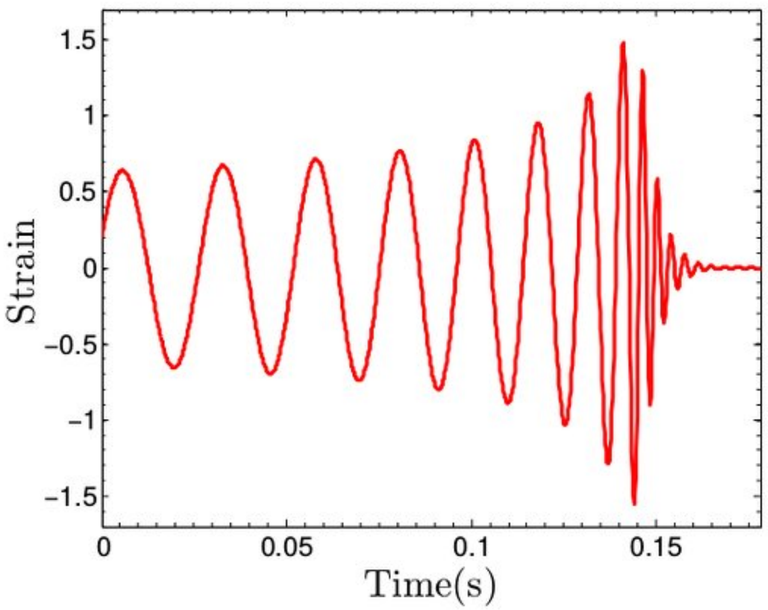
LIGO-Virgo data analysis

Various levels of search and analysis of signals:

Online trigger searches:

CoherentWaveBurst Time-frequency
(Wilson, Meyer, Daubechies-Jaffard-Journe, Klimenko et al.)
Omicron-LALInference sine-Gaussians
Gabor-type wavelet analysis **(Gabor,...,Lynch et al.)**
Matched-filter:
PyCBC (f-domain), gstLAL (t-domain)





Offline data analysis:

Generic transient searches
Binary coalescence searches

Here: focus on matched-filter definition

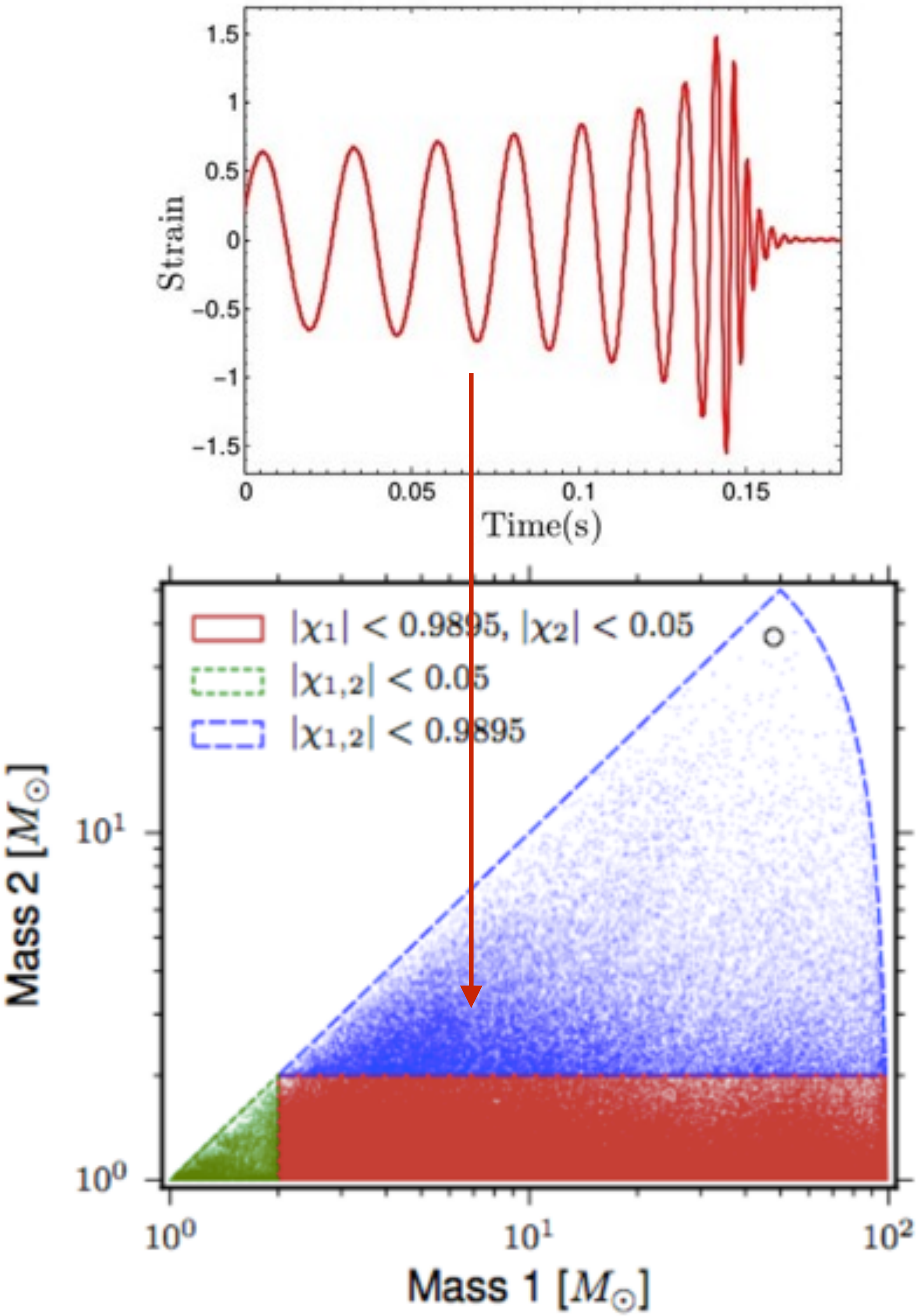
(crucial for high SNR, significance assessment, and parameter estimation)

Matched Filtering

$$\langle output|h_{\rm template}\rangle = \int \frac{df}{S_n(f)} o(f) h^*_{\rm template}(f)$$

MATCHED FILTERING SEARCH AND DATA ANALYSIS

Banks of templates (e.g. 250 000 EOBNR templates in O1) for search inspiralling and coalescing BBH GW waveforms: m1, m2, chi1=S1/m1^2, chi2=S2/m2^2 for m1+m2> 4Msun; + ~ 50 000 PN inspiralling templates for m1+m2< 4 Msun;
O2: ~ 325 000 EOB templates + 75 000 PN templates



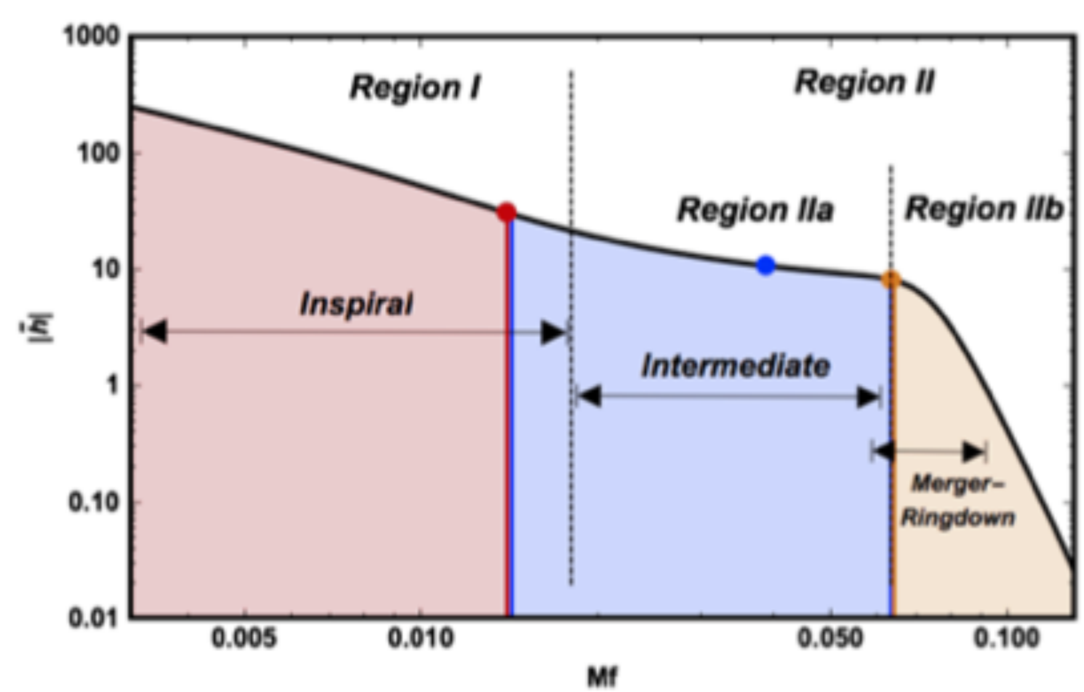
Two types of templates:

Bank of spinning EOB[NR] templates
(Taracchini et al. 14, Bohé et al'17) in ROM form (Puerrer et al.’14); Nagar et al…

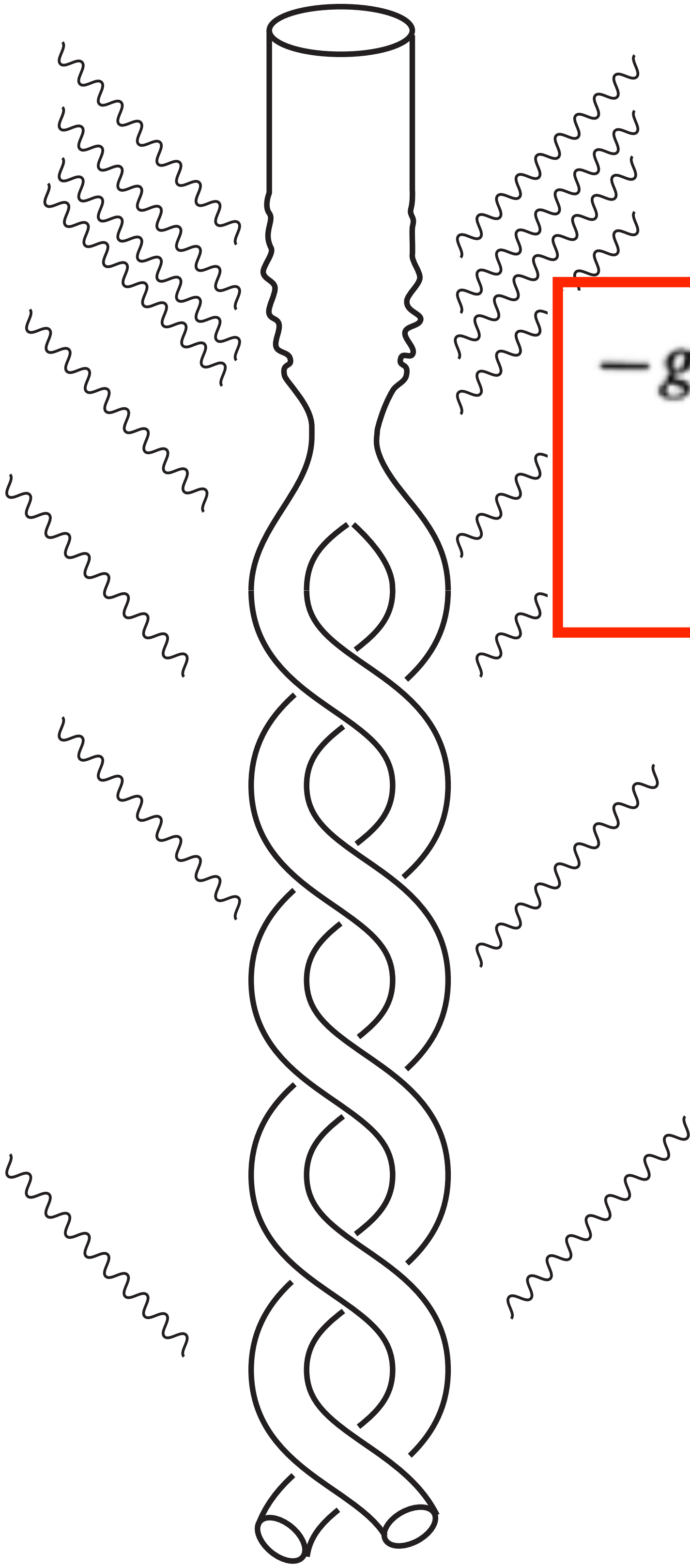
Bank of Phenom[EOB+NR] templates
(Ajith…'07, Hannam…'14, Husa…’16, Khan…’16)

$$h(f) = A(f)e^{i\Psi(f)}$$

$$\Psi(f) = \sum_n c_n v^n(f)\ ;\ v(f) \equiv (\pi M f)^{\frac{1}{3}}$$

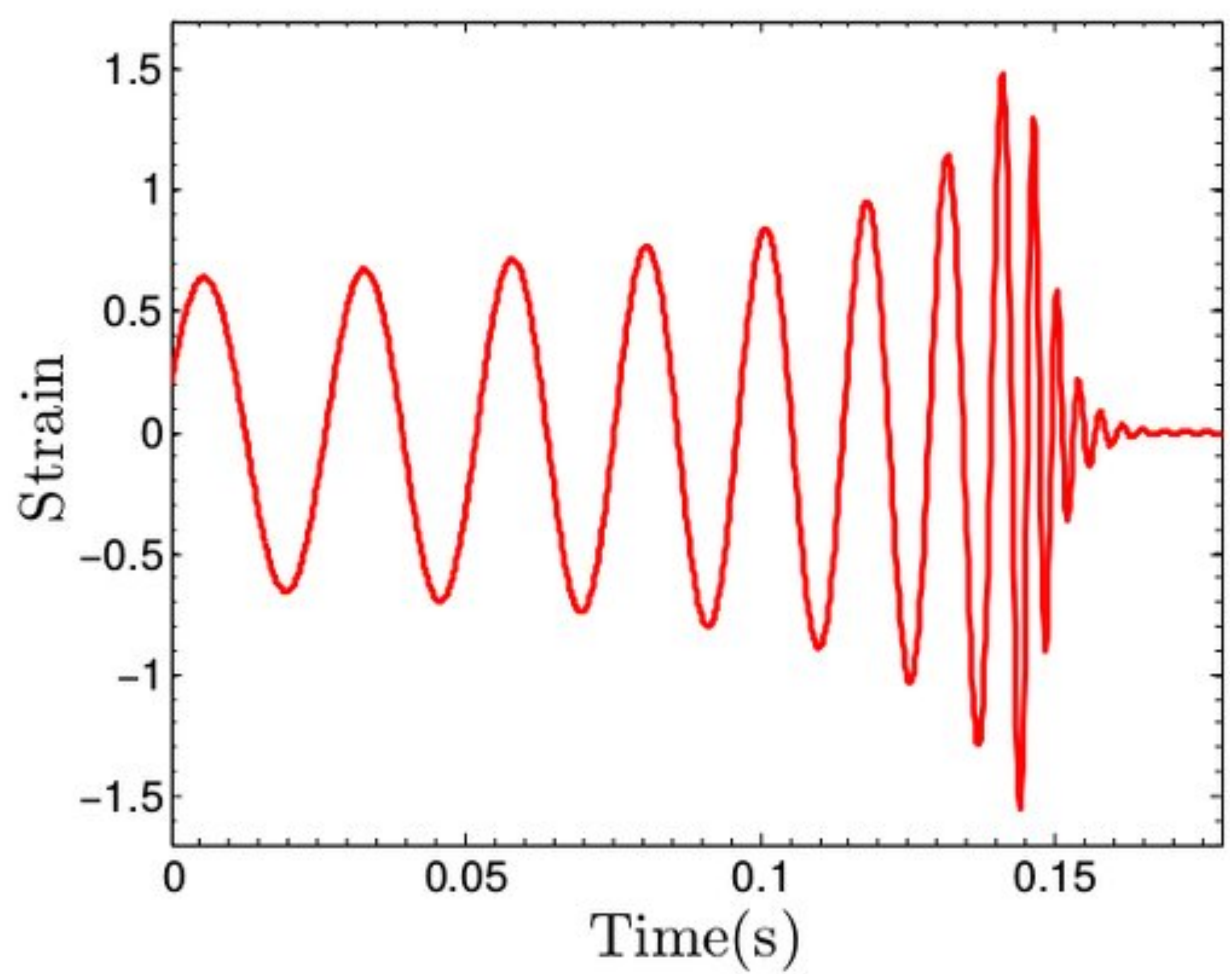


$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad R_{\mu\nu} = 0$$



$$ds^2 = g_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu$$

$$-g^{\mu\nu} g_{\alpha\beta,\,\mu\nu} + g^{\mu\nu} g^{\rho\sigma} \big(g_{\alpha\mu,\,\rho}\, g_{\beta\nu,\,\sigma} - g_{\alpha\mu,\,\rho}\, g_{\beta\sigma,\,\nu} + g_{\alpha\mu,\,\rho}\, g_{\nu\sigma,\,\beta} + g_{\beta\mu,\,\rho}\, g_{\nu\sigma,\,\alpha} - \frac{1}{2} g_{\mu\rho,\,\alpha}\, g_{\nu\sigma,\,\beta}\big) = 0$$



Basics of Gravitational Waves

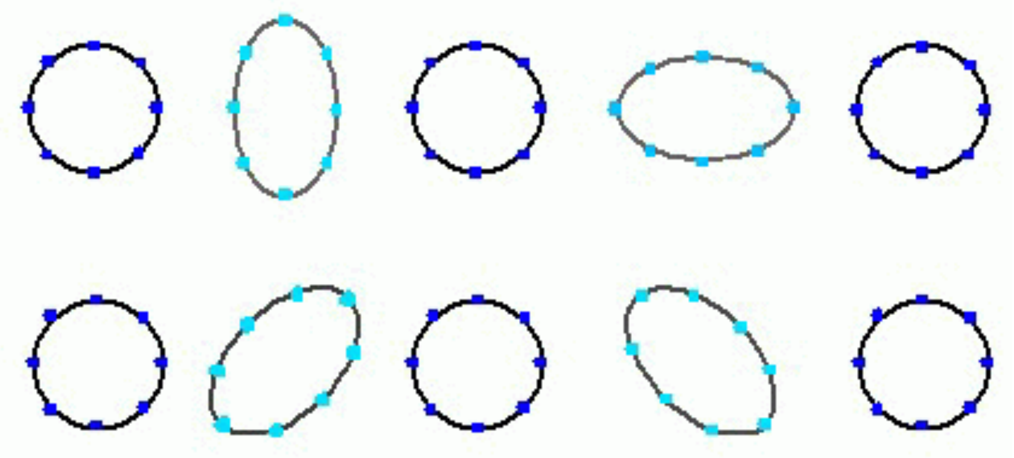
In linearized GR (Einstein 1916,1918):

$$g_{ij} = \delta_{ij} + h_{ij}$$

Two Transverse-Traceless (TT) tensor polarizations propagating at v=c

$$h_{ij} = h_+(t - \frac{z}{c})(e^i_x e^j_x - e^i_y e^j_y) + h_\times(t - \frac{z}{c})(e^i_x e^j_y + e^j_x e^i_y)$$

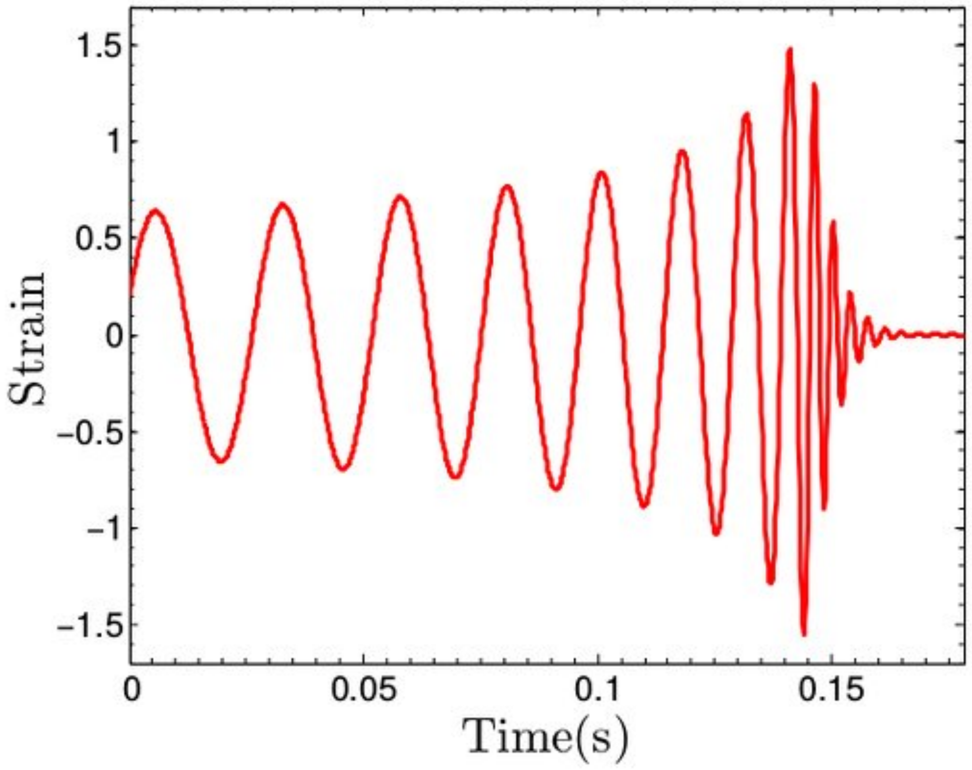
$$\frac{\delta L}{L} = \frac{1}{2} h_{ij} n^i n^j$$



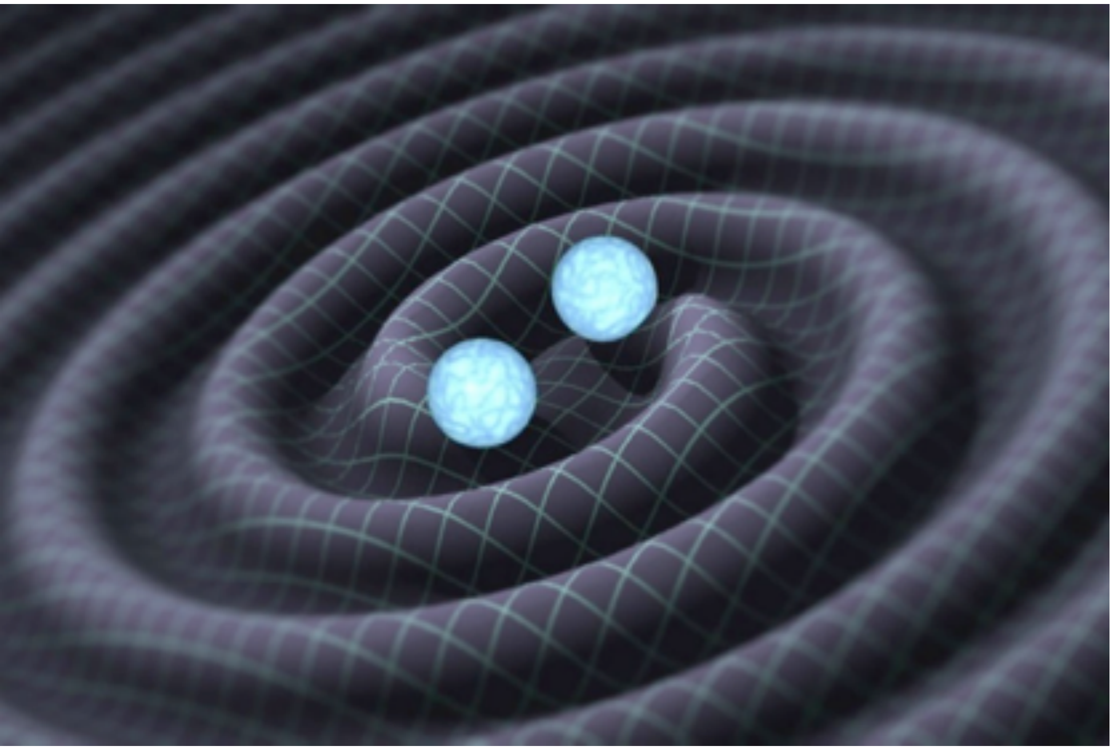
Lowest-order generation: quadrupole formula

$$h_{ij} \simeq \frac{2G}{c^4 r} \ddot{Q}^{TT}_{ij}(t - r/c)$$

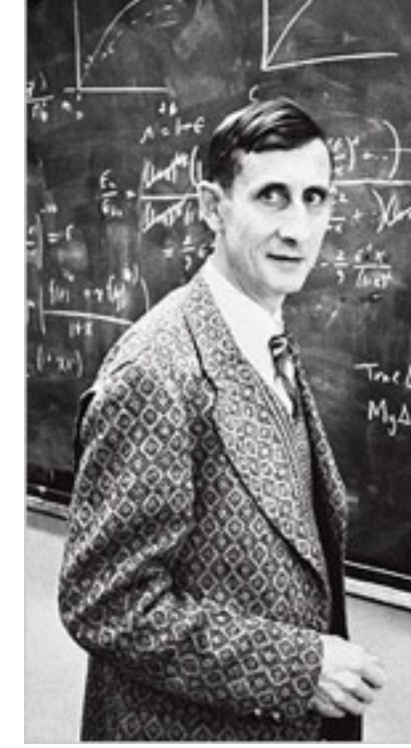
$$Q_{ij} = \int d^3x \frac{T^{00}}{c^2} \left(x^i x^j - \frac{1}{3} x^s x^s \delta_{ij} \right)$$



Pioneering the GWs from coalescing compact binaries



Freeman Dyson 1963

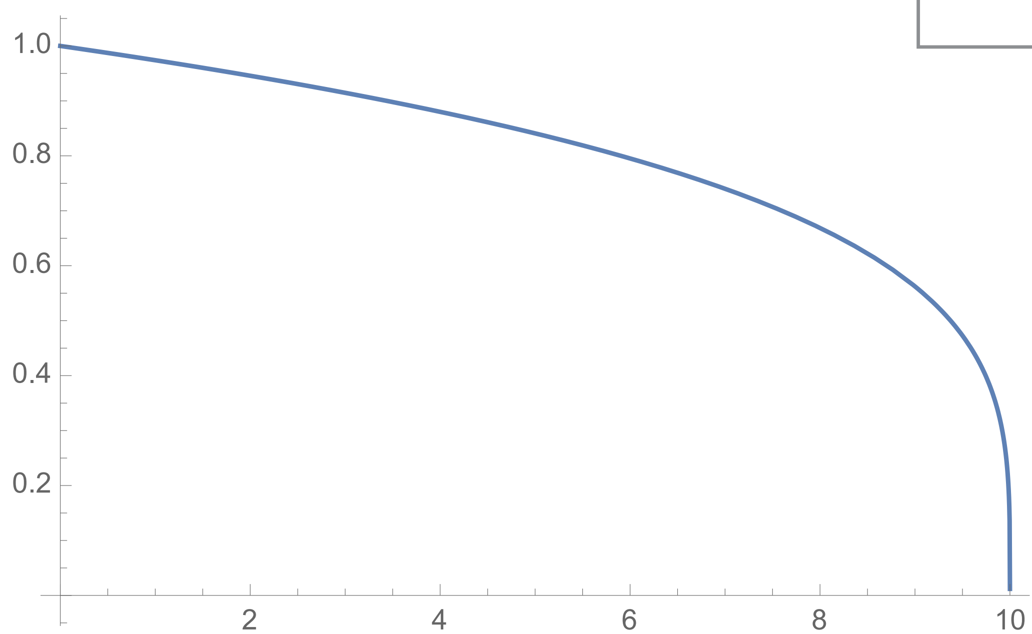


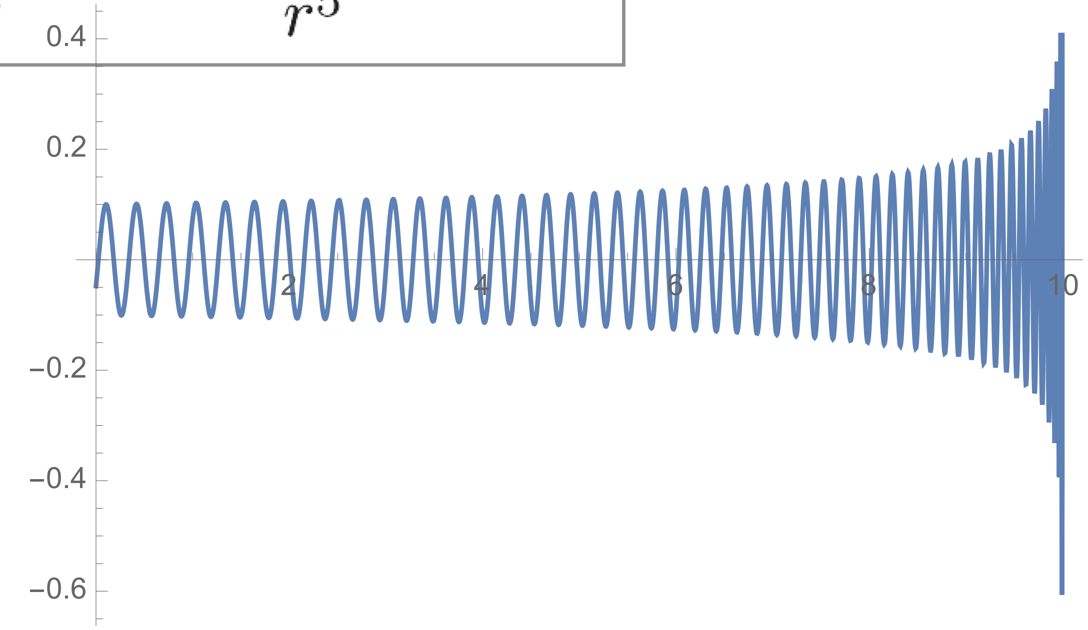
$$E = -\frac{G\, m_1\, m_2}{2r}$$

$$\frac{d}{dt} E = -F$$

Einstein 1918 + Landau-Lifshitz 1941

$$F = \frac{32\, G^4}{5\, c^5} \frac{m_1^2\, m_2^2 (m_1 + m_2)}{r^5}$$

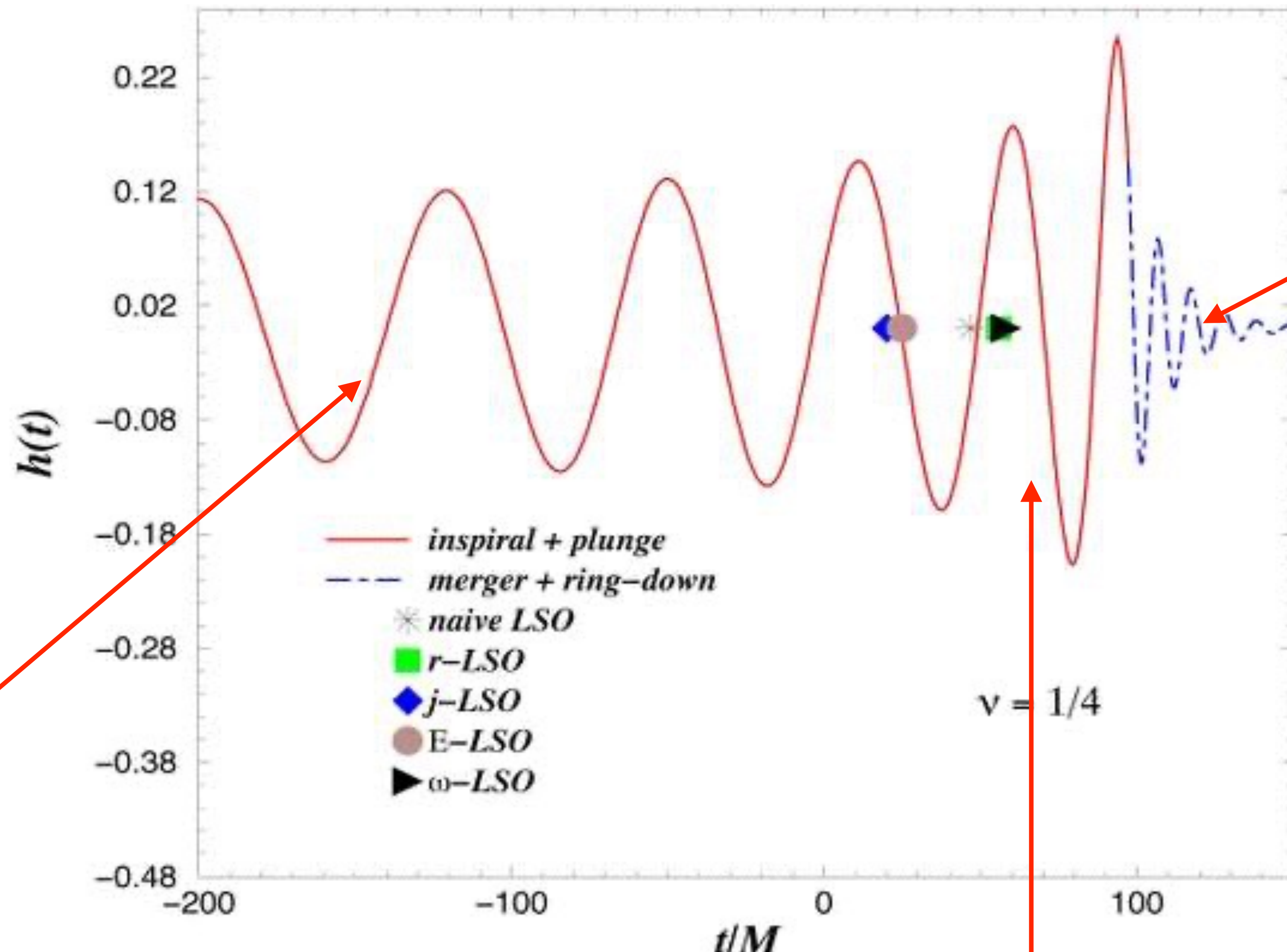


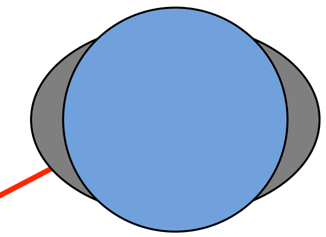


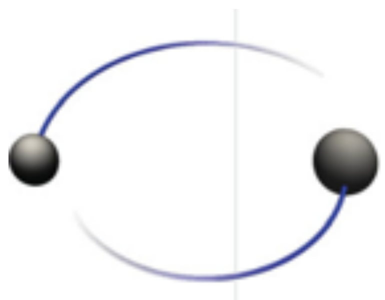
Freeman Dyson's challenge: describe the intense flash of GWs emitted by the last orbits and the merger of a binary BH, when v~c and r~GM/c^2

Gravitational Waves emitted by the Merger of two Black Holes

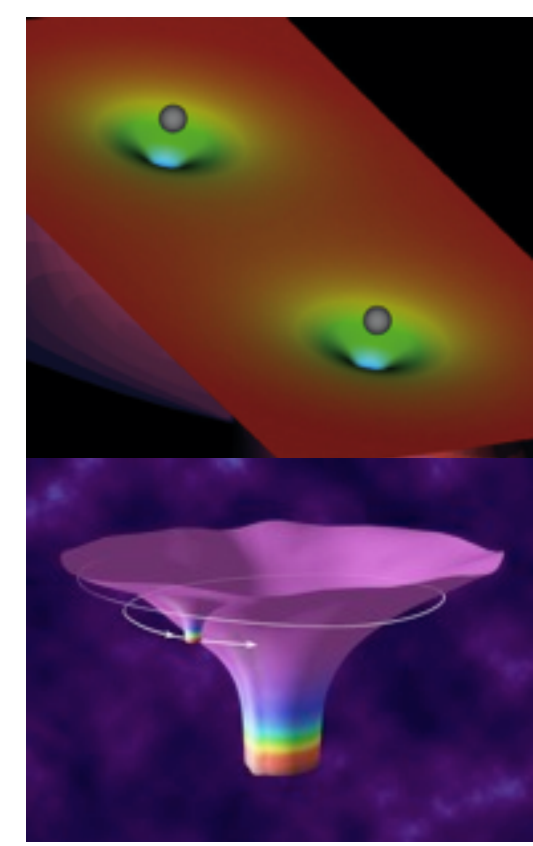
Buonanno-TD'2000

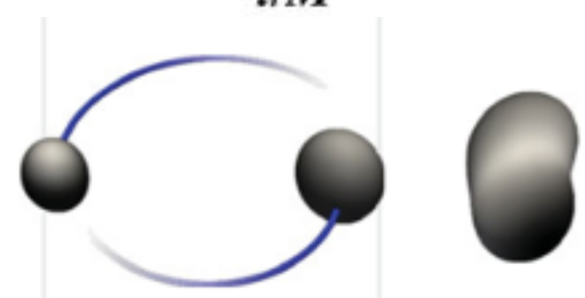






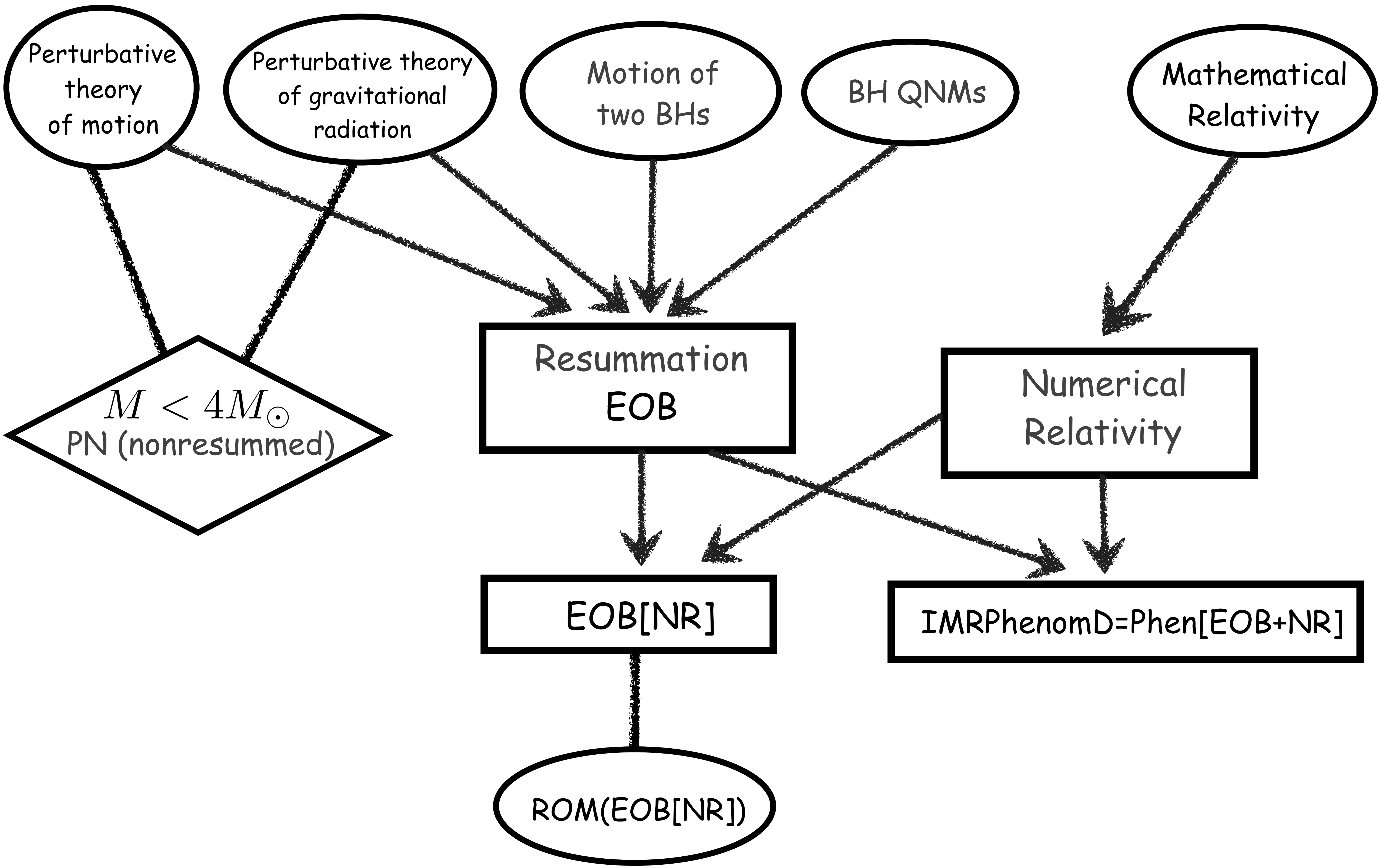
Ringdown (BBH):
« vibration modes »
of final BH (QNM);
perturbation
of BHs à la
Regge-Wheeler-Zerilli-
Teukolsky
+Vishveshwara





Inspiral:
perturbative
computation
of higher-order
contributions
to E=H and F
(expansion in v^2/c^2
+ tidal polarizability
+ of NS)

Late inspiral, « plunge » and merger:
first estimated by the Effective One-Body method (AB-TD 2000)
later confirmed and improved by using
numerical simulations (Pretorius…2005)



Perturbative theory of motion
Perturbative theory of gravitational radiation
Motion of two BHs
BH QNMs
Mathematical Relativity
Resummation EOB
Numerical Relativity
$M < 4M_\odot$
PN (nonresummed)
EOB[NR]
IMRPhenomD=Phen[EOB+NR]
ROM(EOB[NR])

Tools used for the 2-body pb

Post-Newtonian (PN) approximation (**expansion in 1/c**)

Post-Minkowskian (PM) approximation (**expansion in G**)

Multipolar post-Minkowskian (MPM) approximation
theory to the GW emission of binary systems

Matched Asymptotic Expansions useful both for the motion of strongly self-gravitating bodies, and for the nearzone-wavezone matching

Gravitational Self-Force (SF): expansion in m1/m2

Effective One-Body (EOB) Approach

Numerical Relativity (NR)

Recent

Effective Field Theory (EFT)

Quantum scattering amplitude —> classical PM approximation theory
aided by Double-Copy, « Feynman-integral Calculus », Experimental Mathematics, …

Tutti Frutti method

Long History of the GR Problem of Motion

Einstein 1912 : **geodesic principle**

$$-\int m\sqrt{-g_{\mu\nu}\, dx^\mu\, dx^\nu}$$

Einstein 1913-1916 **post-Minkowskian**

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\ ,\ h_{\mu\nu} \ll 1$$

Einstein, Droste : **post-Newtonian**

$$h_{00} \sim h_{ij} \sim \frac{v^2}{c^2}\ ,\ h_{0i} \sim \frac{v^3}{c^3}\ ,\ \partial_0 h \sim \frac{v}{c}\partial_i h$$

Weakly self-gravitating extended bodies:

$$\nabla_\nu T^{\mu\nu} = 0 \quad ; \quad T^{\mu\nu} = \rho'\, u^\mu u^\nu + p\, g^{\mu\nu} \Rightarrow \nabla_u\, u^\mu = O(\nabla p)$$

Einstein-Grossmann ’13,
1916 post-Newtonian: Droste, Lorentz, Einstein (visiting Leiden), De Sitter ;
Lorentz-Droste ‘17, Chazy ‘28, Levi-Civita ’37 ….,

Eddington’ 21, …, Lichnerowicz ‘39, Fock ‘39, Papapetrou ‘51,
… Dixon ‘64, Bailey-Israël ‘75, Ehlers-Rudolph ‘77….

Difficulties at higher PN approximations ca.1970:
Chandrasekhar-Nutku’69, Chandraskhar-Esposito’70,
Burke’69-70,Thorne’69, Ohta-Okamura-Kimura-Hiida’73

BASICS OF BLACK HOLES

1916 Schwarzschild (non rotating) Black Hole (BH)

$$ds^2 = -(1 - \frac{2GM}{c^2 r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

Schwarzschild radius (singularity ?): r_S= 2GM/c^2

radial potential

$$A_S(r) = 1 - \frac{2GM}{c^2 r}$$

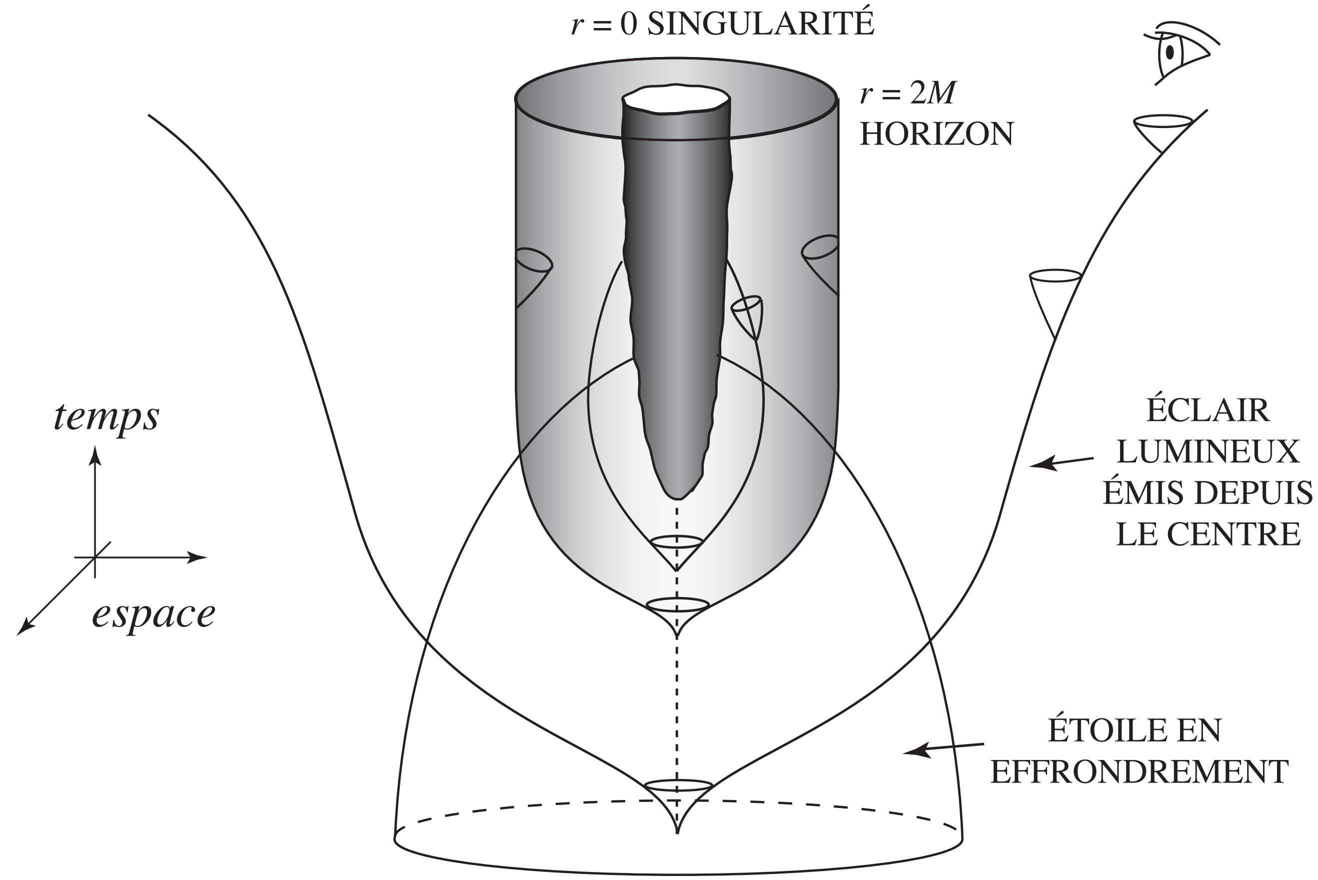
1939 Oppenheimer-Snyder « continued collapse »

1963 Kerr Rotating BH: M, S

1965 Doroshkevich, Zel'dovich, Novikov

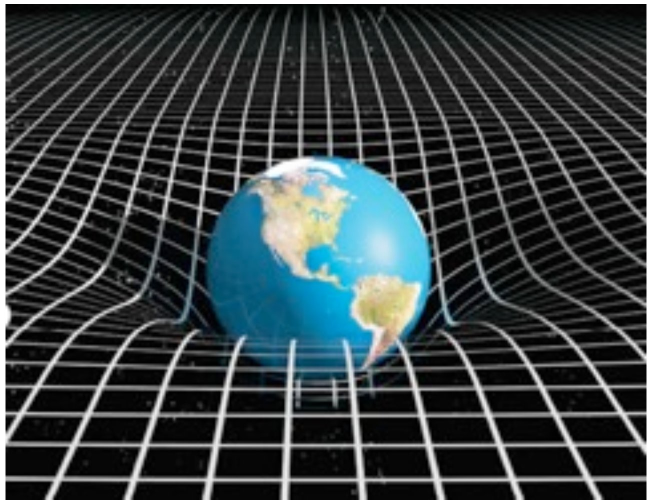
1969 Penrose

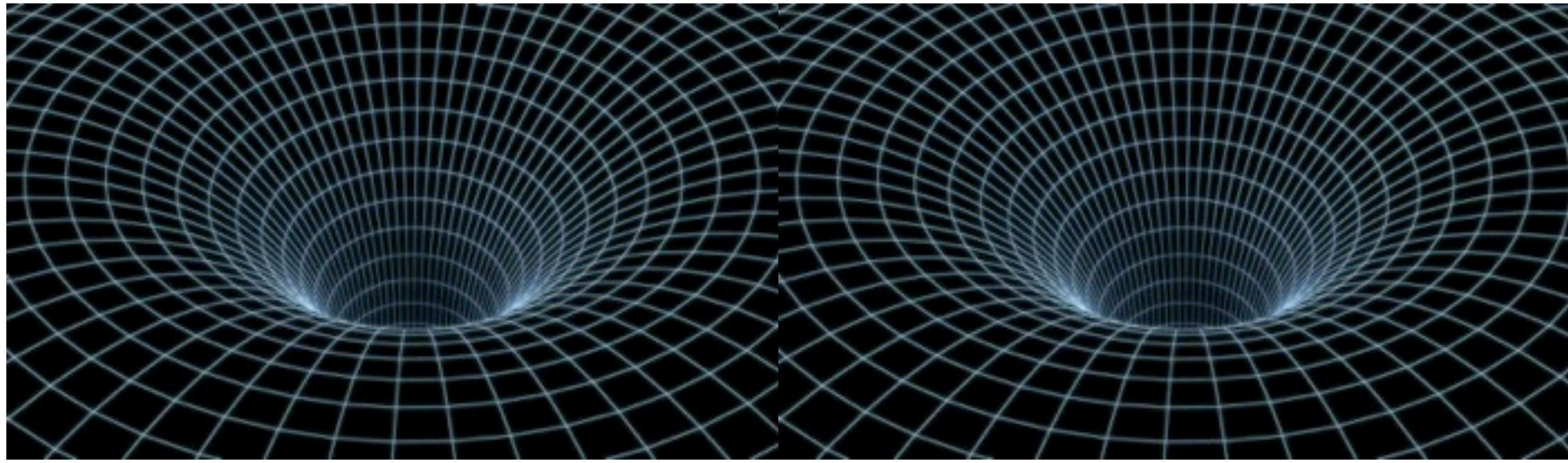
Horizon: cylindrical-like regular null hyper-surface whose sectional area is nearly constant, and actually slowly increasing (Christodoulou '70, Christodoulou-Ruffini '71, Hawking '71)



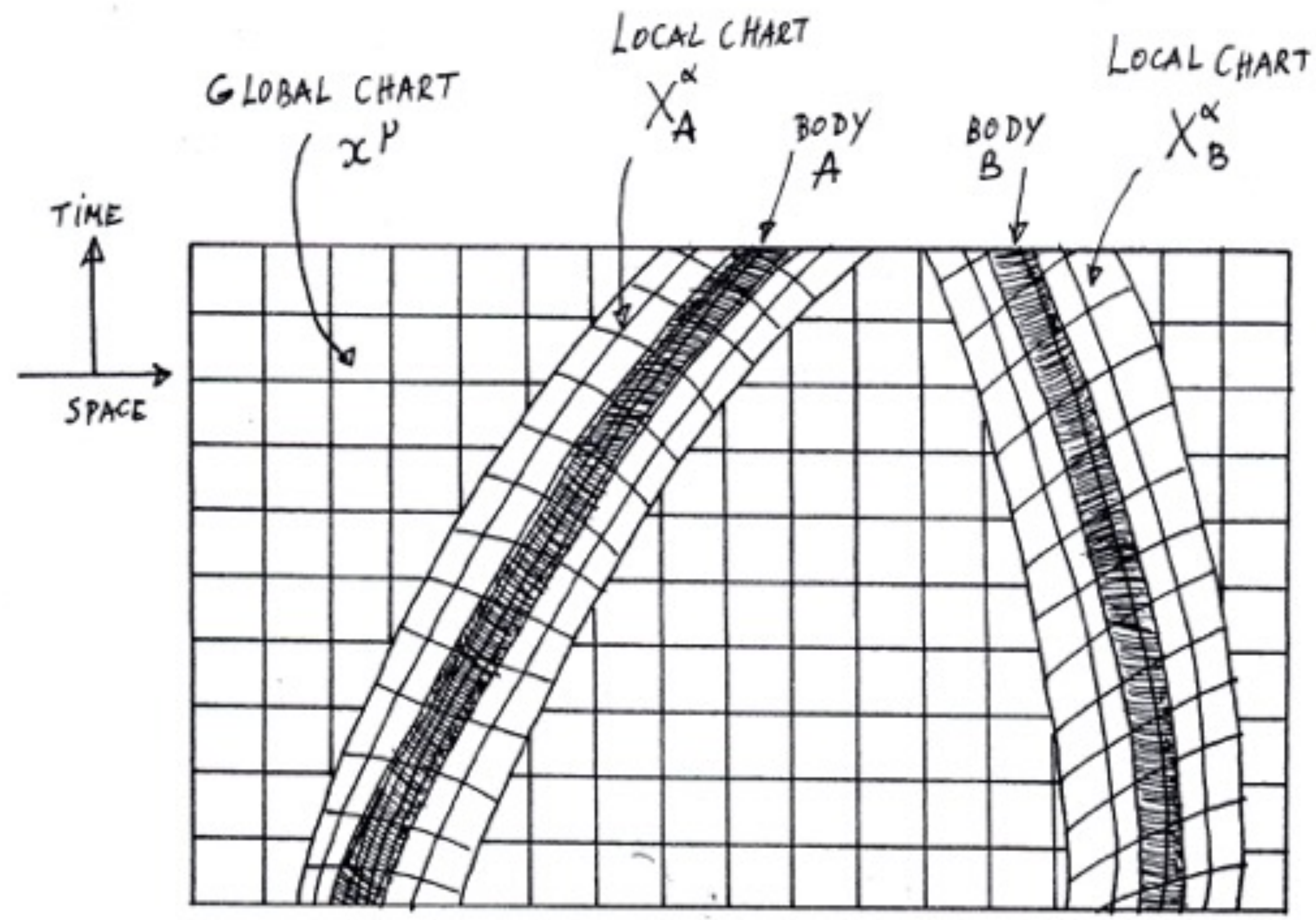
No hair property in D=4

Challenge: Motion of Strongly Self-gravitating Bodies (NS, BH)





Multi-chart approach to motion of strong-self-gravity bodies, and matched asymptotic expansions [**EIH '38**], Manasse '63, Demianski-Grishchuk '74, D'Eath'75, Kates '80, Damour '82



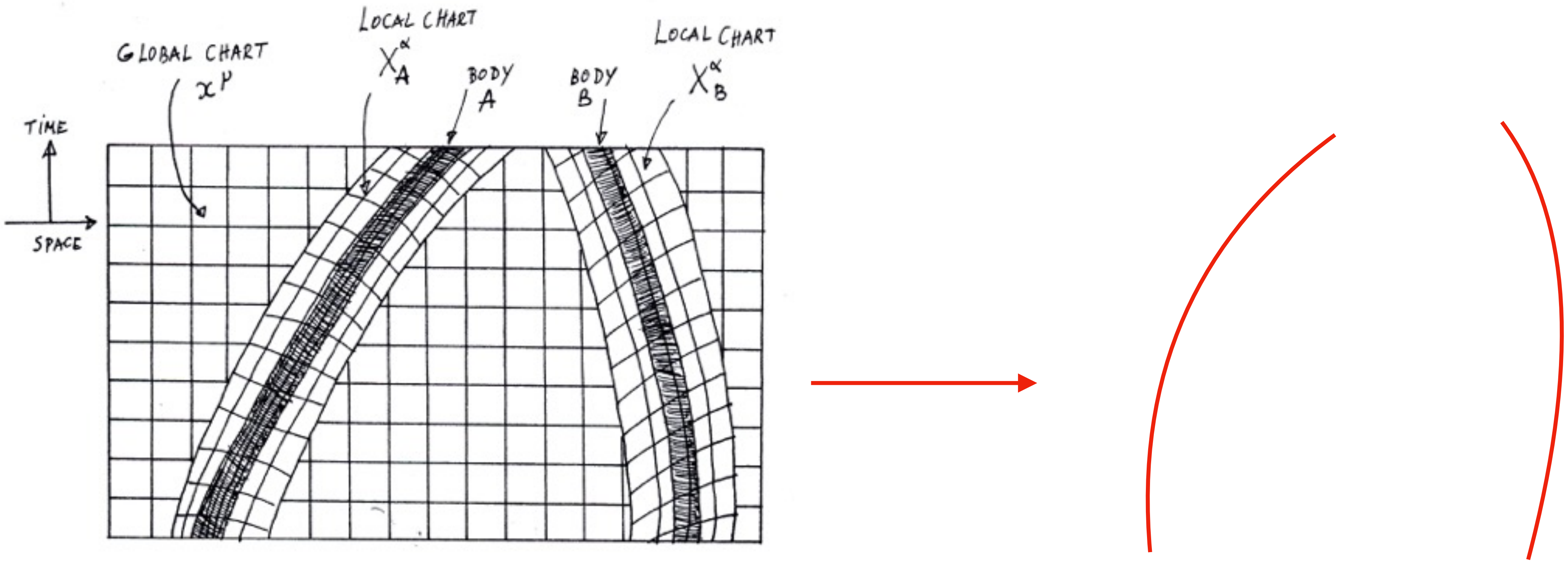
Useful even for weakly self-gravitating bodies, i.e."relativistic celestial mechanics", Brumberg-Kopeikin '89, Damour-Soffel-Xu '91-94

Combine two expansions in two charts:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + G h^{(1)}_{\mu\nu}(x) + G^2 h^{(2)}_{\mu\nu}(x) + \dots \quad G_{\alpha\beta}(x) = G^{(0)}_{\alpha\beta}(x) + G^{(1)}_{\alpha\beta}(x) + \cdots$$

Practical Techniques for Computing the Motion of Compact Bodies (NS or BH)

Skeletonization : $T_{\mu\nu} \longrightarrow$ **point-masses** (Mathisson '31, Infeld'54,…)



Perturbatively solving Einstein eqs: Post-Minkowskian or Post-Newtonian

$$\Box \equiv \Delta - \frac{1}{c^2}\partial_t^2$$

$$(-g)^{1/2} g^{\mu\nu} \equiv \eta^{\mu\nu} + h^{\mu\nu} \text{ with } \partial_\nu h^{\mu\nu} = 0$$

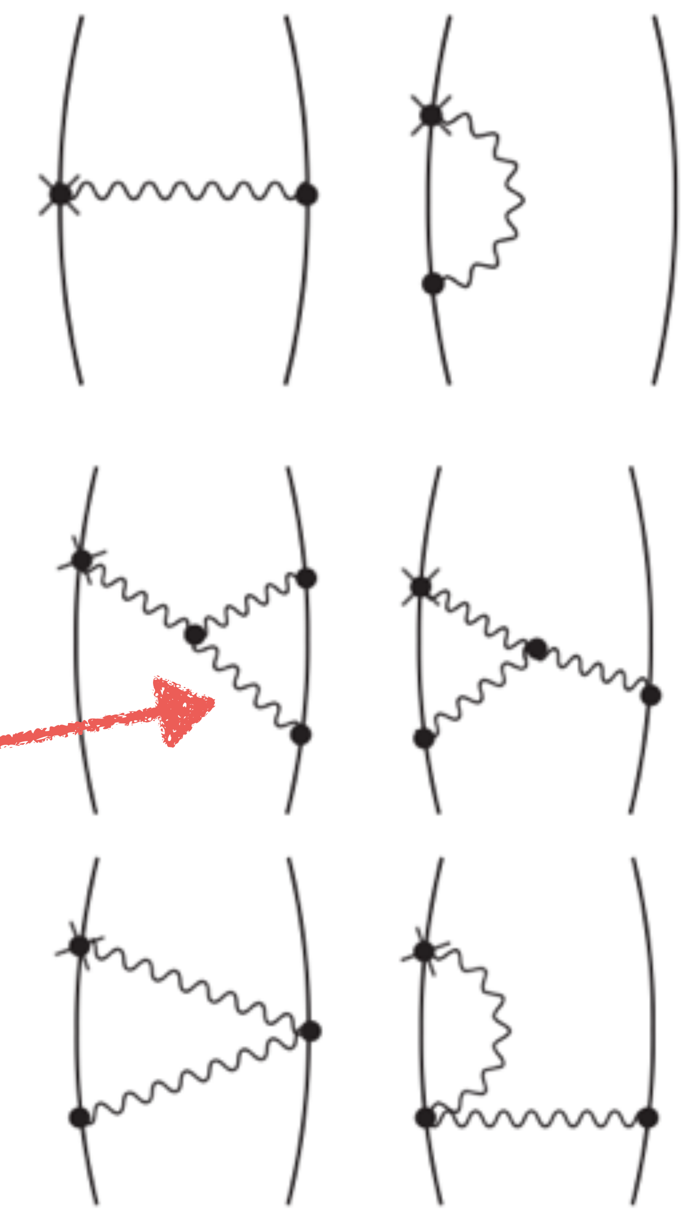
$$\Box h^{\alpha\beta} = 16\pi G \sum_a m_a \int ds_a g^{1/2} \delta_4(x - z_a(s_a) u_a^\alpha u_a^\beta + N^{\alpha\beta} + \cdots$$

$$\begin{aligned} N^{\alpha\beta} = & -h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \tfrac{1}{8}\eta^{\alpha\beta}\partial_\mu h \partial^\mu h - \tfrac{1}{4}\partial^\alpha h \partial^\beta h - \tfrac{1}{4}\eta^{\alpha\beta}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \\ & + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \tfrac{1}{2}\partial^\alpha h^{\mu\nu}\partial^\beta h_{\mu\nu} - \partial^\alpha h^{\mu\nu}\partial_\mu h^\beta_\nu - \partial^\beta h^{\mu\nu}\partial_\mu h^\alpha_\nu \\ & + \partial_\nu h^{\alpha\mu}\partial^\nu h^\beta_\mu + \partial_\nu h^{\alpha\mu}\partial^\mu h^\beta_\nu \end{aligned} \qquad \text{(A16)}$$

Post-Minkowskian expansion used to solve PN difficulties:

Poincaré covariant retarded eqs of motion

(Rosenblum'78,Westpfahl-Goller'79, Bel-Damour-Deruelle-Ibanez-Martin'81)



$$\dot{u}^\alpha = G\underset{(1)}{\Gamma^\alpha} + G^2\left(\Gamma^\alpha_{\bullet} + \Gamma^\alpha_{\cap} + \Gamma^\alpha_{T} + \Gamma^\alpha_{S} + \Gamma^\alpha_{\times}\right) + \mathcal{O}(G^3)$$

$$\underset{(1)}{\Gamma^\alpha} = m'\rho^{-2}\left[(1-2\omega^2)A^\alpha - (1+2\omega^2+4\omega A)v^\alpha\right]$$

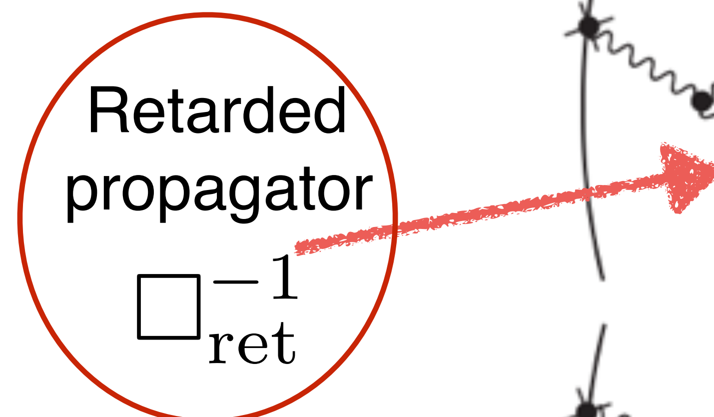
$$\frac{\Gamma^\alpha_{\bullet}}{mm'} = \frac{A^\alpha}{\rho^3}\left[\frac{-4\omega^4+12\omega^2-1}{A^3} + \frac{-4\omega^5+12\omega^3-\omega}{A^2} + \frac{4\omega^4-1}{A}\right.$$
$$\left. - 11\omega + 22\omega^3 + 11A(2\omega^2-1)\right] + \frac{v^\alpha}{\rho^3}\left[\frac{-8\omega^3+12\omega}{A^2}\right.$$
$$\left. + \frac{-8\omega^4+28\omega^2}{A} + 26\omega + \frac{4\times 17}{3}\omega^3 + 11A(1+6\omega^2+4\omega A)\right]$$

$$\Gamma^\alpha_{\cap} = \frac{2m'^2}{\rho^3}(2\omega^2-1)(A^\alpha - v^\alpha)$$

$$\frac{\Gamma^\alpha_{T}}{mm'} = \frac{A^\alpha}{\rho^3}\left[\frac{4\omega^4-1}{A^3} + \frac{4\omega^5-\omega}{A^2} - \frac{4\omega^4-1}{A}\right]$$
$$+ \frac{v^\alpha}{\rho^3}\left[\frac{8\omega^3+4\omega}{A^2} + \frac{4\omega^4-1}{A} - 4\omega - 8\omega^3\right]$$

$$\Gamma^\alpha_{S} = \frac{2m'^2}{\rho^3}\left[A^\alpha(1+\omega^2+2A\omega+A^2) - v^\alpha(1+A\omega+\right.$$

$$\frac{\Gamma^\alpha_{\times}}{mm'} = -4\left[\frac{d}{ds}\underset{(1)}{\Gamma^\alpha}\right]\frac{\ln A}{m'} - \frac{A^\alpha}{\rho^3}\left[\frac{2}{A^5} + \frac{5\omega}{A^4} + \frac{(22\omega^2-7)}{A^3}\right.$$
$$+ \frac{4\omega}{A^2}(5\omega^2-2) + \frac{8(1-2\omega^2-\omega^4)}{A}$$
$$\left. + 17\omega(1-2\omega^2) + 5A(1-2\omega^2)\right] - \frac{v^\alpha}{\rho^3}\left[\frac{47}{3A^3} + \frac{48\omega}{A^2}\right.$$
$$\left. + \frac{4(\omega^2-2)}{A} - 62\omega - \frac{4\times 31}{3}\omega^3 - A(5+78\omega^2) - 20\omega A^2\right]$$



These eqs of motion involve retarded gravitational interactions and thereby contain both conservative and dissipative effects, as exhibited by expanding them in a Post-Newtonian series in powers of 1/c (retardation)

2.5PN (1/c^5) Eqs of Motion (Damour-Deruelle'81, Damour'82)

$$
\begin{aligned}
\boldsymbol{a}_1 = &-\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} \quad \text{(Newtonian acceleration)}\\
&+\frac{1}{c^2}\Bigg\{\Bigg[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\frac{Gm_2}{r_{12}^2}\left(\frac{3}{2}(n_{12}v_2)^2-v_1^2+4(v_1v_2)-2v_2^2\right)\Bigg]\boldsymbol{n}_{12}\\
&\qquad+\frac{Gm_2}{r_{12}^2}\left(4(n_{12}v_1)-3(n_{12}v_2)\right)\boldsymbol{v}_{12}\Bigg\}\\
&+\frac{1}{c^4}\Bigg\{\Bigg[-\frac{57G^3m_1^2m_2}{4r_{12}^4}-\frac{69G^3m_1m_2^2}{2r_{12}^4}-\frac{9G^3m_2^3}{r_{12}^4}\\
&\qquad+\frac{Gm_2}{r_{12}^2}\left(-\frac{15}{8}(n_{12}v_2)^4+\frac{3}{2}(n_{12}v_2)^2v_1^2-6(n_{12}v_2)^2(v_1v_2)-2(v_1v_2)^2+\frac{9}{2}(n_{12}v_2)^2v_2^2\right.\\
&\qquad\qquad\left.+4(v_1v_2)v_2^2-2v_2^4\right)\\
&\qquad+\frac{G^2m_1m_2}{r_{12}^3}\left(\frac{39}{2}(n_{12}v_1)^2-39(n_{12}v_1)(n_{12}v_2)+\frac{17}{2}(n_{12}v_2)^2-\frac{15}{4}v_1^2-\frac{5}{2}(v_1v_2)+\frac{5}{4}v_2^2\right)\\
&\qquad+\frac{G^2m_2^2}{r_{12}^3}\left(2(n_{12}v_1)^2-4(n_{12}v_1)(n_{12}v_2)-6(n_{12}v_2)^2-8(v_1v_2)+4v_2^2\right)\Bigg]\boldsymbol{n}_{12}\\
&\qquad+\Bigg[\frac{G^2m_2^2}{r_{12}^3}\left(-2(n_{12}v_1)-2(n_{12}v_2)\right)+\frac{G^2m_1m_2}{r_{12}^3}\left(-\frac{63}{4}(n_{12}v_1)+\frac{55}{4}(n_{12}v_2)\right)\\
&\qquad+\frac{Gm_2}{r_{12}^2}\left(-6(n_{12}v_1)(n_{12}v_2)^2+\frac{9}{2}(n_{12}v_2)^3+(n_{12}v_2)v_1^2-4(n_{12}v_1)(v_1v_2)\right.\\
&\qquad\qquad\left.+4(n_{12}v_2)(v_1v_2)+4(n_{12}v_1)v_2^2-5(n_{12}v_2)v_2^2\right)\Bigg]\boldsymbol{v}_{12}\Bigg\}
\end{aligned}
$$

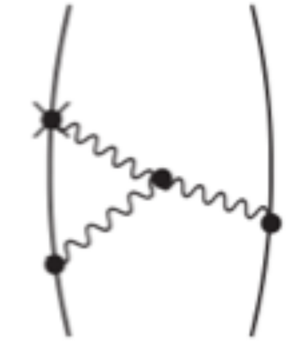
$$
\begin{aligned}
&+\frac{1}{c^5}\Bigg\{\Bigg[\frac{208G^3m_1m_2^2}{15r_{12}^4}(n_{12}v_{12})-\frac{24G^3m_1^2m_2}{5r_{12}^4}(n_{12}v_{12})+\frac{12G^2m_1m_2}{5r_{12}^3}(n_{12}v_{12})v_{12}^2\Bigg]\boldsymbol{n}_{12}\\
&\qquad+\Bigg[\frac{8G^3m_1^2m_2}{5r_{12}^4}-\frac{32\,G^3m_1m_2^2}{5r_{12}^4}-\frac{4G^2m_1m_2}{5r_{12}^3}v_{12}^2\Bigg]\boldsymbol{v}_{12}\Bigg\}
\end{aligned}
$$

1PN
v^2/c^2
+Gm/(c^2 r)

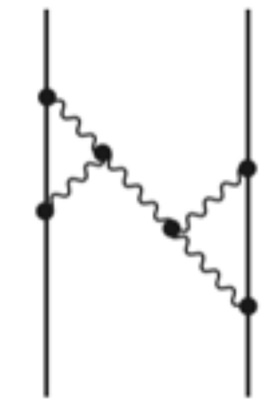
2 PN
1/c^4

2.5 PN

G^2

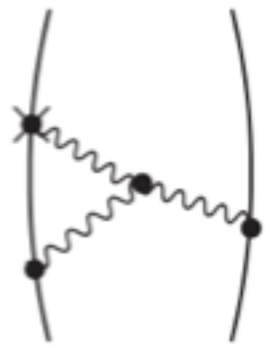


adding some
3PM, G^3, two-loop
contributions

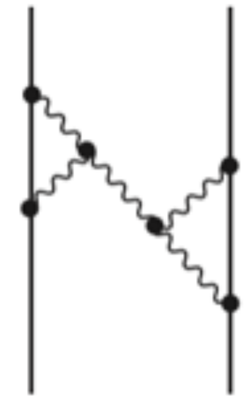


Technical bottleneck of PM approach (at the time and, partly, still today): computing spacetime integrals involving retarded propagators and the nonlinear gravitational interactions of GR

2 PM
1 loop



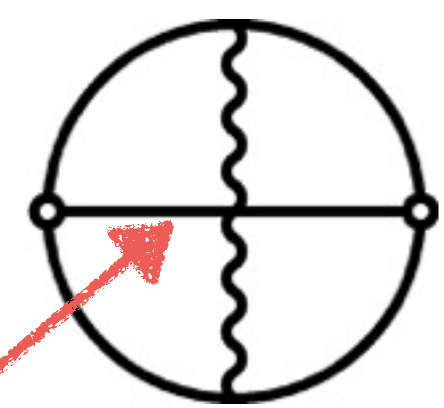
3 PM
2 loop



Beyond 2.5PN : Separate Conservative and Radiation Damping Effects

Conservative Dynamics: use the post-Newtonian (PN) approximation method

$$\Box^{-1} = (\Delta - \frac{1}{c^2}\partial_t^2)^{-1} = \Delta^{-1} + \frac{1}{c^2}\partial_t^2\Delta^{-2} + ...$$



Simpler to iterate instantaneous Newtonian 1/r=1/|x-y| potential

1/|x-y|

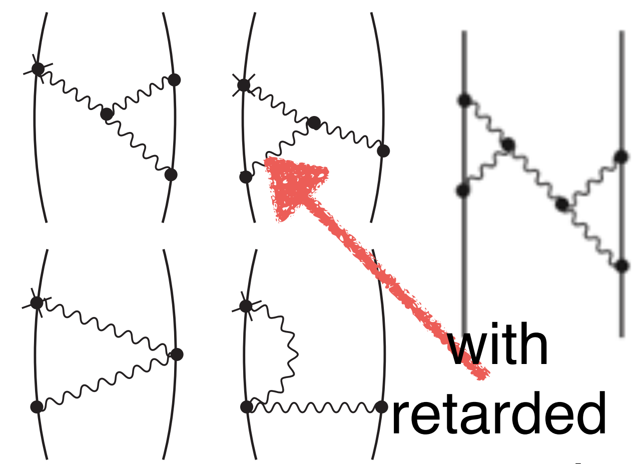
Radiation Damping: assume balance and compute radiation losses at infinity using a Multipolar Post-Minkowskian (MPM) approach

State of the art for PN dynamics

- 1PN (including v^2/c^2) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38
- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
  Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
  LO-radiation-reaction Kopeikin '85

First complete 2PN and 2.5PN dynamics obtained by using 2PM (G^2) EOM of Bel et al.'81

- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
  Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
  Blanchet-Damour-Esposito-Farèse' 04, Foffa-Sturani '11
- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
  Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09
- **4PN** (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,'16
  Bini-Damour '13, Damour-Jaranowski-Schäfer '14, Marchand+'18, Foffa+'19



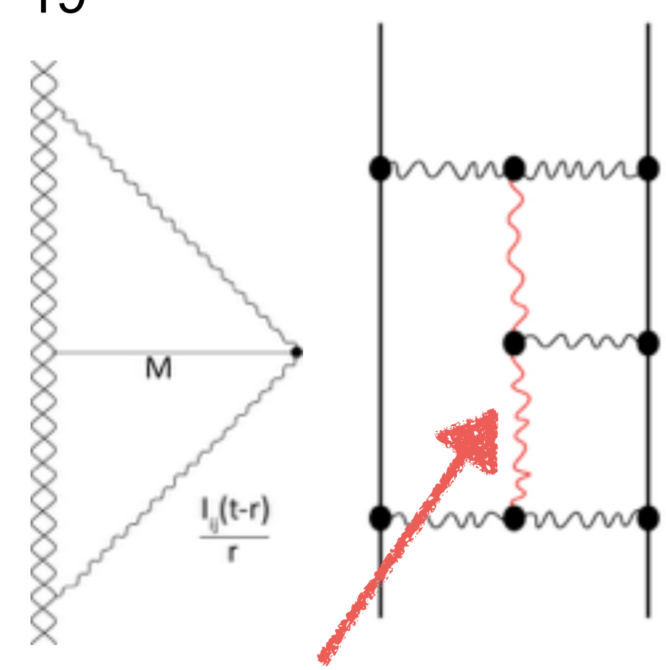
with retarded propagator

New feature at G^4/c^8 (4PN and 4PM) : **non-locality in time** (linked to IR divergences of formal PN-expansion) (Blanchet,TD '88)

- **5PN** (inc. v^{10}/c^{10} and **G^6**) Bini-Damour-Geralico'19: complete **modulo two**
- **numerical** parameters; **Bluemlein et al'21**: potential-graviton contrib. and
- partial determination of radiation-graviton contrib. used QGRAF to generate

545812 4-loop diagrams, and 332020 5-loop diagrams

- **6PN** (inc. v^{12}/c^{12} and **G^7**) Bini-Damour-Geralico'20: complete **modulo four**
- additional parameters



soft (radiation) gravitons

Inclusion of **spin-dependent effects**: Barker-O' Connell'75, Faye-Blanchet-Buonanno'06, Damour-Jaranowski-Schaefer'08, Porto-Rothstein '06, Levi '10, Steinhoff-Hergt-Schaefer '10, Steinhoff'11, Levi-Steinhoff'15-18, Bini-TD, Vines , Guevara-Ochirov-Vines,….

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2}\frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.$$
$$\left.-6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)$$
$$+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2) + 6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

2-body Taylor-expanded 3PN Hamiltonian [DJS 01]

$$
\begin{aligned}
c^6 H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{G m_1 m_2}{r_{12}}\Bigg(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4 m_2^2} \\
& - 10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4 m_2^2} \\
& + 2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2\mathbf{p}_2^2 - 10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2^3} \\
& + \frac{(\mathbf{p}_1^2\mathbf{p}_2^2 - 2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2^3} + 15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3} \\
& - 18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3}\Bigg) + \frac{G^2 m_1 m_2}{r_{12}^2}\Bigg(\frac{1}{16}(m_1 - 27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4} \\
& - \frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} + \frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3} \\
& - \frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} \\
& + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} \\
& - \frac{1}{48}(220m_1 + 193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\Bigg) + \frac{G^3 m_1 m_2}{r_{12}^3}\Bigg(-\frac{1}{48}\left(425m_1^2 + \left(473 - \frac{3}{4}\pi^2\right)m_1 m_2 + 150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2} \\
& + \frac{1}{16}\left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{1}{16}\left(20m_1^2 - \left(43 + \frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} \\
& + \frac{1}{16}\left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\Bigg) \\
& + \frac{1}{8}\frac{G^4 m_1 m_2^3}{r_{12}^4}\left(\left(\frac{227}{3} - \frac{21}{4}\pi^2\right)m_1 + m_2\right) + (1 \leftrightarrow 2).
\end{aligned}
$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014, JS 2015]

$$\begin{aligned} c^8 H^{\mathrm{local}}_{\mathrm{4PN}}(\mathbf{x}_a,\mathbf{p}_a) &= \frac{7(\mathbf{p}_1^2)^5}{256m_1^9} + \frac{Gm_1m_2}{r_{12}}H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2m_1m_2}{r_{12}^2}m_1H_{46}(\mathbf{x}_a,\mathbf{p}_a) \\ &+ \frac{G^3m_1m_2}{r_{12}^3}(m_1^2H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1m_2H_{442}(\mathbf{x}_a,\mathbf{p}_a)) \\ &+ \frac{G^4m_1m_2}{r_{12}^4}(m_1^3H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2m_2H_{422}(\mathbf{x}_a,\mathbf{p}_a)) \\ &+ \frac{G^5m_1m_2}{r_{12}^5}H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1\leftrightarrow 2). \end{aligned} \tag{A3}$$

$$\begin{aligned} H_{48}(\mathbf{x}_a,\mathbf{p}_a) =& \frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16m_1^6m_2^2} \\ &- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256m_1^5m_2^3} \\ &+ \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^5m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^5m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} \\ &- \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} \\ &+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} \\ &- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} \\ &+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^4} \\ &- \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^4m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^4m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^4m_2^4} \\ &- \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^4m_2^4} \\ &- \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^4m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^4m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^4m_2^4}. \end{aligned} \tag{A4a}$$

$$\begin{aligned} H_{46}(\mathbf{x}_a,\mathbf{p}_a) =& \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128m_1^5m_2} \\ &+ \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16m_1^5m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^5m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2} \\ &+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^4m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^4m_2^2} \\ &- \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320m_1^4m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480m_1^4m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} \\ &- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840m_1^4m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^4m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^4m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^4m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8m_1^3m_2^3} \\ &+ \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^3m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} \\ &+ \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192m_1^3m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96m_1^3m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3} \\ &+ \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4} \\ &- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2m_1^2m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^4} \\ &- \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24m_1^2m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^2m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8m_2^6}. \end{aligned} \tag{A4b}$$

$$\begin{aligned} H_{441}(\mathbf{x}_a,\mathbf{p}_a) =& \frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640m_1^3m_2} \\ &+ \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800m_1^3m_2} \\ &- \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960m_1^2m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^2m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400m_1^2m_2^2} \\ &+ \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400m_1^2m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^2m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^2m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^4}. \end{aligned} \tag{A4c}$$

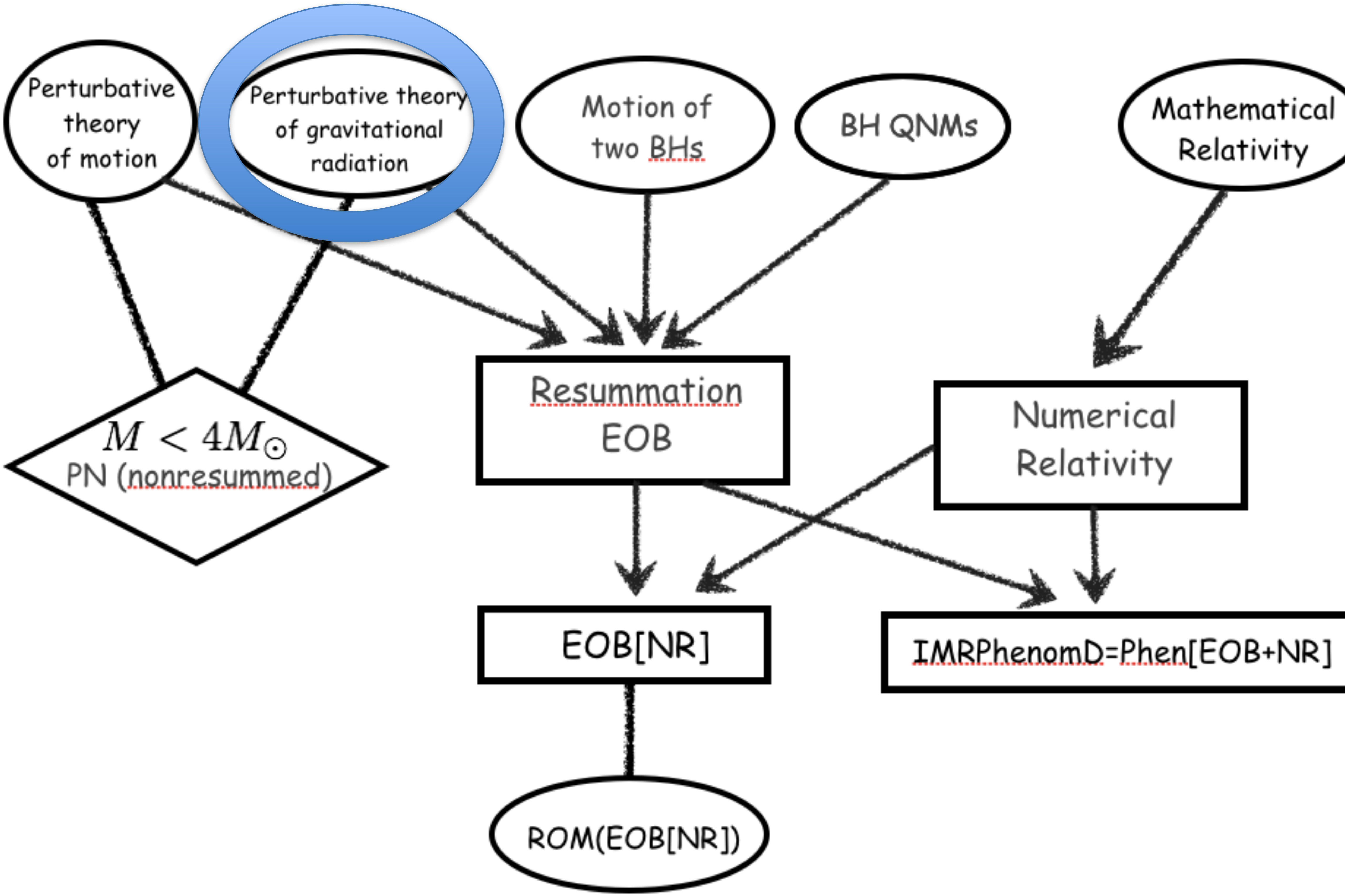
$$\begin{aligned} H_{442}(\mathbf{x}_a,\mathbf{p}_a) =& \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\ &+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} \\ &+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2m_2^2} \\ &- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} \\ &+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2} \\ &+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}, \end{aligned} \tag{A4d}$$

$$H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2m_1m_2}, \tag{A4e}$$

$$\begin{aligned} H_{422}(\mathbf{x}_a,\mathbf{p}_a) =& \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2} \\ &+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2} \\ &+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}. \end{aligned} \tag{A4f}$$

$$H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2m_2^2. \tag{A4g}$$

$$\begin{aligned} H^{\mathrm{nonloc}}_{\mathrm{4PN}}(t) &= -\frac{1}{5}\frac{G^2M}{c^8}I^{(3)}_{ij}(t) \\ &\times \mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}I^{(3)}_{ij}(t+v). \end{aligned}$$



Perturbative theory of motion
Perturbative theory of gravitational radiation
Motion of two BHs
BH QNMs
Mathematical Relativity
$M < 4M_\odot$
PN (nonresummed)
Resummation EOB
Numerical Relativity
EOB[NR]
IMRPhenomD=Phen[EOB+NR]
ROM(EOB[NR])

Perturbative Theory of the Generation of Gravitational Radiation

Einstein '16, '18 (+ Landau-Lifshitz 41, and Fock '55) : h_+, h_x and **quadrupole formula**

Relativistic, **multipolar extensions** of LO quadrupole radiation :

Sachs-Bergmann '58, Sachs '61, Mathews '62, Peters-Mathews '63, Pirani '64

Campbell-Morgan '71,

Campbell et al '75,

nonlinear effects:

Bonnor-Rotenberg '66,

Epstein-Wagoner-Will '75-76

Thorne '80, .., Will et al 00

MPM Formalism:

Blanchet-Damour '86,

Damour-Iyer '91,

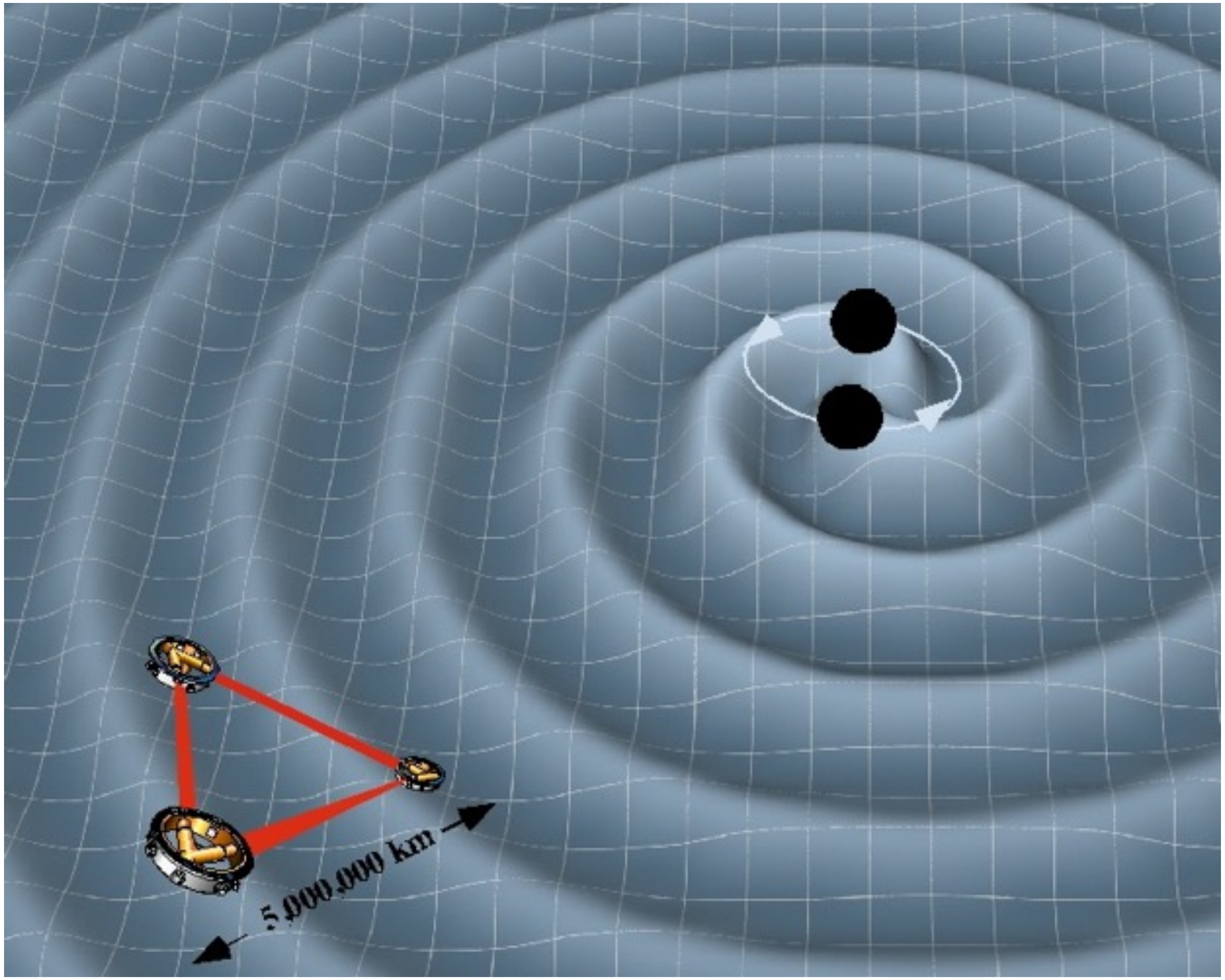
Blanchet '95 '98

Combines multipole exp. ,

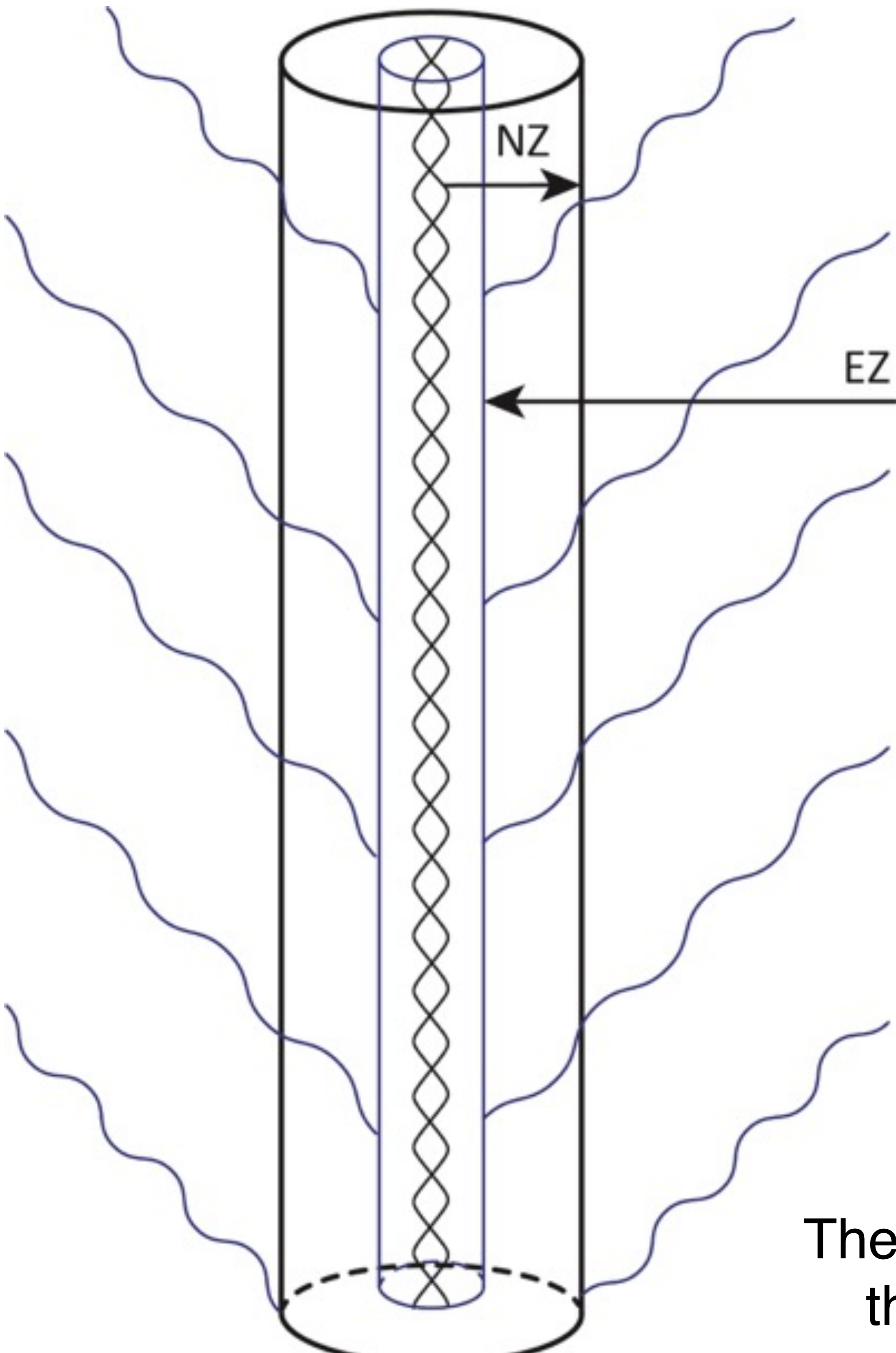
Post Minkowkian exp.,

analytic continuation,

and PN matching



MULTIPOLAR POST-MINKOWSKIAN FORMALISM (BLANCHET-DAMOUR-IYER)



Decomposition of space-time in various overlapping regions:

1. **near-zone: r << lambda : PN**
2. **exterior zone: r >> r_source: MPM**
3. **far wave-zone: Bondi-type expansion**

then matching between the zones

in exterior zone, iterative solution of Einstein's vacuum field equations by means of a double expansion in non-linearity and in multipoles, with crucial use of analytic continuation (complex B) for dealing with formal UV divergences at r=0

$$g = \eta + G h_1 + G^2 h_2 + G^3 h_3 + ...,$$
$$\Box h_1 = 0,$$
$$\Box h_2 = \partial\partial h_1 h_1,$$
$$\Box h_3 = \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2,$$
$$h_1 = \sum_\ell \partial_{i_1 i_2 \ldots i_\ell}\left(\frac{M_{i_1 i_2 \ldots i_\ell}(t-r/c)}{r}\right) + \partial\partial\ldots\partial\left(\frac{\epsilon_{j_1 j_2 k} S_{k j_3 \ldots j_\ell}(t-r/c)}{r}\right),$$
$$h_2 = FP_B \Box^{-1}_{\rm ret}\left(\left(\frac{r}{r_0}\right)^B \partial\partial h_1 h_1\right) + ...,$$
$$h_3 = FP_B \Box^{-1}_{\rm ret} \ldots.$$

STF tensors encoding multipole moments

mass-type and spin-type multipole moments

The PN-matched MPM formalism has allowed to compute the GW emission to very high accuracy (Blanchet et al)

Link radiative multipoles <-> source variables

(Blanchet-Damour '89'92, Damour-Iyer'91, Blanchet '95…)

radiative quadrupole seen at infinity

$$
\begin{aligned}
\mathrm{U}_{ij}(U) = {} & \mathrm{M}^{(2)}_{ij}(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}^{(4)}_{ij}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{11}{12}\right] \\
& + \frac{G}{c^5}\left\{-\frac{2}{7}\int_0^{+\infty} d\tau\, \mathrm{M}^{(3)}_{a\langle i}(U-\tau)\mathrm{M}^{(3)}_{j\rangle a}(U-\tau)\right. \\
& \left. \qquad -\frac{2}{7}\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(2)}_{j\rangle a} - \frac{5}{7}\mathrm{M}^{(4)}_{a\langle i}\mathrm{M}^{(1)}_{j\rangle a} + \frac{1}{7}\mathrm{M}^{(5)}_{a\langle i}\mathrm{M}_{j\rangle a} + \frac{1}{3}\varepsilon_{ab\langle i}\mathrm{M}^{(4)}_{j\rangle a}\mathrm{S}_b\right\} \\
& + \frac{2G^2\mathrm{M}^2}{c^6}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ij}(U-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right)+\frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right)+\frac{124627}{44100}\right] \\
& + \mathcal{O}\left(\frac{1}{c^7}\right).
\end{aligned}
$$

tail

memory

instant.

tail-of-tail

$$\mathrm{M}_{ij} = \mathrm{I}_{ij} - \frac{4G}{c^5}\left[\mathrm{W}^{(2)}\mathrm{I}_{ij} - \mathrm{W}^{(1)}\mathrm{I}^{(1)}_{ij}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

$$\Sigma = \frac{\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2},$$

$$\Sigma_i = \frac{\overline{\tau}^{0i}}{c},$$

$$\Sigma_{ij} = \overline{\tau}^{ij}$$

integral over effective source

$$
\begin{aligned}
\mathrm{I}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz & \left\{\delta_l\hat{x}_L\Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)}\delta_{l+1}\hat{x}_{iL}\Sigma^{(1)}_i\right. \\
& \left. + \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{ijL}\Sigma^{(2)}_{ij}\right\}(\mathbf{x}, u+z|\mathbf{x}|/c), \qquad (85)
\end{aligned}
$$

$$\mathrm{J}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\, \varepsilon_{ab\langle i_l}\left\{\delta_l\hat{x}_{L-1\rangle a}\Sigma_b - \frac{2l+1}{c^2(l+2)(2l+3)}\delta_{l+1}\hat{x}_{L-1\rangle ac}\Sigma^{(1)}_{bc}\right\}(\mathbf{x}, u+z|\mathbf{x}|/c).$$

tau^munu hides nonlinear effects

Perturbative (3.5PN) GW flux from (circular) binary system

- lowest order : Einstein 1918 Peters-Mathews 63
- 1 + (v^2/c^2) : Wagoner-Will 76
- … + (v^3/c^3) : Blanchet-Damour 92, Wiseman 93
- … + (v^4/c^4) : Blanchet-Damour-Iyer Will-Wiseman 95
- … + (v^5/c^5) : Blanchet 96
- … + (v^6/c^6) : Blanchet-Damour-Esposito-Farèse-Iyer 2004
- … + (v^7/c^7) : Blanchet
- … + most of (v^8/c^8) : Blanchet et al

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$x = \left(\frac{v}{c}\right)^2 = \left(\frac{G(m_1+m_2)\Omega}{c^3}\right)^{\frac{2}{3}} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{2}{3}}$$

$$\begin{aligned}\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + 4\pi x^{3/2} \\ & + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\ & + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16\,x) \right. \\ & \left. \quad + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right] x^3 \\ & + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned}$$

LO quadrupole radiation

3.5PN

4PN still incomplete

Analytical GW Templates for BBH Coalescences ?

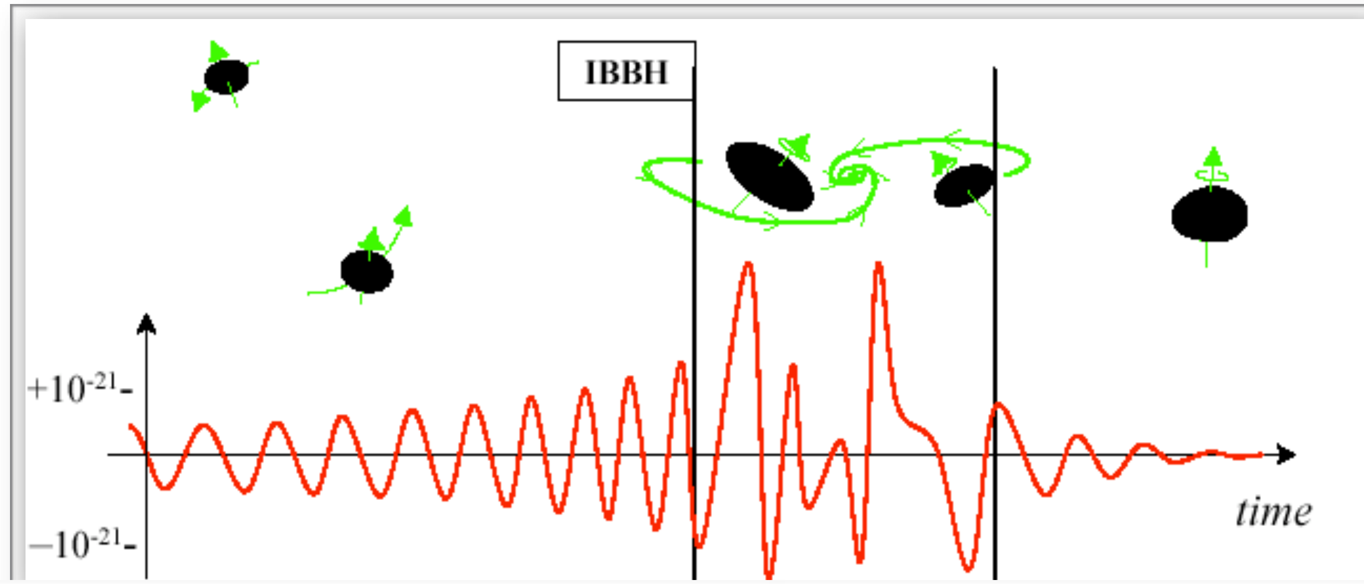
PN corrections to Einstein's quadrupole frequency « chirping »
from PN-improved balance equation dE(f)/dt = - F(f)

$$\frac{d\phi}{d\ln f} = \frac{\omega^2}{d\omega/dt} = Q_\omega^N \widehat{Q}_\omega \qquad \frac{v}{c} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{1}{3}}$$

$$Q_\omega^N = \frac{5\,c^5}{48\,\nu\,v^5}\,;\ \widehat{Q}_\omega = 1 + c_2\left(\frac{v}{c}\right)^2 + c_3\left(\frac{v}{c}\right)^3 + \cdots \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

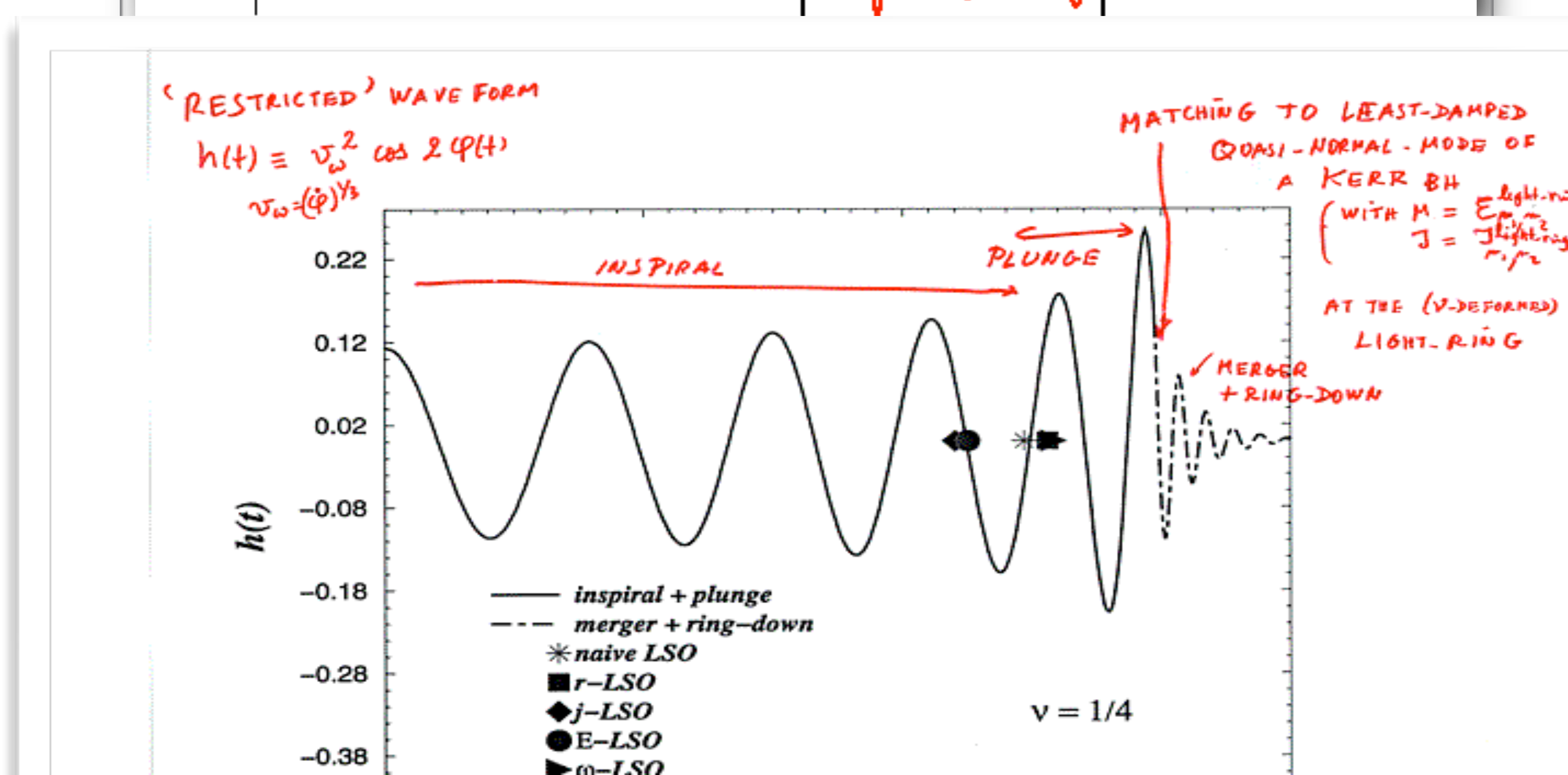
Cutler et al. '93:
« slow convergence of PN »

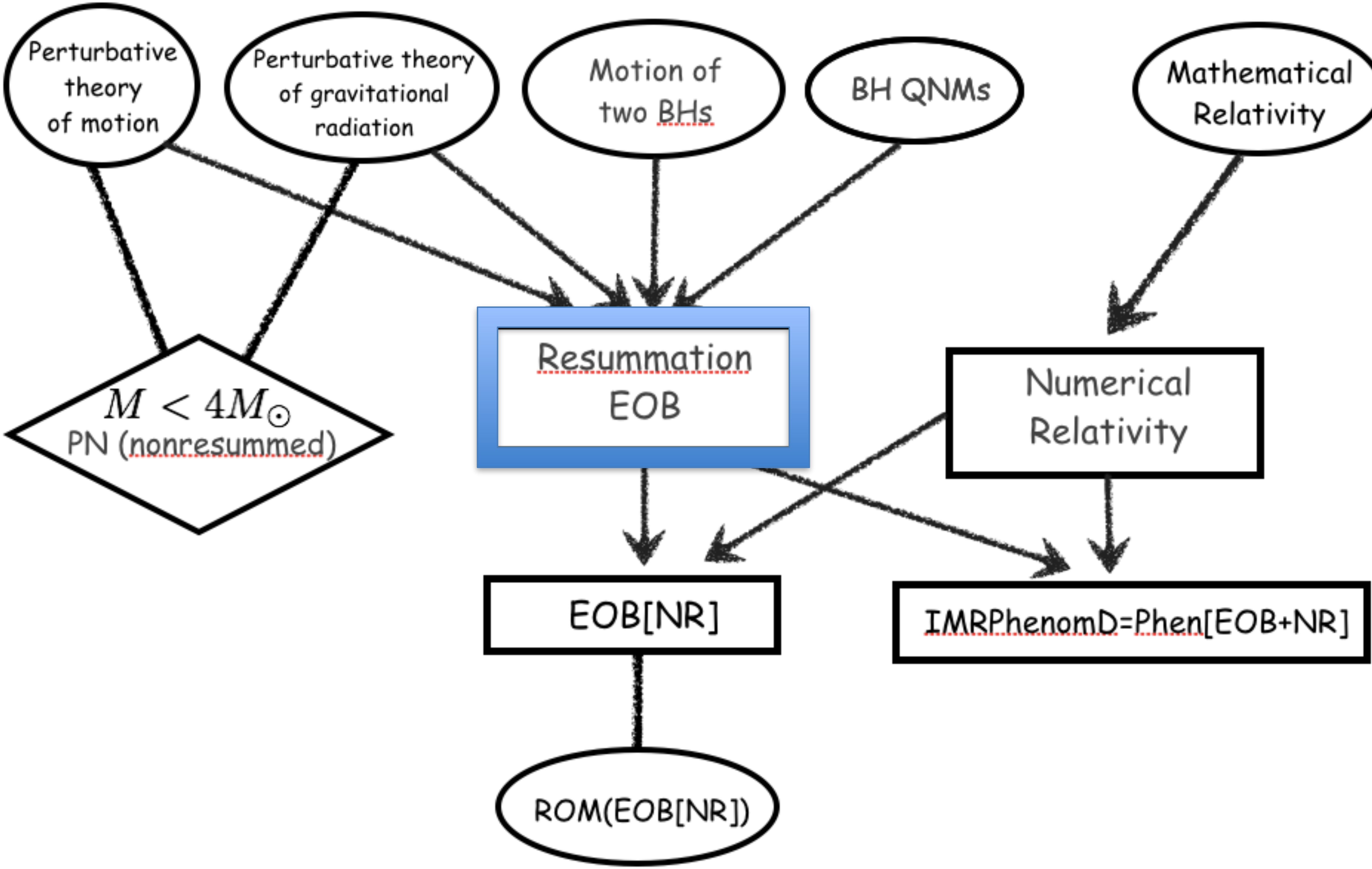
Brady-Creighton-Thorne'98:
« inability of current computational techniques to evolve a BBH through its last ~10 orbits of inspiral » and to compute the merger



Damour-Iyer-Sathyaprakash'98:
use resummation methods for E and F

Buonanno-Damour '99-00:
novel, resummed approach:
Effective-One-Body
analytical formalism



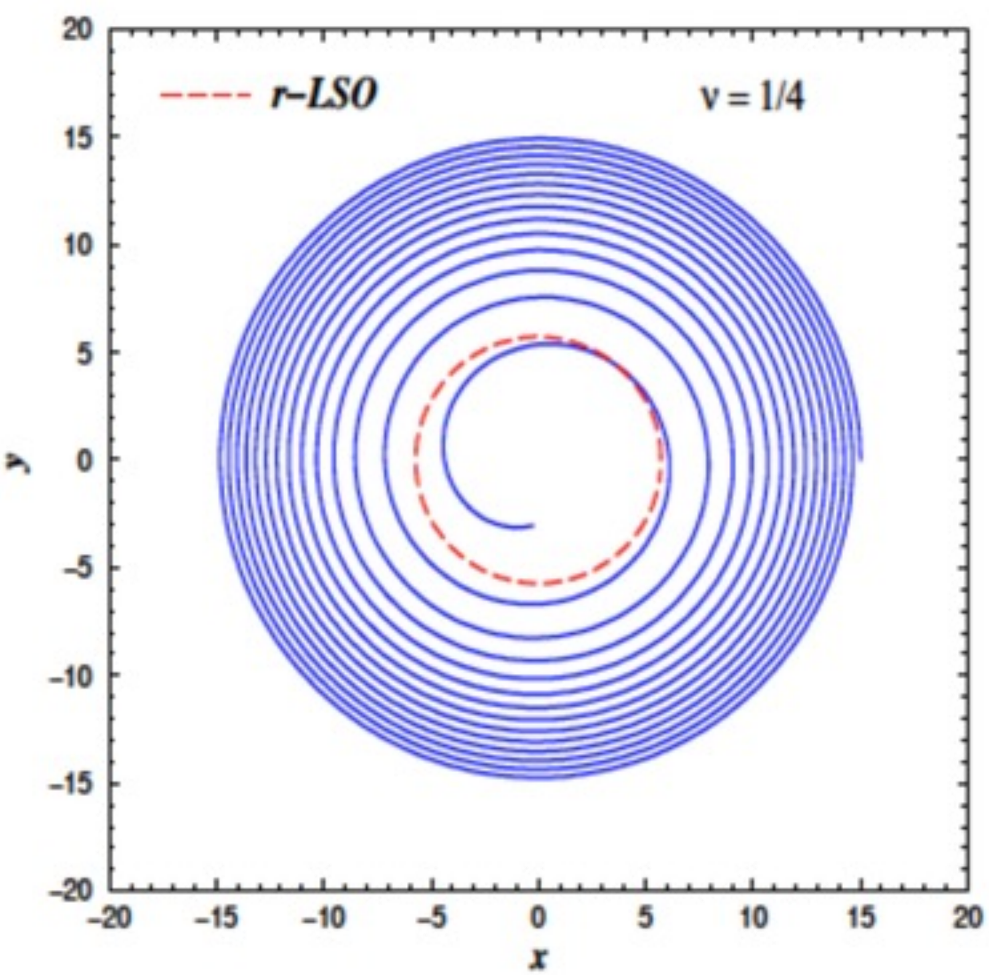

Perturbative theory of motion
Perturbative theory of gravitational radiation
Motion of two BHs
BH QNMs
Mathematical Relativity
$M < 4M_{\odot}$
PN (nonresummed)
Resummation EOB
Numerical Relativity
EOB[NR]
IMRPhenomD=Phen[EOB+NR]
ROM(EOB[NR])

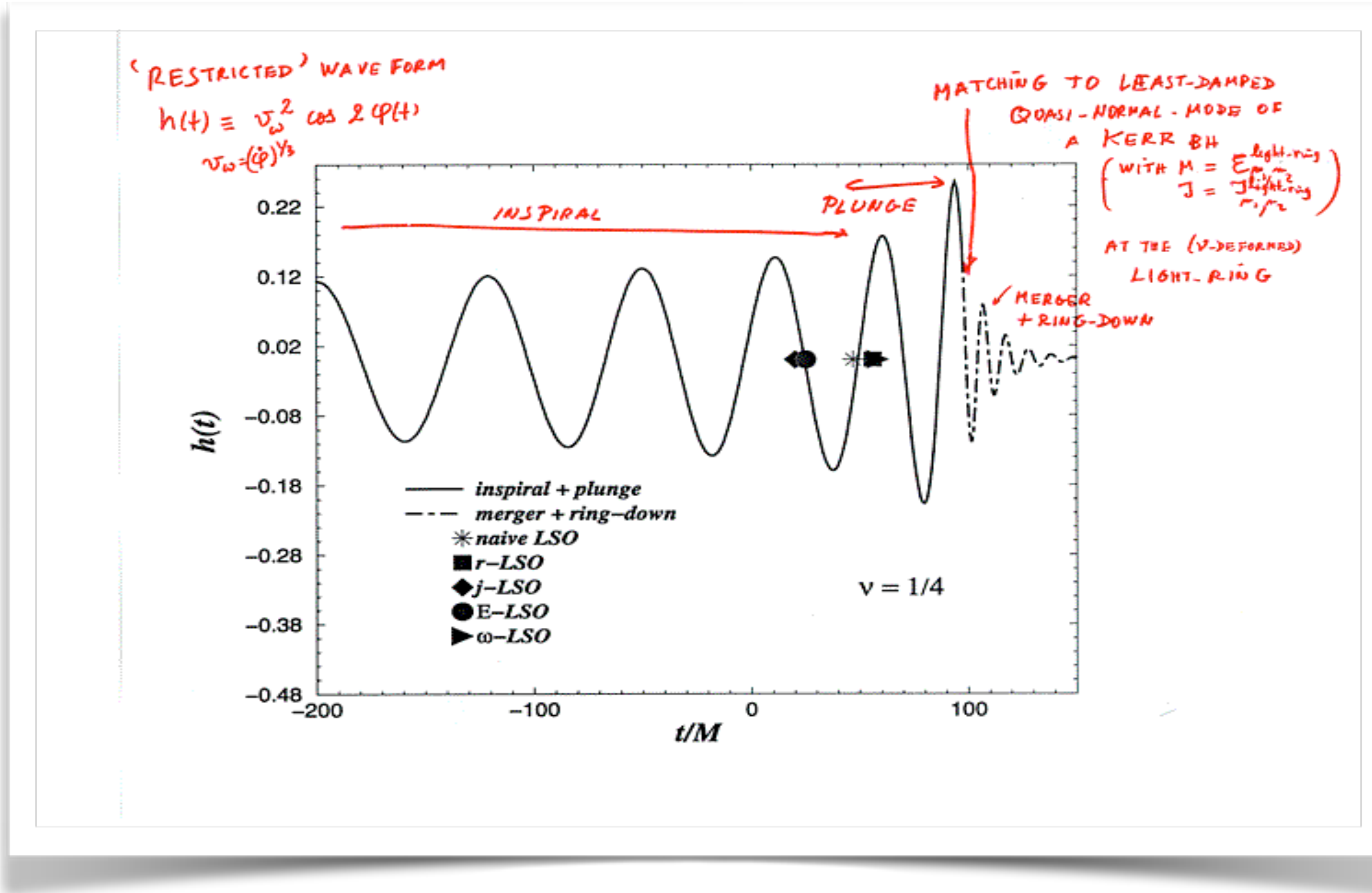
Effective One Body (EOB) Method

Buonanno-Damour 1999, 2000; Damour-Jaranowski-Schaefer 2000; Damour 2001 (SEOB)
[developed by: Barausse, Bini, Buonanno, Damour, Jaranowski, Nagar, Pan, Schaefer, Taracchini, …]

Resummation of perturbative PN results → **approximate description of the merger**
+ addition of **ringdown** (Vishveshwara 70, Davis-Ruffini-Tiomno 1972) [+ CLAP (Price-Pullin'94)]

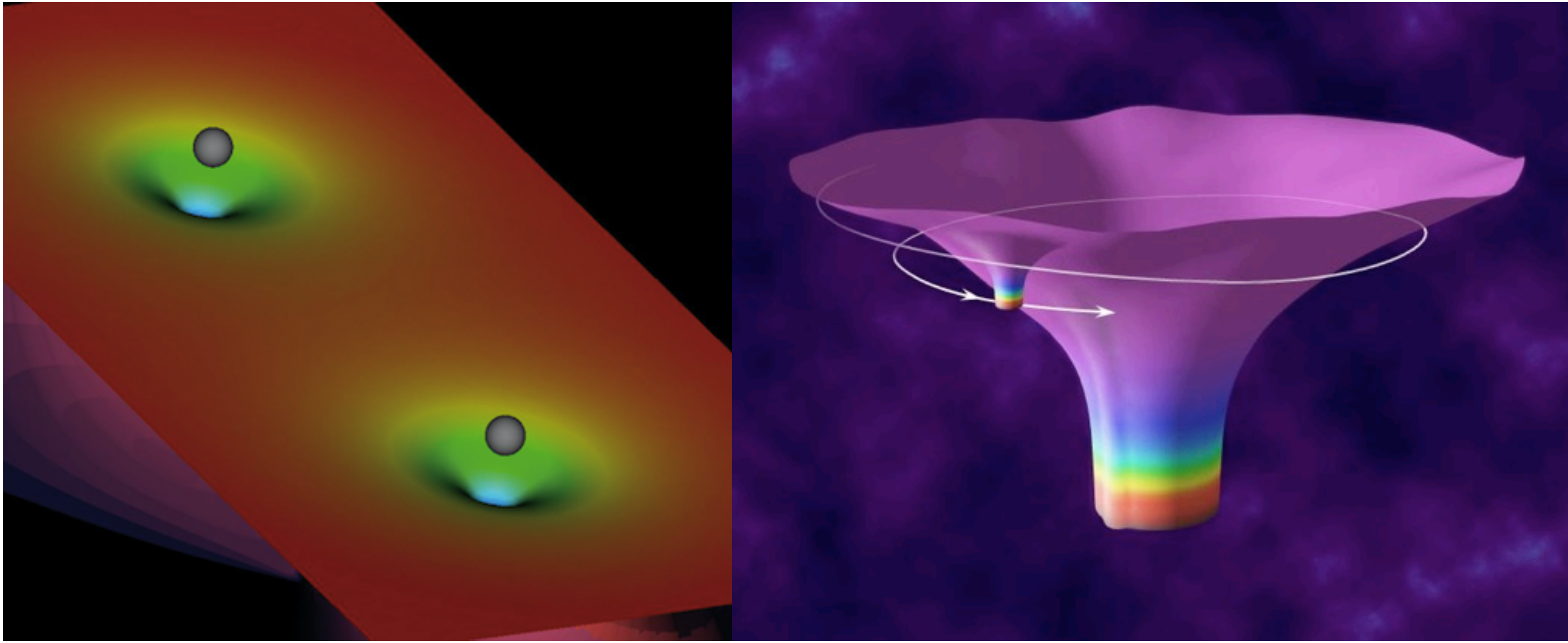
Buonanno-Damour 2000





Predictions as early as 2000 :
continued transition, non adiabaticity, first complete waveform, final spin (OK within 10%), final mass

EOB: resumming the dynamics of a two-body system (m_1,m_2,S_1,S_2) in terms of the dynamics of a particle of mass mu and spin S* moving in some effective metric g(M,S)



Effective metric for non-spinning bodies: a nu-deformation of Schwarzschild

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \nu = \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$ds^2_{\text{eff}} = -A(r;\nu)\, dt^2 + B(r;\nu)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

2-body Taylor-expanded N + 1PN + 2PN+ 3PN Hamiltonian

$$H_{\mathrm{N}}(\mathbf{x}_a,\mathbf{p}_a)=\frac{\mathbf{p}_1^2}{2m_1}-\frac{1}{2}\frac{Gm_1m_2}{r_{12}}+(1\leftrightarrow 2)$$

$$c^2H_{1\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2}+14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}+(1\leftrightarrow 2),$$

$$c^4H_{2\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a)=\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}-\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.$$
$$\left.-6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-\frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)$$
$$+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2}+19\frac{\mathbf{p}_2^2}{m_2^2}\right)-\frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2)+6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}+(1\leftrightarrow 2),$$

$$c^6H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7}+\frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6}+4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2}+6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\right.$$
$$-10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2}+24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}$$
$$+2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}+\frac{(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}$$
$$+\frac{(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}$$
$$\left.-18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}+5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right)+\frac{G^2m_1m_2}{r_{12}^2}\left(\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right.$$
$$-\frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3}+\frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}$$
$$-\frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2}-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}$$
$$+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}$$
$$\left.-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)+\frac{G^3m_1m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.$$
$$+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(20m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}$$
$$\left.+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right)+(1\leftrightarrow 2).$$

Explicit 3PN EOB dynamics (Damour-Jaranowski-Schaefer '01)

post-geodesic effective mass-shell:

$$g_{\rm eff}^{\mu\nu}\, P'_\mu\, P'_\nu + \mu^2\, c^2 + Q(P'_\mu) = 0\,,$$

$$ds^2_{\rm eff} = -A(R;\nu)dt^2 + B(R;\nu)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$u \equiv \frac{GM}{R\,c^2}$$

$$M = m_1 + m_2\,, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}\,, \qquad \nu = \frac{m_1 m_2}{(m_1 + m_2)^2} = \frac{\mu}{M}$$

$$-P'_0 = \mathcal{E}_{\rm eff}$$

$$P'_\varphi = J_{\rm eff} = J_{\rm real}$$

$$A^{\rm 3PN}(u) = 1 - 2u + 2\nu\, u^3 + \left(\frac{94}{3} - \frac{41}{32}\,\pi^2\right)\nu u^4,$$

$$\overline{D}^{\rm 3PN}(u) = 1 + 6\nu\, u^2 + (52\nu - 6\nu^2)\, u^3,$$

$$\widehat{Q}^{\rm 3PN} \equiv \frac{Q}{\mu^2 c^2} = (8\nu - 6\nu^2)\, u^2\, \frac{p_r^4}{c^4}.$$

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}.$$

Spinning EOB effective Hamiltonian

$$H_{\rm eff} = H_{\rm orb} + H_{\rm so} \quad \rightarrow H_{\rm EOB} = Mc^2\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu c^2} - 1\right)}$$

$$\hat{H}^{\rm eff}_{\rm orb} = \sqrt{A\left(1 + B_p \boldsymbol{p}^2 + B_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 - \frac{1}{1+\frac{(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}{r^2}}\frac{(r^2+2r+(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2)}{\mathcal{R}^4+\Delta(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}((\boldsymbol{n}\times\boldsymbol{p})\cdot\boldsymbol{\chi}_0)^2 + Q_4\right)}.$$

$$H_{\rm so} = G_S \boldsymbol{L}\cdot\boldsymbol{S} + G_{S^*}\boldsymbol{L}\cdot\boldsymbol{S}^*,$$

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}\,;\; \mathbf{S}_* = \frac{m_2}{m_1}\mathbf{S_1} + \frac{m_1}{m_2}\mathbf{S_2}\,,$$

Gyrogravitomagnetic ratios (when neglecting spin^2 effects)

$$r^3 G_S^{\rm PN} = 2 - \frac{5}{8}\nu u - \frac{27}{8}\nu p_r^2 + \nu\left(-\frac{51}{4}u^2 - \frac{21}{2}up_r^2 + \frac{5}{8}p_r^4\right) + \nu^2\left(-\frac{1}{8}u^2 + \frac{23}{8}up_r^2 + \frac{35}{8}p_r^4\right)$$

$$r^3 G_{S_*}^{\rm PN} = \frac{3}{2} - \frac{9}{8}u - \frac{15}{8}p_r^2 + \nu\left(-\frac{3}{4}u - \frac{9}{4}p_r^2\right) - \frac{27}{16}u^2 + \frac{69}{16}up_r^2 + \frac{35}{16}p_r^4 + \nu\left(-\frac{39}{4}u^2 - \frac{9}{4}up_r^2 + \frac{5}{2}p_r^4\right) + \nu^2\left(-\frac{3}{16}u^2 + \frac{57}{16}up_r^2 + \frac{45}{16}p_r^4\right)$$

Resummed EOB waveform

(Damour-Iyer-Sathyaprakash '98) Damour-Nagar '07, Damour-Iyer -Nagar '08, Pan et al. '10

$$h_{\ell m} \equiv h_{\ell m}^{(N,\epsilon)} \hat{h}_{\ell m}^{(\epsilon)} \hat{h}_{\ell m}^{\rm NQC}$$

$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\rm eff}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{\hat{k}})}{\Gamma(\ell+1)} e^{\pi \hat{\hat{k}}} e^{2i\hat{\hat{k}} \ln(2kr_0)}$$

NB: T_lm resums an infinite number of terms and already contains, eg, 4.5PN tail^3 terms (Messina-Nagar17)

$$\rho_{22}(x;\nu) = 1 + \left(\frac{55\nu}{84} - \frac{43}{42}\right) x + \left(\frac{19583\nu^2}{42336} - \frac{33025\nu}{21168} - \frac{20555}{10584}\right) x^2$$
$$+ \left(\frac{10620745\nu^3}{39118464} - \frac{6292061\nu^2}{3259872} + \frac{41\pi^2\nu}{192} - \frac{48993925\nu}{9779616} - \frac{428}{105}\mathrm{eulerlog}_2(x) + \frac{1556919113}{122245200}\right) x^3$$
$$+ \left(\frac{9202}{2205}\mathrm{eulerlog}_2(x) - \frac{387216563023}{160190110080}\right) x^4 + \left(\frac{439877}{55566}\mathrm{eulerlog}_2(x) - \frac{16094530514677}{533967033600}\right) x^5 + \mathcal{O}(x^6),$$

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=1}^{\ell} (m\Omega)^2 |R h_{\ell m}^{(\epsilon)}|^2$$

EOB

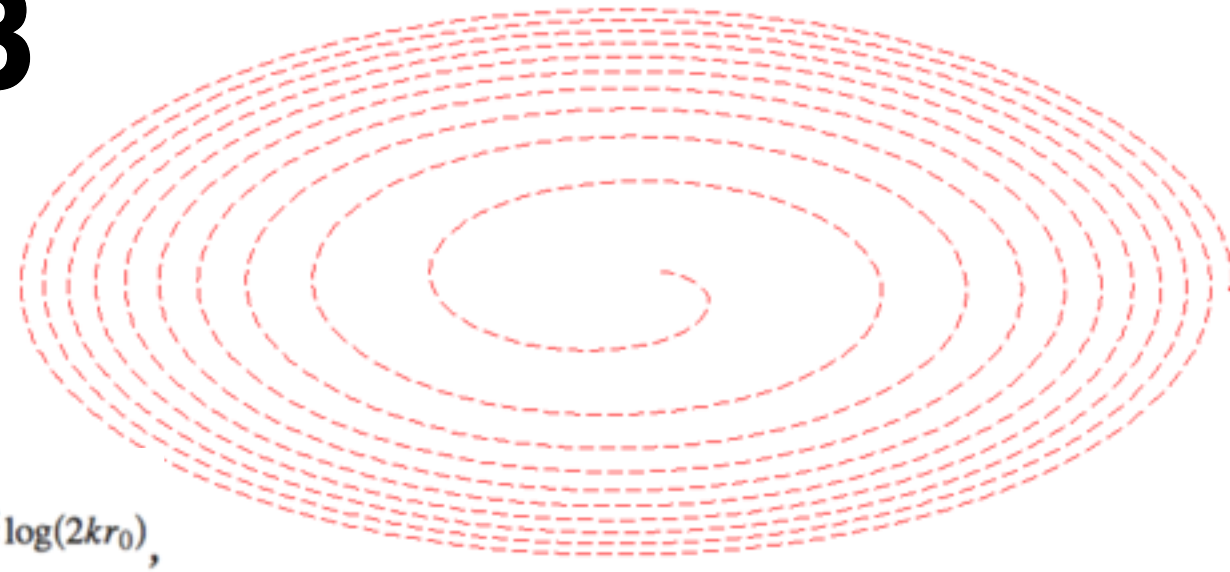
$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}},$$

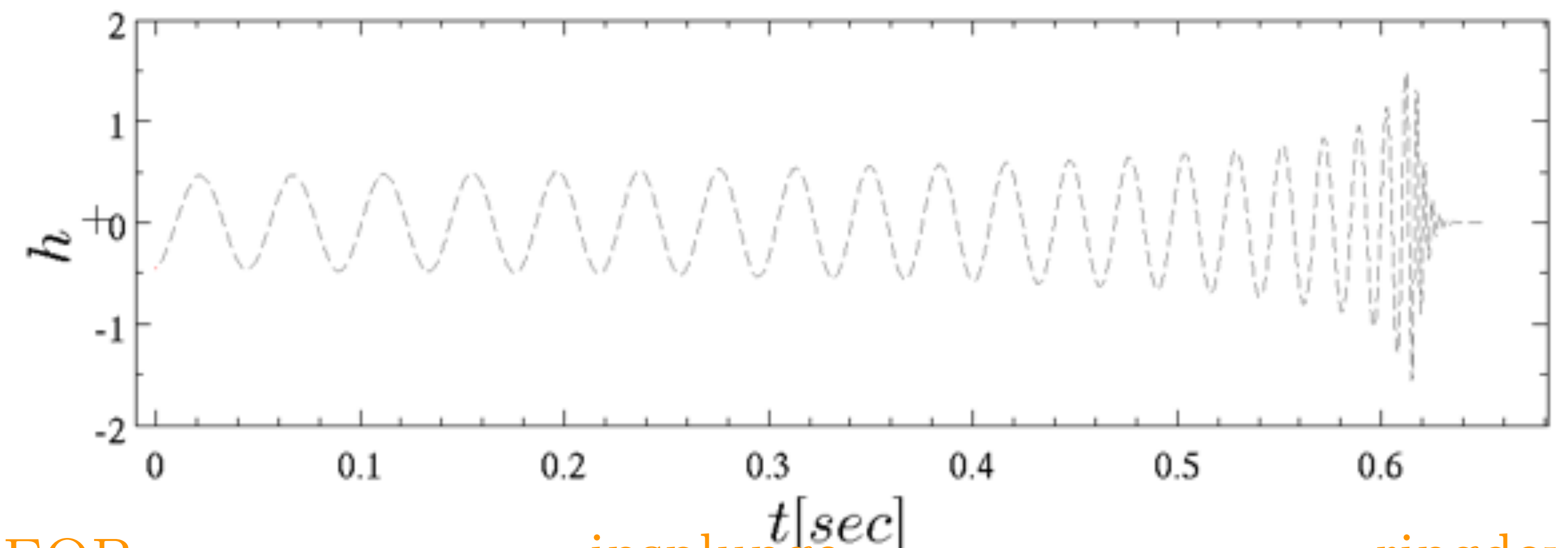
$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r},$$

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi},$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi .$$

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{\hat{k}})}{\Gamma(\ell+1)} e^{\pi \hat{\hat{k}}} e^{2i\hat{\hat{k}} \log(2kr_0)},$$





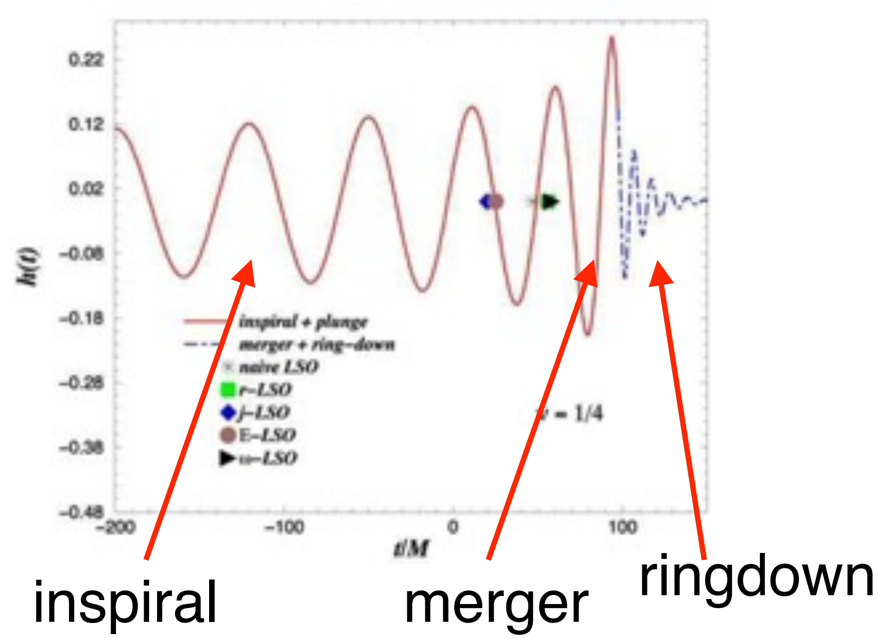
$$h_{\ell m} \equiv h_{\ell m}^{(N,\epsilon)} \hat{h}_{\ell m}^{(\epsilon)} \hat{h}_{\ell m}^{\rm NQC}$$

$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\rm eff}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\rm max}} \sum_{m=1}^{\ell} (m\Omega)^2 |R h_{\ell m}^{(\epsilon)}|^2$$

$$h_{\ell m}^{\rm EOB} = \theta(t_m - t) h_{\ell m}^{\rm insplunge}(t) + \theta(t - t_m) h_{\ell m}^{\rm ringdown}$$

$$h_{\ell m}^{\rm ringdown}(t) = \sum_N C_N^+ e^{-\sigma_N^+ (t - t_m)}$$

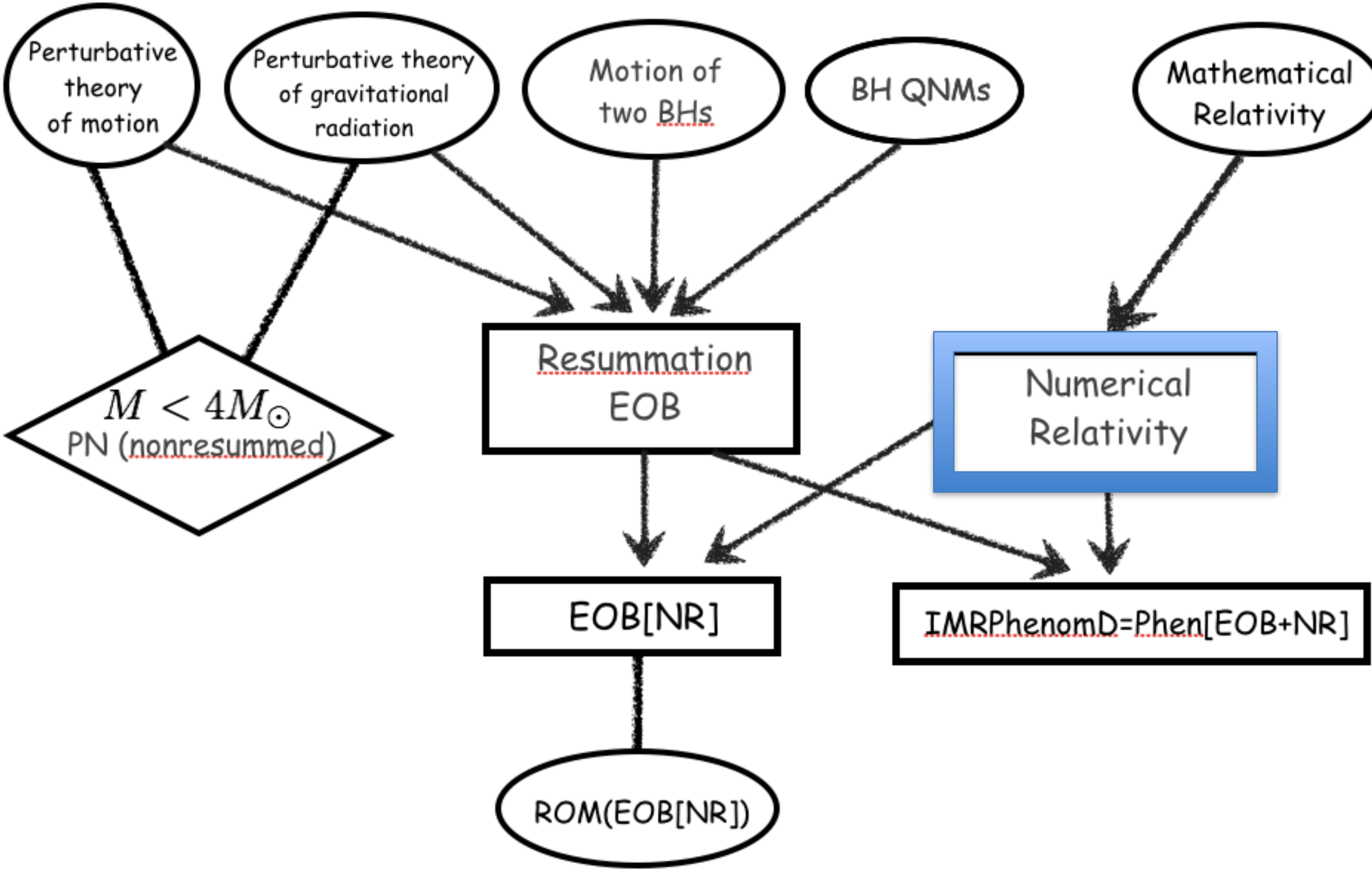


First complete waveforms for BBH coalescences:

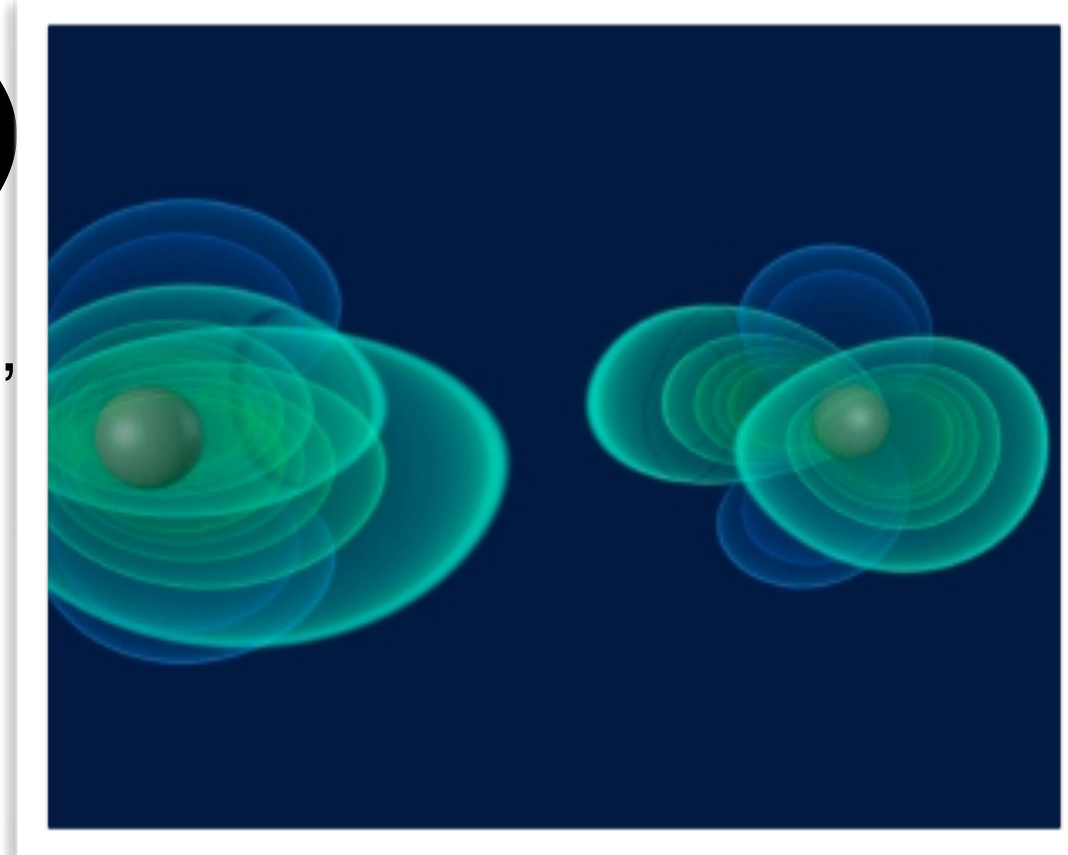
analytical EOB (Buonanno-Damour'00,Buonanno-Chen-Damour'05)

After the 2005 NR breakthrough (Pretorius,…)

development of the NR-completed EOB waveforms

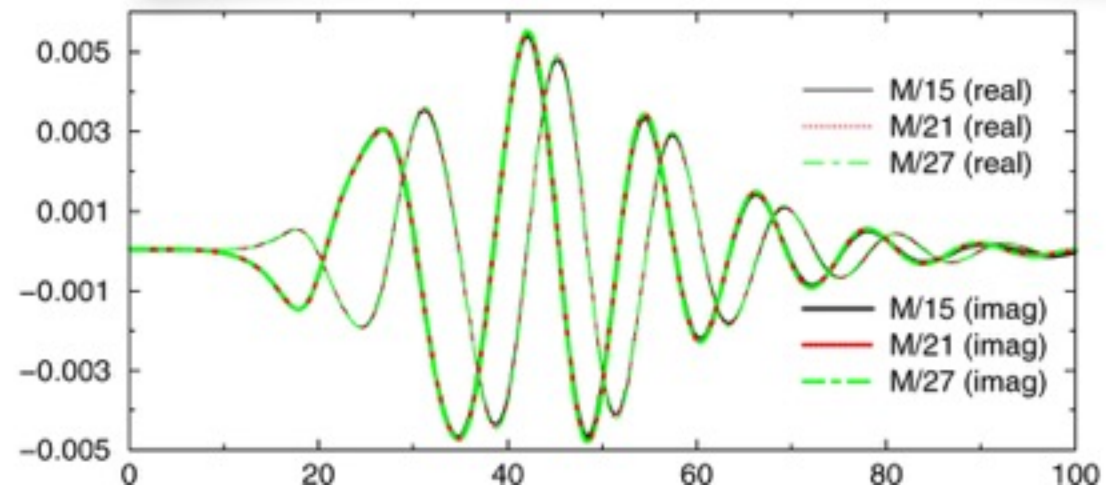

Perturbative theory of motion
Perturbative theory of gravitational radiation
Motion of two BHs
BH QNMs
Mathematical Relativity
$M < 4M_\odot$
PN (nonresummed)
Resummation EOB
Numerical Relativity
EOB[NR]
IMRPhenomD=Phen[EOB+NR]
ROM(EOB[NR])

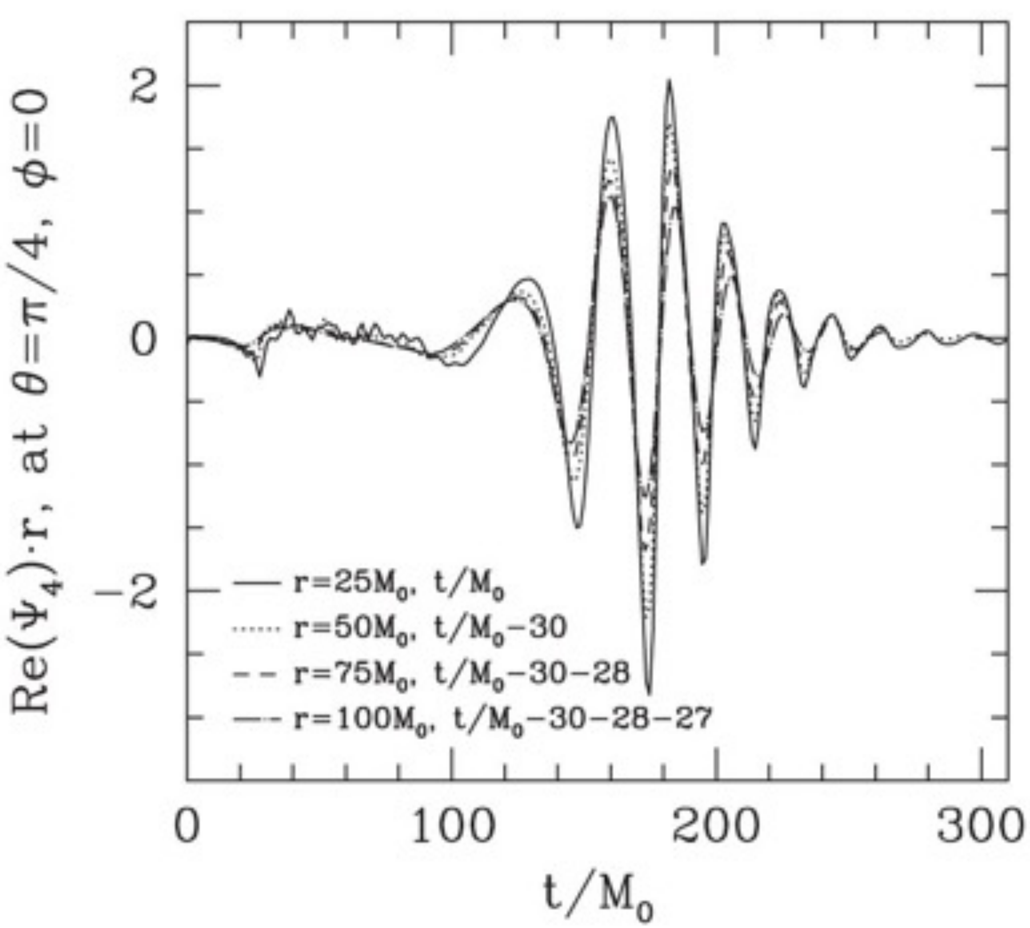
Numerical Relativity (NR)

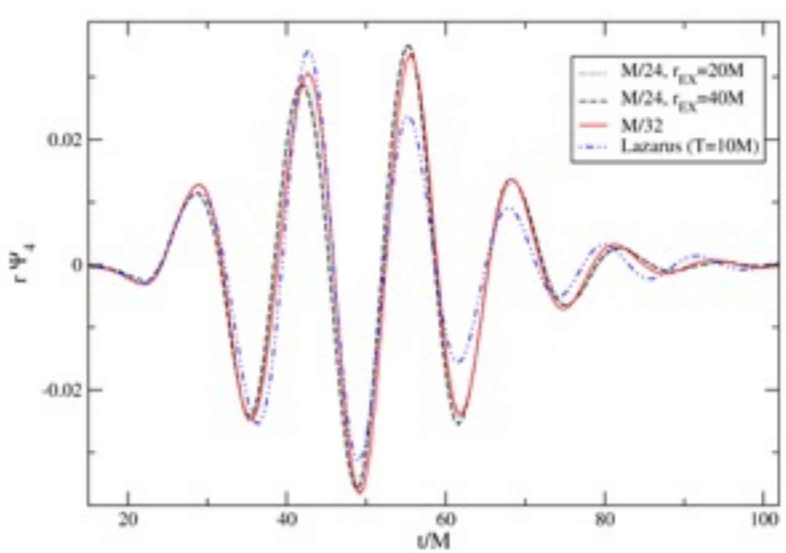


Mathematical foundations :
Darmois 27, Lichnerowicz 43, Choquet-Bruhat 52-,
new hyperbolic formulations of Einstein's eqs,…

Breakthrough:
Pretorius 2005 generalized harmonic coordinates,
constraint damping, excision
Moving punctures:
Campanelli-Lousto-Maronetti-Zlochover 2006
Baker-Centrella-Choi-Koppitz-van Meter 2006







The first EOB vs NR comparisons

Buonanno-Cook-Pretorius 2007

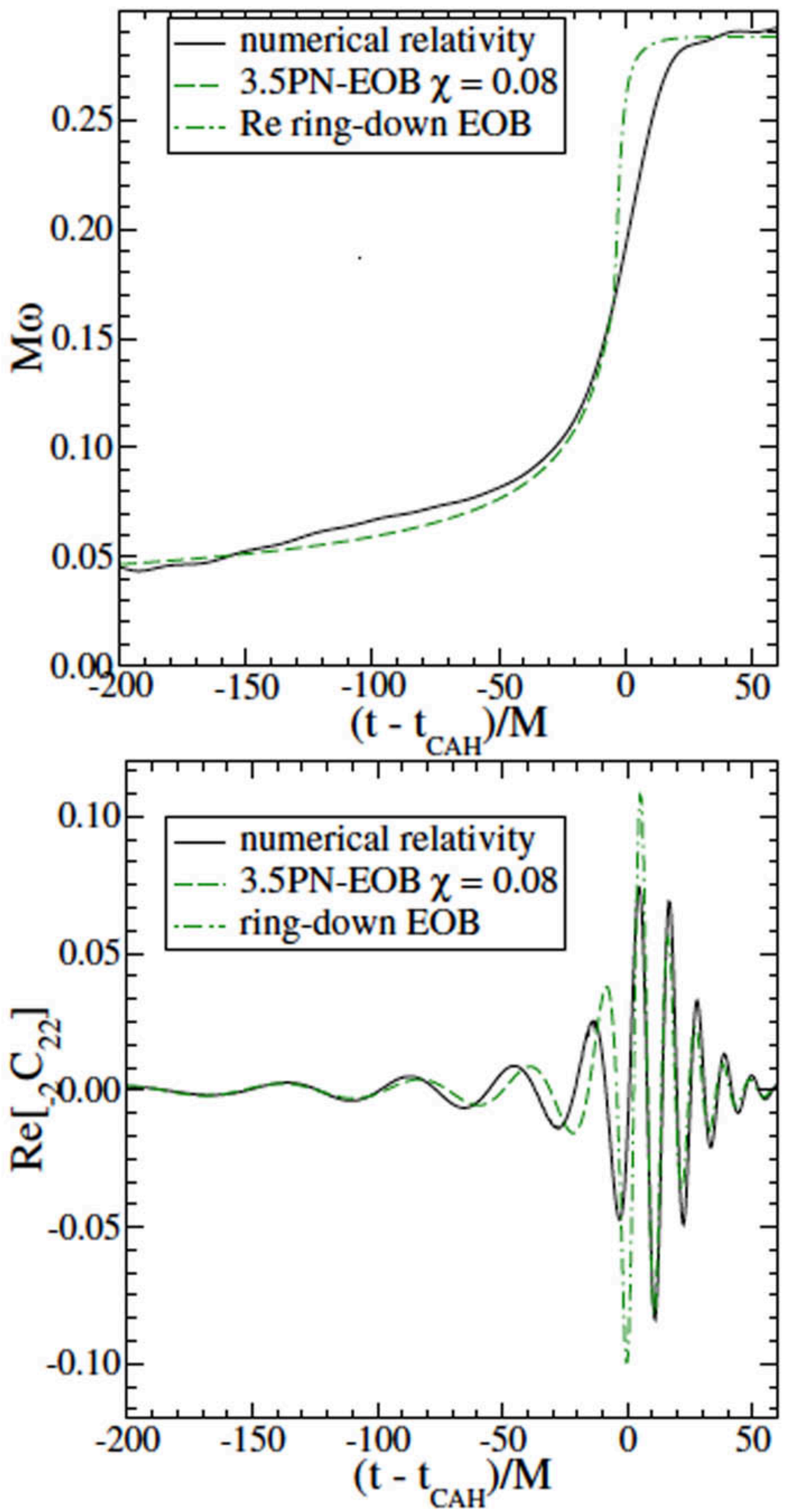
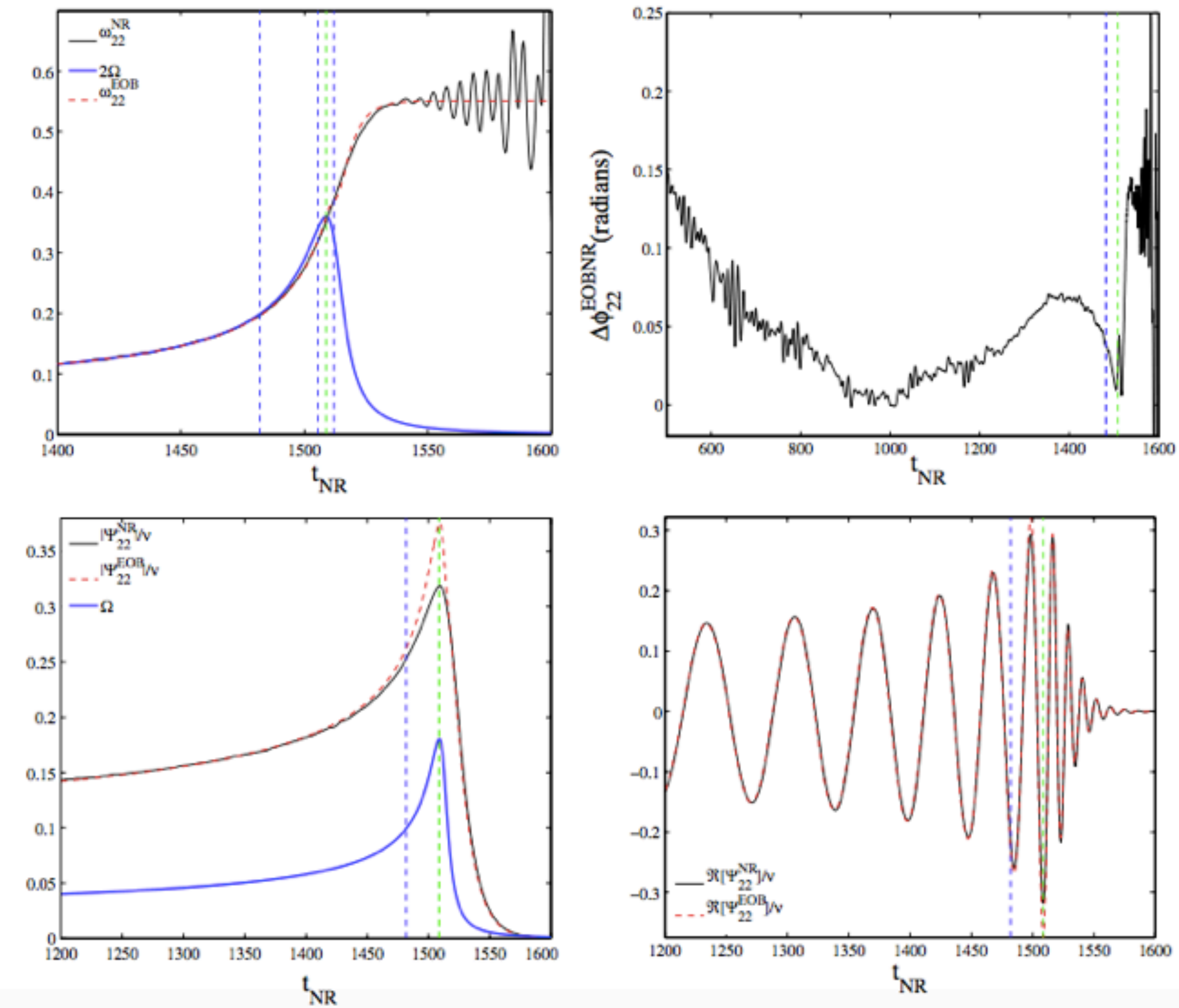


FIG. 21 (color online). We compare the NR and EOB frequency and Re$[{}_{-2}C_{22}]$ waveforms throughout the entire inspiral–merger–ring-down evolution. The data refers to the $d = 16$ run.

DAMOUR, NAGAR, DORBAND, POLLNEY, AND REZZOLLA PHYSICAL REVIEW D **77**, 084017 (2008)



SXS COLLABORATION NR CATALOG

(Mroué et al.'13, Boyle et al. '19)

• www.blackholes.org

The last version contains 2018 simulated waveforms including 1426 spin-precessing configurations

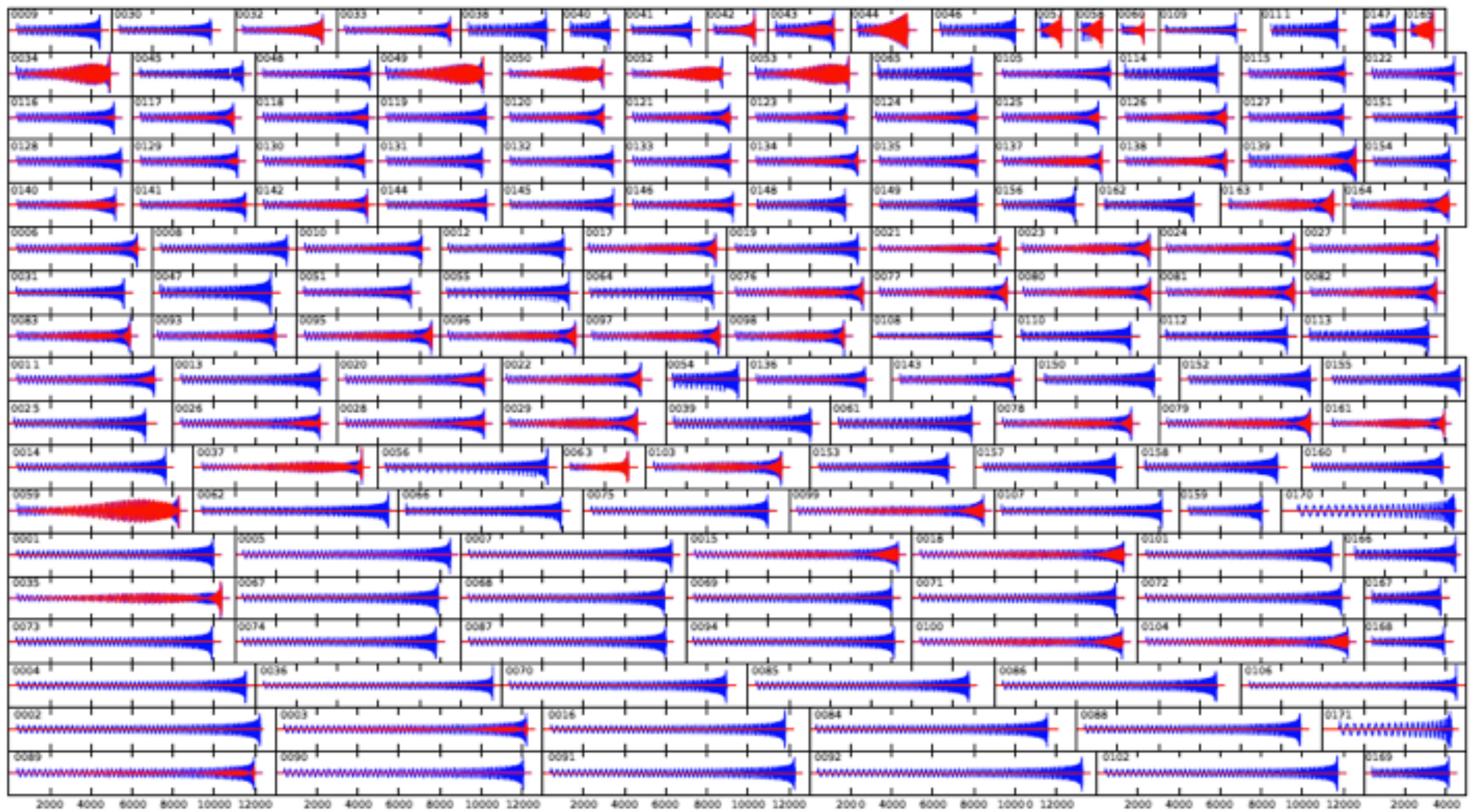
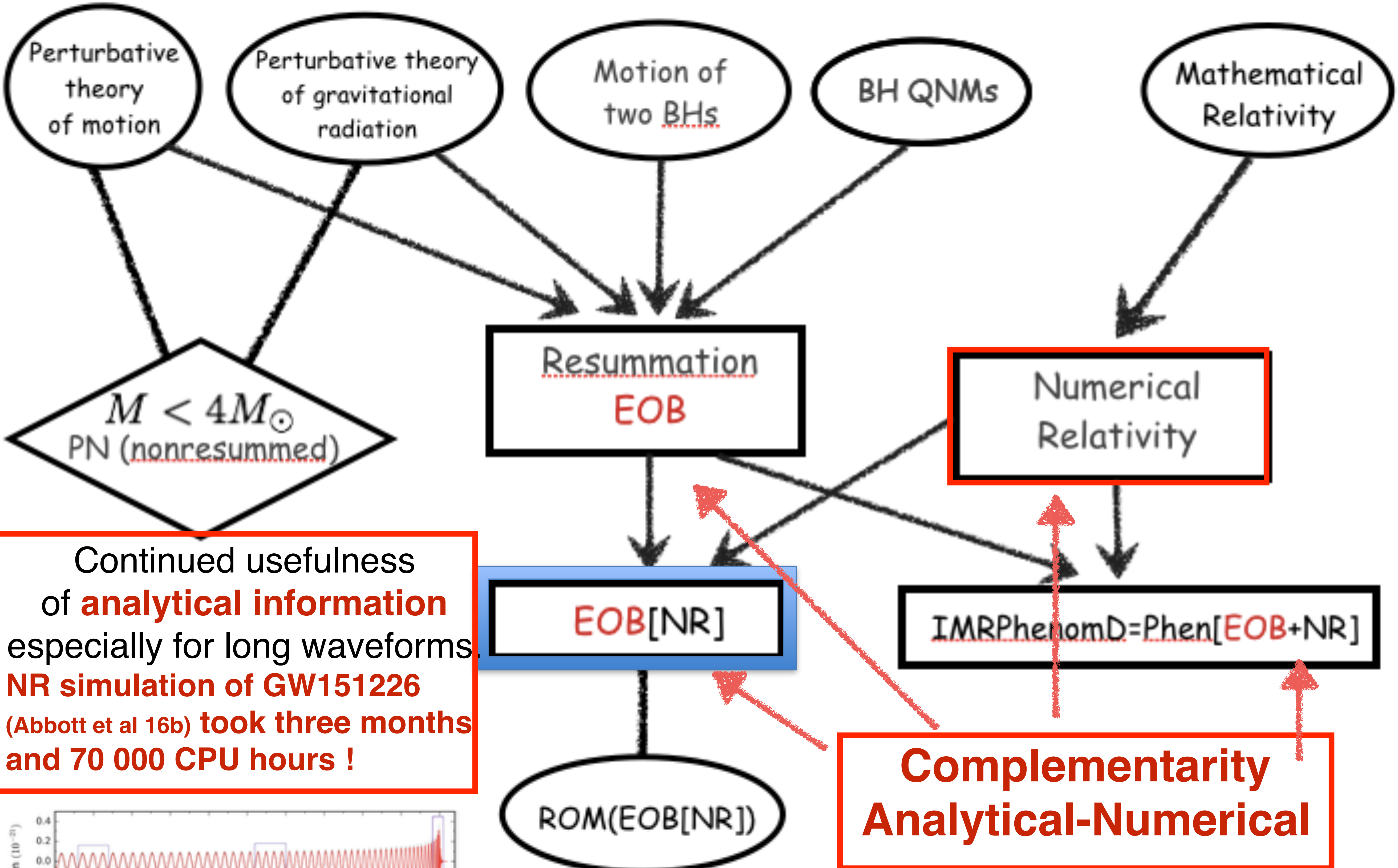
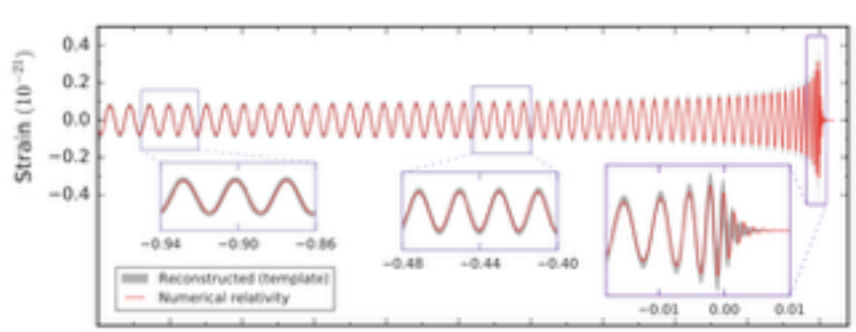


FIG. 3: Waveforms from all simulations in the catalog. Shown here are h_+ (blue) and $h_\times$ (red) in a sky direction parallel to the initial orbital plane of each simulation. All plots have the same horizontal scale, with each tick representing a time interval of $2000M$, where M is the total mass.

NR waveforms play a crucial role for an accurate description of the merger, and of large-spin effects. But they have limitations: number of orbits, sampling of the multi-parameter space.



Continued usefulness of **analytical information** especially for long waveforms.
NR simulation of GW151226 (Abbott et al 16b) took three months and 70 000 CPU hours !



NR-completed resummed 5PN EOB radial A potential

(here Damour-Nagar-Bernuzzi '13, Nagar-et al '16; alternative: Taracchini et al '14, Bohe et al '17)

4PN analytically complete + 5 PN logarithmic term in the A(u, nu) function,
With u = GM/R and nu = m1 m2 / (m1 + m2)^2
[Damour 09, Blanchet et al 10, Barack-Damour-Sago 10, Le Tiec et al 11, Barausse et al 11, Akcay et al 12, Bini-Damour 13, Damour-Jaranowski-Schäfer 14, Nagar-Damour-Reisswig-Pollney 15]

$$A(u;\nu,a_6^c) = P_5^1\Big[1 - 2u + 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41}{32}\pi^2\right)u^4$$

$$+ \nu\left[-\frac{4237}{60} + \frac{2275}{512}\pi^2 + \left(-\frac{221}{6} + \frac{41}{32}\pi^2\right)\nu + \frac{64}{5}\ln(16e^{2\gamma}u)\right]u^5$$

$$+ \nu\left[a_6^c(\nu) - \left(\frac{7004}{105} + \frac{144}{5}\nu\right)\ln u\right]u^6\Big]$$

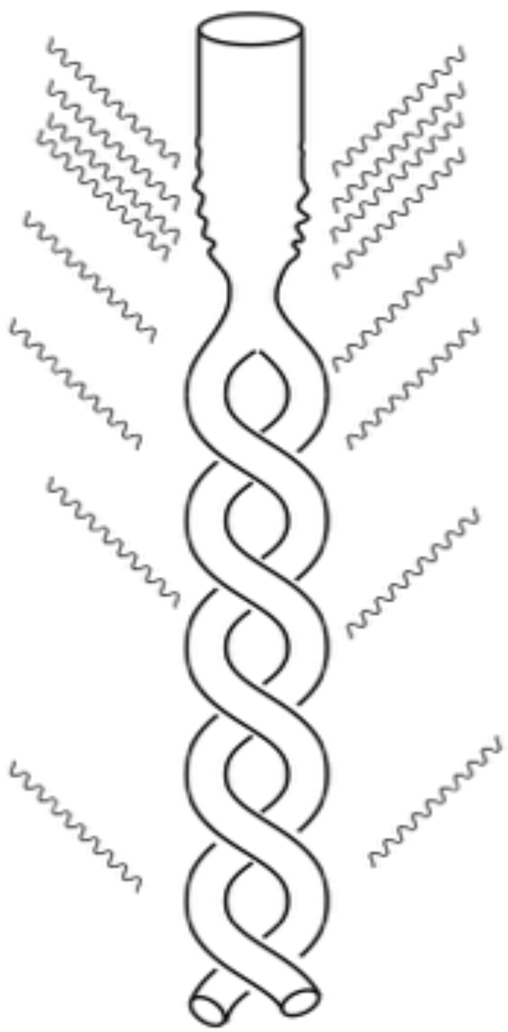
$$u = \frac{GM}{c^2 R}$$

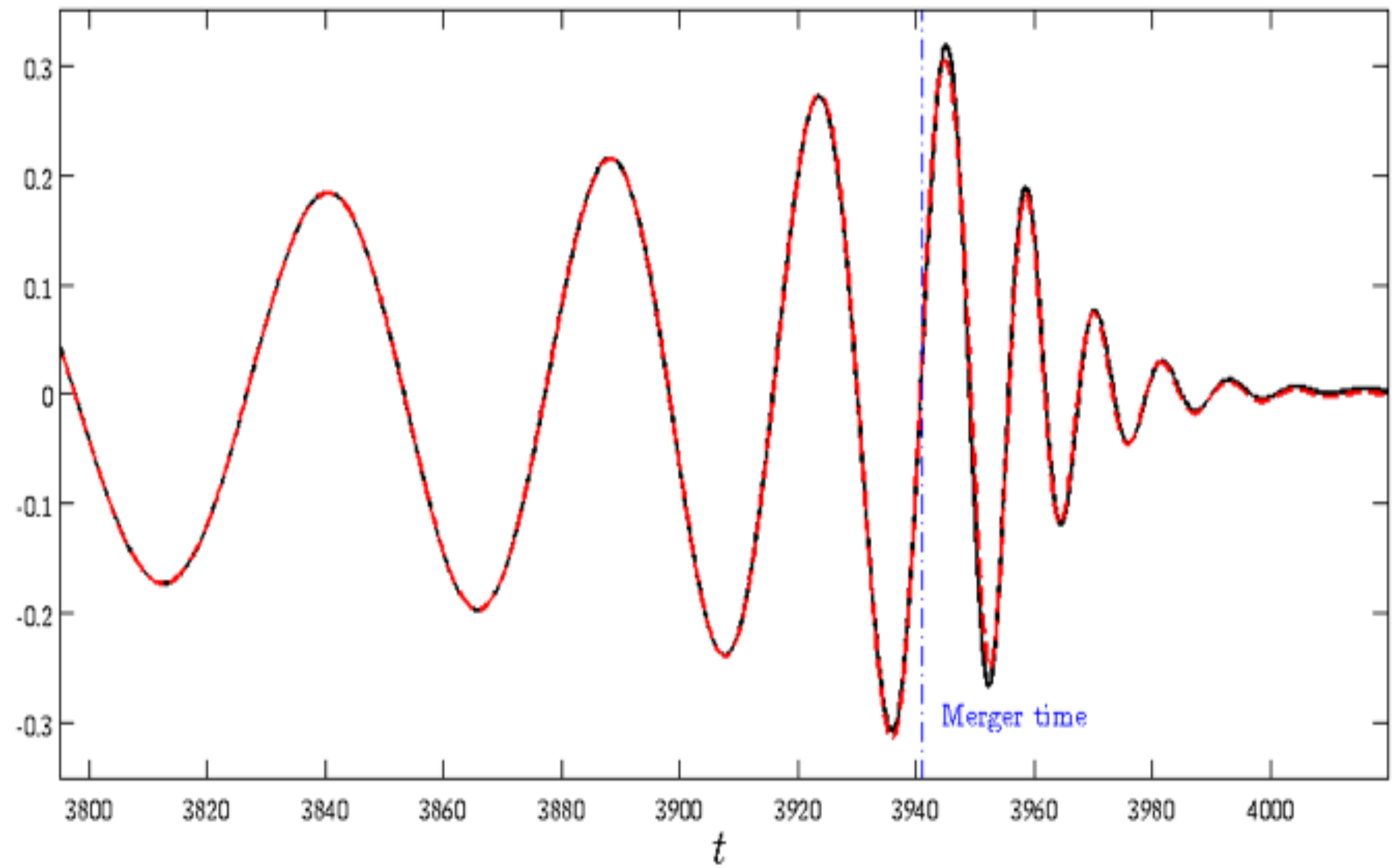
$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$a_6^{c\,\text{NR-tuned}}(\nu) = 81.38 - 1330.6\,\nu + 3097.3\,\nu^2$$

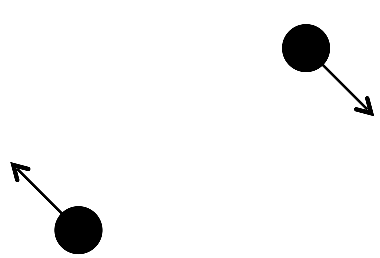
NR-calibrated 5PN contribution

EOB[NR] / NR Comparison



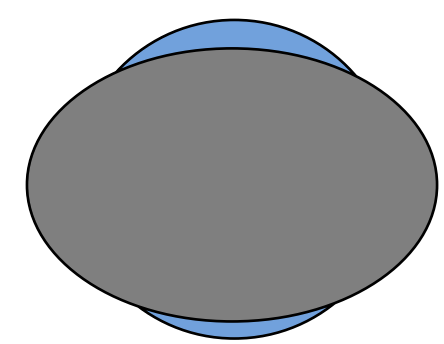


Inspiral + « plunge »



Two orbiting point-masses:
Resummed dynamics

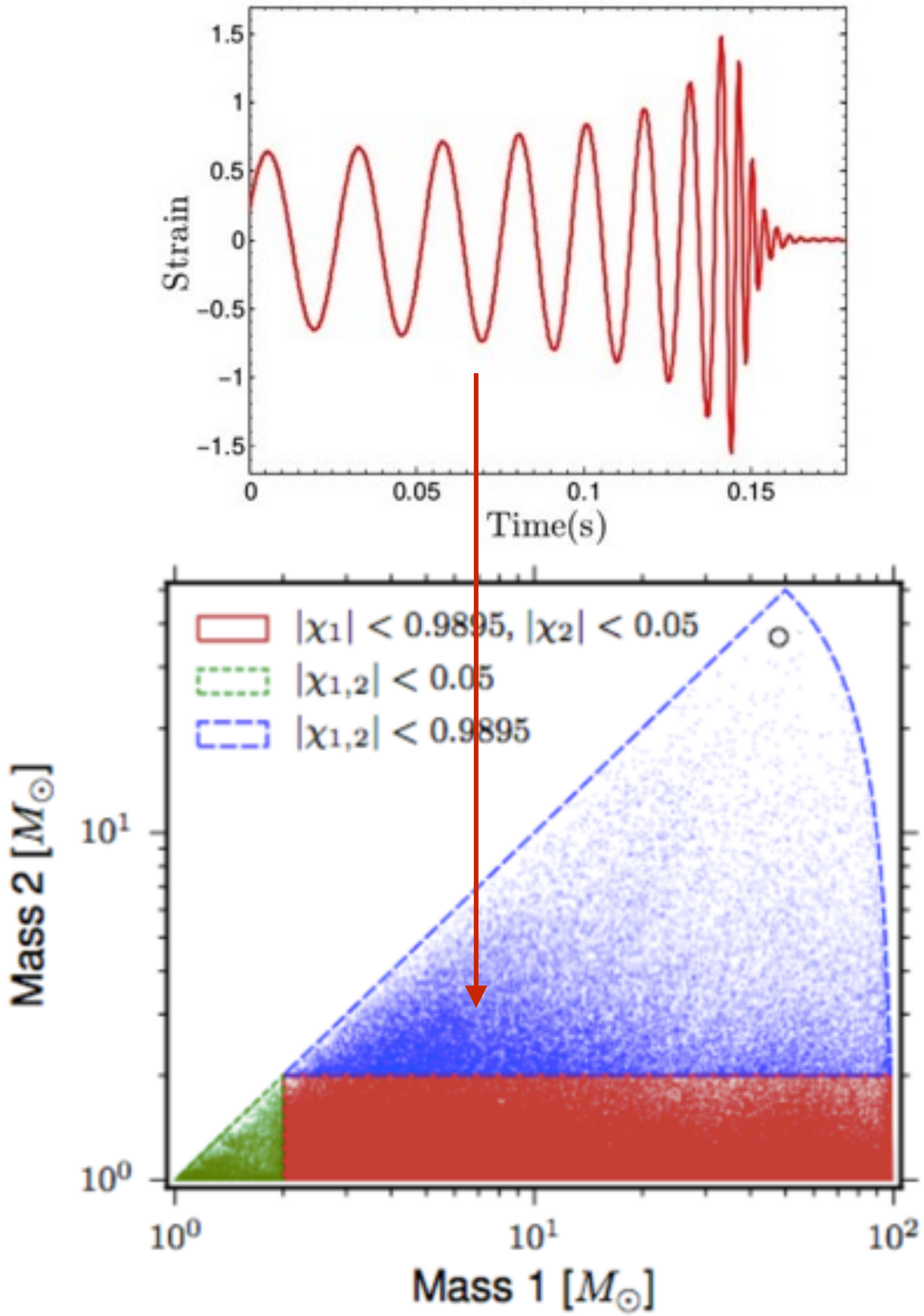
Ringing BH



Instantaneous GW power at coalescence ~ 10^56 erg/s ~ 10^-3 c^5/G

TEMPLATE BANKS USED FOR SEARCH AND DATA ANALYSIS

Banks of templates (e.g. 250 000 EOBNR templates in O1) for search inspiralling and coalescing BBH GW waveforms: m1, m2, chi1=S1/m1^2, chi2=S2/m2^2 for m1+m2> 4Msun; + ~ 50 000 PN inspiralling templates for m1+m2< 4 Msun;
O2: ~ 325 000 EOB templates + 75 000 PN templates



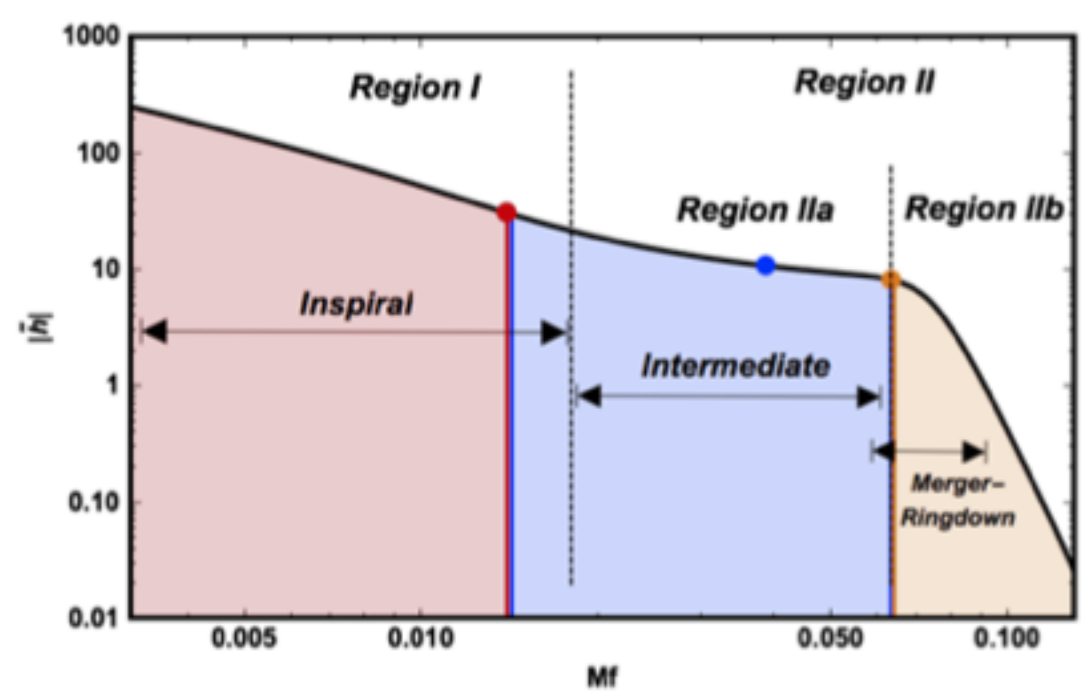
Two types of templates:

Bank of spinning EOB[NR] templates
(Taracchini et al. 14, Bohé et al'17) in ROM form (Puerrer et al.'14); Nagar et al…

Bank of Phenom[EOB+NR] templates
(Ajith…'07, Hannam…'14, Husa…'16, Khan…'16)

$$h(f) = A(f)e^{i\Psi(f)}$$

$$\Psi(f) = \sum_n c_n v^n(f)\ ;\ v(f) \equiv (\pi M f)^{\frac{1}{3}}$$

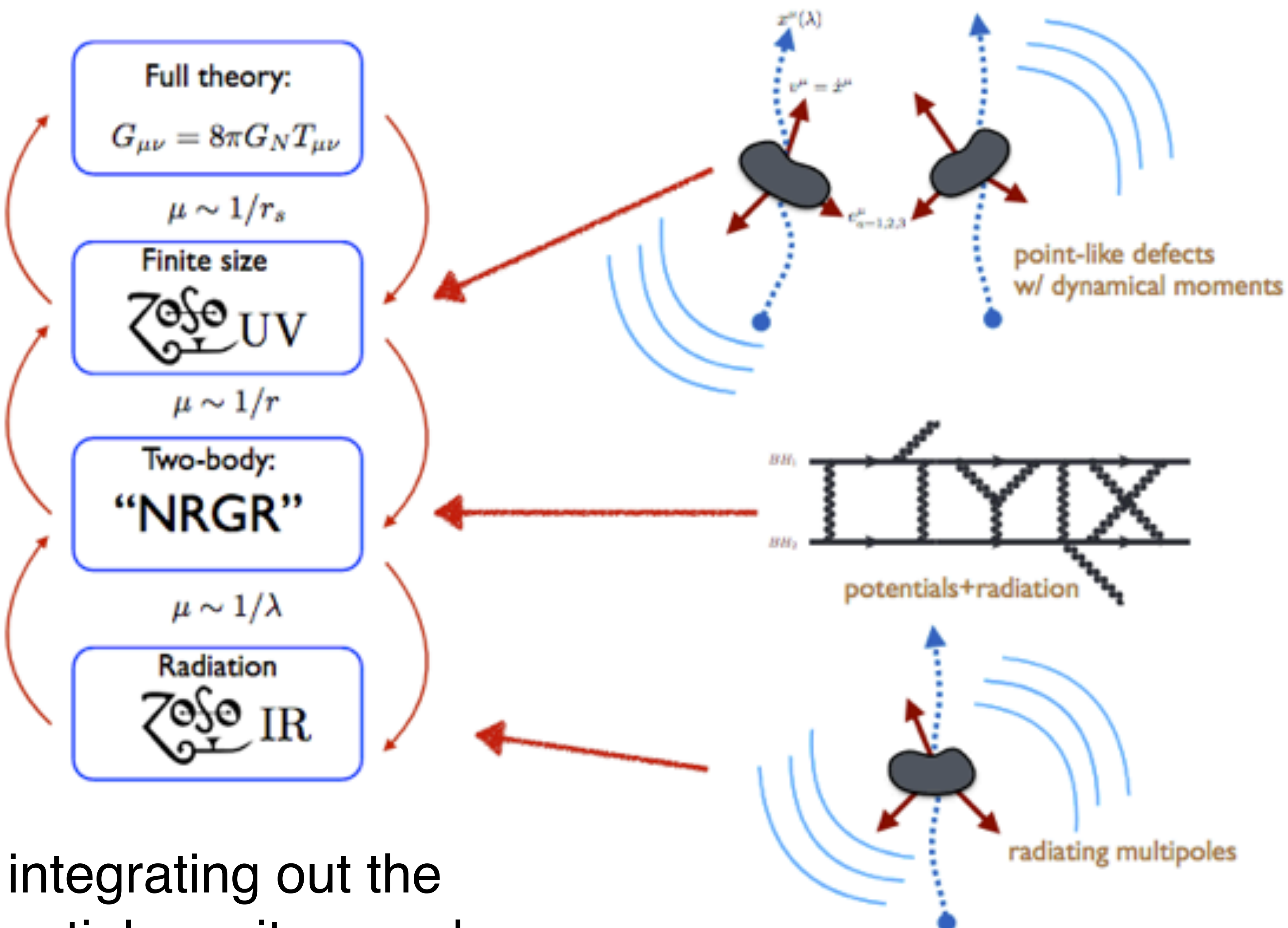


A glimpse of new analytical methods under active development:

effective field theory,
tutti frutti,
classical and quantum scattering.

EFT approach (Goldberger-Rothstein'06, Goldberger-Ros,Porto, Levi,Foffa-Sturani,Bluemlein..)

separating scales: r_s << r << lambda



Equivalent to usual methods (with some technical differences)

Similar but different from MPM method

integrating out the potential graviton modes

$$\exp[iS_{\rm NRGR}[x_a, \bar{h}]] = \int \mathcal{D}H_{\mu\nu} \exp[iS[\bar{h} + H, x_a] + iS_{\rm GF}],$$

Use QFT methods and codes
p-space integration
method of regions
reached 5PN (with ambiguities)

$$S_{\rm mult} = -\frac{1}{\Lambda}\left\{\int {\rm d}\tau\left[E + \frac{1}{2}\dot{x}^\mu L_{\alpha\beta}\omega_\mu^{\alpha\beta} - \frac{1}{2}\sum_{n\geq 0}\left(c_n^{\mathcal{I}}\mathcal{I}^{\mu_1\cdots\mu_n\alpha\beta}\mathcal{E}_{\alpha\beta;\mu_1\cdots\mu_n} + c_n^{\mathcal{J}}\mathcal{J}^{\mu_1\cdots\mu_n\alpha\beta}\mathcal{B}_{\alpha\beta;\mu_1\cdots\mu_n}\right)\right]\right\}$$

$$\simeq \frac{1}{\Lambda}\int {\rm d}t\left[\frac{1}{2}Eh_{00} + \frac{1}{2}\epsilon_{ijk}L^i h_{0j,k} + \frac{1}{2}Q^{ij}\mathcal{E}_{ij} + \frac{1}{6}O^{ijk}\mathcal{E}_{ij,k} - \frac{2}{3}J^{ij}\mathcal{B}_{ij} + \cdots\right],$$

6PN dynamics complete at 3PM and 4PM nearly completely determined

Tutti-Frutti strategy
(Bini-TD-Geralico'20)
uses analytical knowledge from PN, MPM and SF

SIXTH POST-NEWTONIAN LOCAL-IN-TIME DYNAMIC

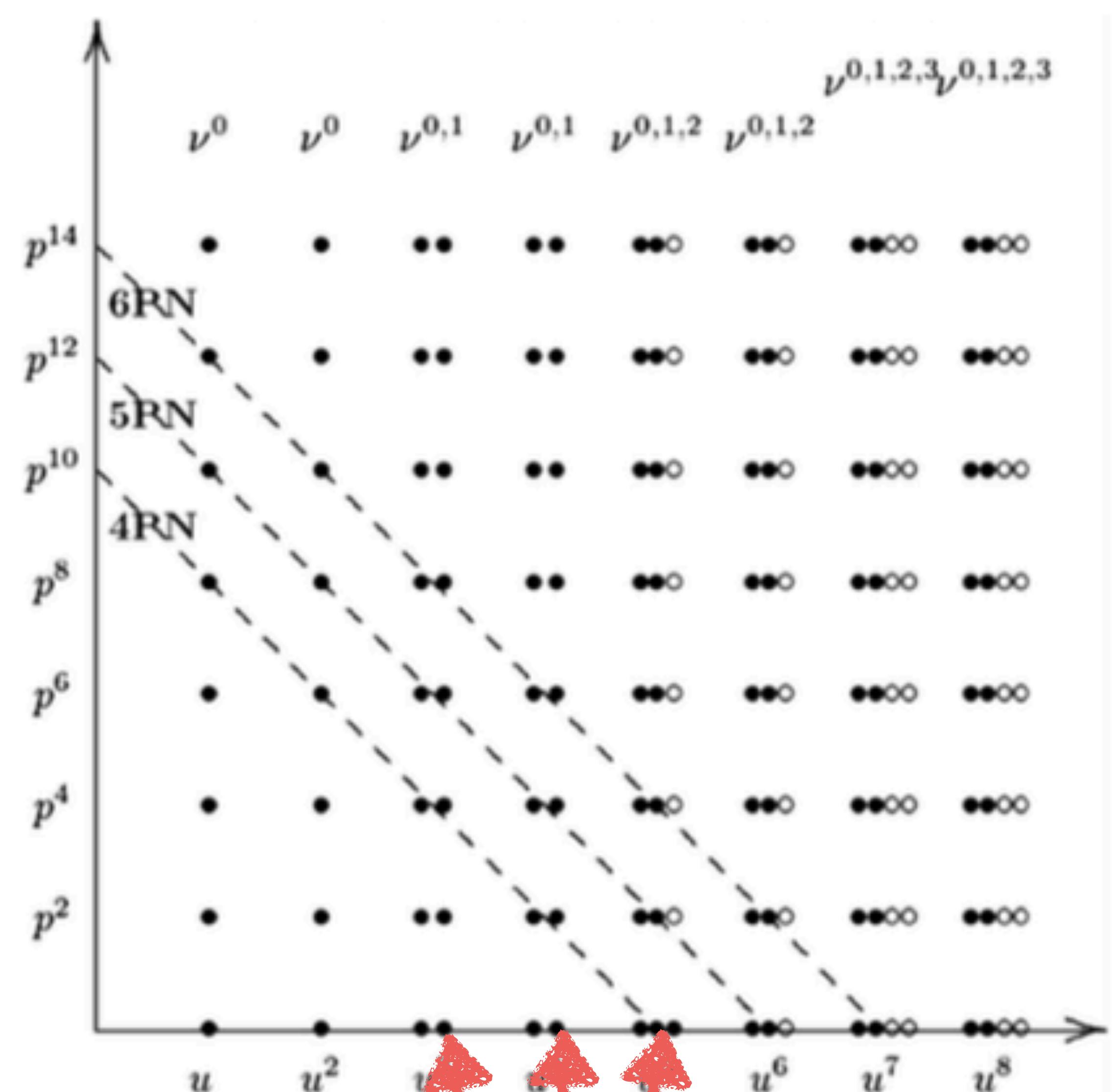


FIG. 1. Schematic representation of the irreducible information contained, at each post-Minkowskian level (keyed by a power of $u = GM/r$), in the local dynamics. Each vertical column of dots describes the post-Newtonian expansion (keyed by powers of p^2) of an energy-dependent function parametrizing the scattering angle. The various columns at a given post-Minkowskian level correspond to increasing powers of the symmetric mass-ratio ν. See text for details.

Nonlocal-in-time interaction

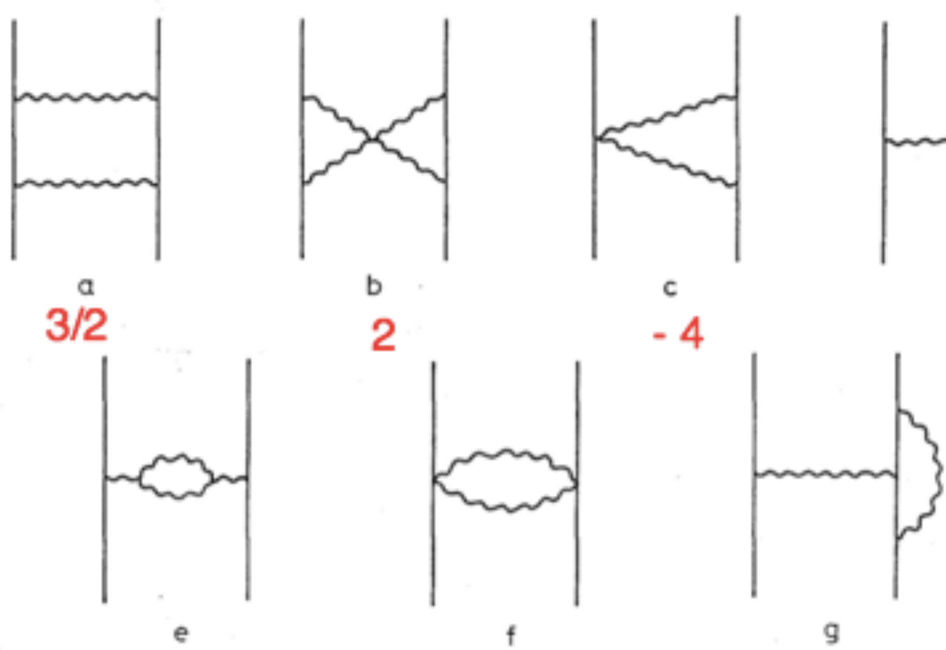
$$S^{4+5\mathrm{PN}}_{\mathrm{nonloc}}[x_1(s_1), x_2(s_2)] = \frac{G^2\mathcal{M}}{c^3}\int dt \mathrm{PF}_{2r^h_{12}(t)/c} \times \int \frac{dt'}{|t-t'|}\mathcal{F}^{\mathrm{split}}_{1\mathrm{PN}}(t,t').$$

$$\mathcal{F}^{\mathrm{split}}_{1\mathrm{PN}}(t,t') = \frac{G}{c^5}\left(\frac{1}{5}I^{(3)}_{ab}(t)I^{(3)}_{ab}(t') + \frac{1}{189c^2}I^{(4)}_{abc}(t)I^{(4)}_{abc}(t') + \frac{16}{45c^2}J^{(3)}_{ab}(t)J^{(3)}_{ab}(t')\right).$$

recent EFT computations
(Foffa-Sturani, Bluemlein et al)
come close to completing the 5PN dynamics

Self-force computation using BH perturbation (Regge-Wheeler-Zerilli) + Detweiler's redshift +First-law BBH (LeTiec-Blanchet-Whiting'11)

Quantum Scattering Amplitudes and 2-body Dynamics



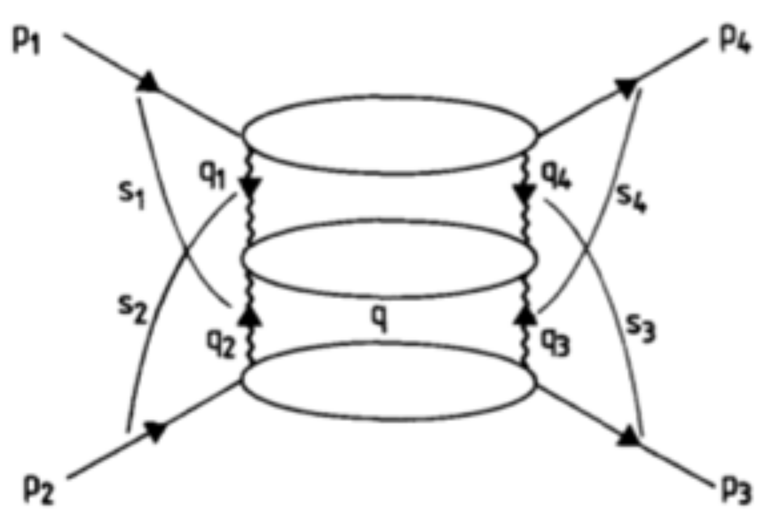


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

- Quantum Scattering Amplitudes —> Potential

one-graviton exchange :
Corinaldesi '56 '71,
Barker-Gupta-Haracz 66,
Barker-O'Connell 70, Hiida-Okamura72

Nonlinear: Iwasaki 71 [First post-Newtonian approx.],
Okamura-Ohta-Kimura-Hiida 73[2 PN]

Amati-Ciafaloni-Veneziano 1987-2008

Ultra-High-Energy (s >> M_Planck^2)
Four-graviton Scattering at 2 loops

Eikonal phase \delta in D=4
with one- and two-loop corrections
using the Regge-Gribov approach

confirmed by DiVecchia+'19

$$\delta = \frac{Gs}{\hbar}\left(\log\left(\frac{L_{IR}}{b}\right) + \frac{6\ell_s^2}{\pi b^2} + \frac{2G^2 s}{b^2}\left(1 + \frac{2i}{\pi}\log(\cdots)\right)\right)$$

Modern techniques for amplitudes (generalized unitarity; double copy; method of regions; IBPs; differential eqs; Bern, Dixon, Dunbar, Carrasco, Johansson, Cachazo et al., Bjerrum-Bohr et al., Cachazo-Guevara,…) can be used (Damour '17CheungRothsteinSolon'18) to improve the classical 2-body dynamics: need a quantum/classical dictionary.

Many recent results: Bern et al; DiVecchia-Heissenberg-Russo-Veneziano, Bjerrum-Bohr, Damgaard, Vanhove; Plefka et al, Riva-Vernizzi; Jakobsen, Mogull, Manohar, Ridgway, Shen,…

Classical scattering perturbation theory

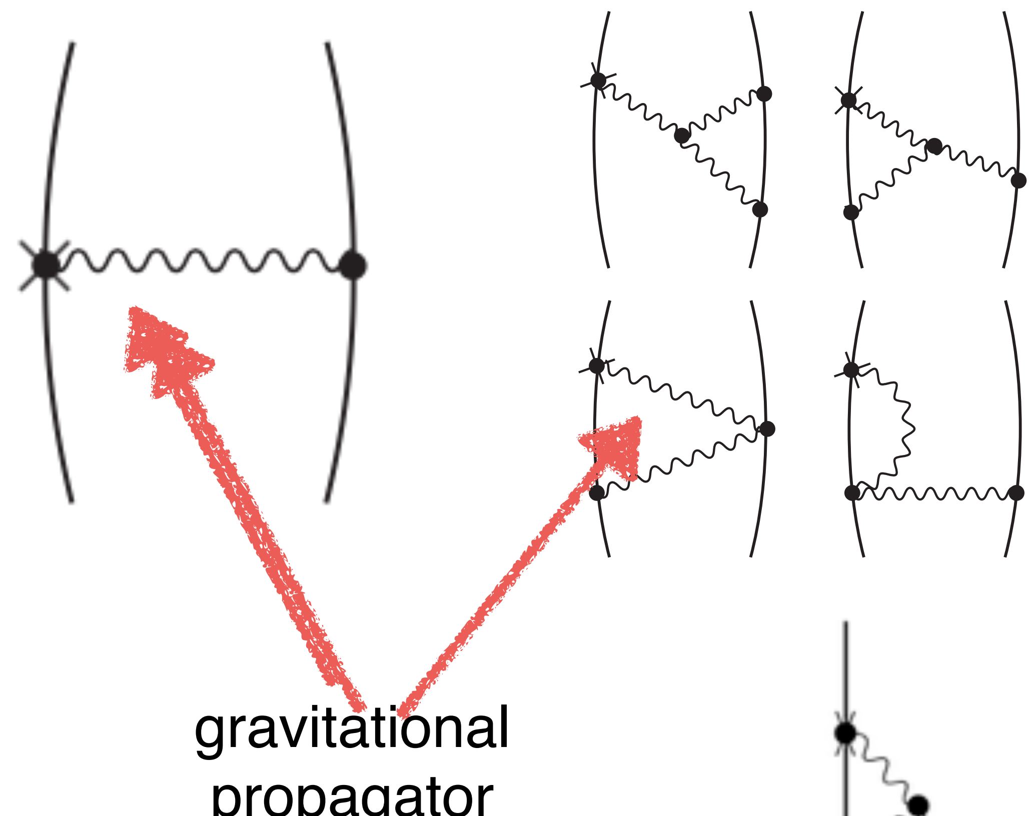
$$\frac{dx_a^\mu}{d\sigma_a} = g^{\mu\nu}(x_a)p_{a\nu},$$
$$\frac{dp_{a\mu}}{d\sigma_a} = -\frac{1}{2}\partial_\mu g^{\alpha\beta}(x_a)p_{a\alpha}p_{a\beta}.$$

$$R^{\mu\nu} - \tfrac{1}{2}Rg^{\mu\nu} = 8\pi G T^{\mu\nu}$$

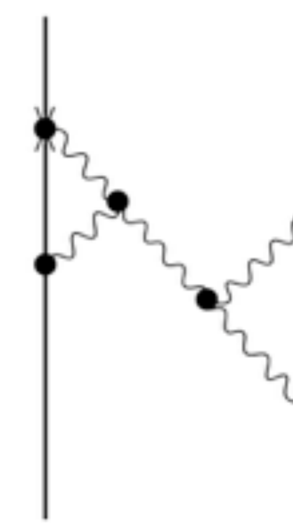
$$T^{\mu\nu}(x) = \sum \int d\sigma_a p_a^\mu p_a^\nu \frac{\delta^4(x - x_a(\sigma_a))}{\sqrt{g}}$$

$$\Delta p_{a\mu} = \int_{-\infty}^{+\infty} d\sigma_a \frac{dp_{a\mu}}{d\sigma_a}$$
$$= -\frac{1}{2}\int_{-\infty}^{+\infty} d\sigma_a \partial_\mu g^{\alpha\beta}(x_a)p_{a\alpha}p_{a\beta}.$$

$$\Delta p_{1\mu} = 2G \int d\sigma_1 d\sigma_2 p_{1\alpha}p_{1\beta}$$
$$\times\, \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'}(x_1(\sigma_1) - x_2(\sigma_2))p_{2\alpha'}p_{2\beta'}$$



$$\mathcal{P}^{\alpha\beta;\alpha'\beta'}(x-y) = \left(\eta^{\alpha\alpha'}\eta^{\beta\beta'} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\alpha'\beta'}\right)\mathcal{G}(x-y)$$



Approach initiated long ago: Rosenblum'78 Westpfahl'79,'85 Portilla'80 Bel et al.'81 limited by the technical difficulty of computing the integrals beyond G^2, ie at G^2=2-loop.

Recently developed to compete with quantum-scattering approach: Kalin-Porto, Porto et al, Plefka et al, Dlapa-Kalin-Liu-Porto,…

two-loop level G^3

Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order

Zvi Bern,[1] Clifford Cheung,[2] Radu Roiban,[3] Chia-Hsien Shen,[1] Mikhail P. Solon,[2] and Mao Zeng[4]
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[2]*Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, California 91125*
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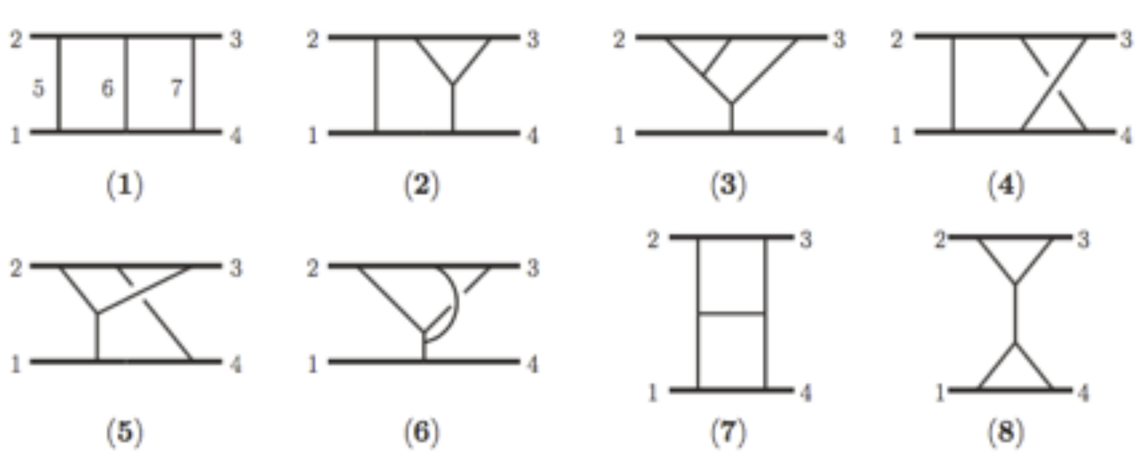
(Received 28 January 2019; published 24 May 2019)

We present the amplitude for classical scattering of gravitationally interacting massive scalars at third post-Minkowskian order. Our approach harnesses powerful tools from the modern amplitudes program such as generalized unitarity and the double-copy construction, which relates gravity integrands to simpler gauge-theory expressions. Adapting methods for integration and matching from effective field theory, we extract the conservative Hamiltonian for compact spinless binaries at third post-Minkowskian order. The resulting Hamiltonian is in complete agreement with corresponding terms in state-of-the-art expressions at fourth post-Newtonian order as well as the probe limit at all orders in velocity. We also derive the scattering angle at third post-Minkowskian order and find agreement with known results.

the eight 2-loop diagrams contributing to the O(G^3/r^3) classical potential



two-loop level

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2 \xi}\Bigg[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu(3+12\sigma^2-4\sigma^4)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} - \frac{18\nu\gamma(1-2\sigma^2)(1-5\sigma^2)}{(1+\gamma)(1+\sigma)}\Bigg] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi}\left[3\gamma(1-2\sigma^2)(1-5\sigma^2)F_1 - 32m^2\nu^2(1-2\sigma^2)^3 F_2\right], \quad (8)$$

arcsinh

potential gravitons only

Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$

Zvi Bern,[1] Julio Parra-Martinez,[2] Radu Roiban,[3] Michael S. Ruf,[4]
Chia-Hsien Shen,[5] Mikhail P. Solon,[1] and Mao Zeng[6]
[1]Mani L. Bhaumik Institute for Theoretical Physics, University of California at Los Angeles

three-loop level G^4

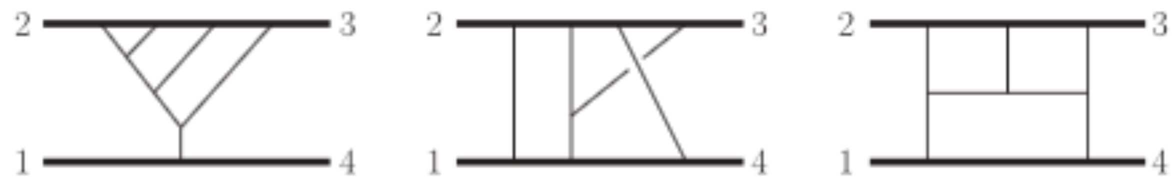


FIG. 2. Sample diagrams at $\mathcal{O}(G^4)$. From left to right: a contribution in the probe limit, a nonplanar diagram that contains iteration terms, and a diagram that contains contributions related to the tail effect.

$$\mathcal{M}_4(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{1/3}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^p + \nu\left(\frac{\mathcal{M}_4^t}{\epsilon} + \mathcal{M}_4^f\right)\right] + \int_\ell \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_\ell \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_\ell \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1} + \int_\ell \frac{\tilde{I}_{r,2}^2}{Z_1},$$

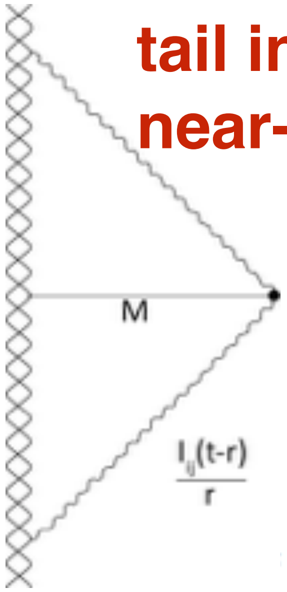
$$\mathcal{M}_4^p = -\frac{35(1-18\sigma^2+33\sigma^4)}{8(\sigma^2-1)}, \quad \mathcal{M}_4^t = h_1 + h_2 \log\left(\frac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}},$$

$$\begin{aligned}
\mathcal{M}_4^f = {} & h_4 + h_5 \log\left(\frac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + h_9 \left[\operatorname{Li}_2\left(\frac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\frac{\sigma+1}{2}\right)\right] \\
& + h_{10}\left[\operatorname{Li}_2\left(\frac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\operatorname{Li}_2\left(\frac{1-\sigma}{1+\sigma}\right) - \operatorname{Li}_2\left(\frac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\operatorname{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - \operatorname{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right)\right] \\
& + \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\operatorname{Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - \operatorname{Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 5\operatorname{Li}_2\left(\sqrt{\frac{\sigma-1}{\sigma+1}}\right) - 5\operatorname{Li}_2\left(-\sqrt{\frac{\sigma-1}{\sigma+1}}\right)\right. \\
& \left. + 2\log\left(\frac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] + h_{12} K^2\left(\frac{\sigma-1}{\sigma+1}\right) + h_{13} K\left(\frac{\sigma-1}{\sigma+1}\right) E\left(\frac{\sigma-1}{\sigma+1}\right) + h_{14} E^2\left(\frac{\sigma-1}{\sigma+1}\right), \qquad (6)
\end{aligned}$$

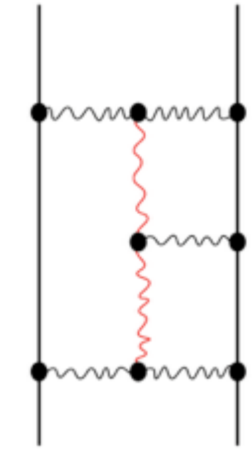
soft (radiationlike) gravitons

Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at $\mathcal{O}(G^4)$

tail in near-zone



Zvi Bern,[1] Julio Parra-Martinez,[2] Radu Roiban,[3] Michael S. Ruf,[1]
Chia-Hsien Shen,[4] Mikhail P. Solon,[1] and Mao Zeng[5]



Conservative Dynamics of Binary Systems at Fourth Post-Minkowskian Order in the Large-eccentricity Expansion

Christoph Dlapa,[1] Gregor Kälin,[1] Zhengwen Liu,[1] and Rafael A. Porto[1]

$$G^4M^7\nu^2|\boldsymbol{q}|\pi^2\left[\mathcal{M}_4^{\rm p}+\nu\left(4\mathcal{M}_4^{\rm t}\log\left(\tfrac{p_\infty}{2}\right)+\mathcal{M}_4^{\pi^2}+\mathcal{M}_4^{\rm rem}\right)\right]+\int_\ell\frac{\tilde{I}^4_{r,1}}{Z_1Z_2Z_3}+\int_\ell\frac{\tilde{I}^2_{r,1}\tilde{I}_{r,2}}{Z_1Z_2}+\int_\ell\frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1}+\int_\ell\frac{\tilde{I}^2_{r,2}}{Z_1},$$

$$\begin{aligned}
\mathcal{M}_4^{\rm p}&=-\frac{35\left(1-18\sigma^2+33\sigma^4\right)}{8\left(\sigma^2-1\right)},\\
\mathcal{M}_4^{\rm t}&=r_1+r_2\log\left(\tfrac{\sigma+1}{2}\right)+r_3\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}},\\
\mathcal{M}_4^{\pi^2}&=r_4\pi^2+r_5\,\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right)+r_6\,\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right)+r_7\,\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right),\\
\mathcal{M}_4^{\rm rem}&=r_8+r_9\log\left(\tfrac{\sigma+1}{2}\right)+r_{10}\tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}+r_{11}\log(\sigma)+r_{12}\log^2\left(\tfrac{\sigma+1}{2}\right)+r_{13}\tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\log\left(\tfrac{\sigma+1}{2}\right)+r_{14}\tfrac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1}\\
&\quad+r_{15}\,\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right)+r_{16}\,\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right)+r_{17}\tfrac{1}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)-\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right].
\end{aligned}\tag{3}$$

$$\begin{aligned}
\mathcal{M}_4^{\rm radgrav,f}=&\frac{12044}{75}p_\infty^2+\frac{212077}{3675}p_\infty^4+\frac{115917979}{793800}p_\infty^6\\
&-\frac{9823091209}{76839840}p_\infty^8+\frac{115240251793703}{1038874636800}p_\infty^{10}\\
&-\frac{411188753665637}{4155498547200}p_\infty^{12}+\cdots,
\end{aligned}\tag{6}$$

whose first three terms match the sixth PN order result in Eq. (6.20) of Ref. [42].

subtleties linked to radiative effects

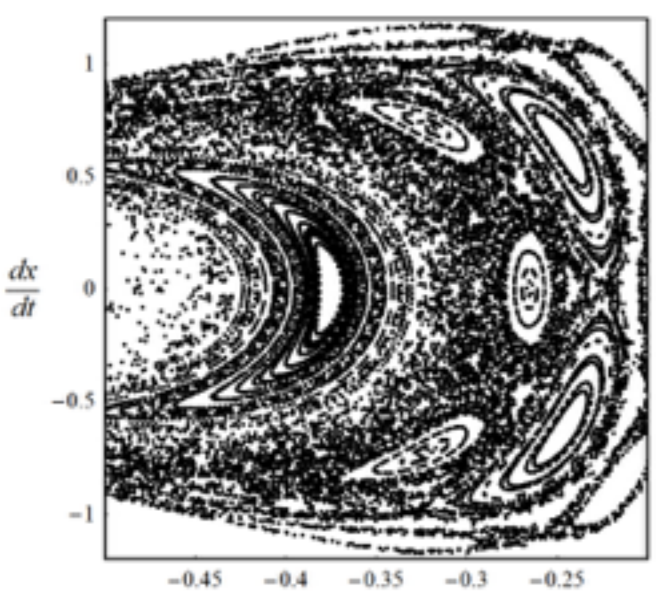
Conclusions

- Analytical approaches to GW signals have played a crucial role (in conjunction with Numerical Relativity simulations) for the detection, interpretation and parameter estimation of coalescing binary systems (BBH and BNS). They will remain very important for future GW detectors: second generation ground-based detectors, space detectors, second generation ground-based detectors.

- The development of new analytical approaches (EFT, classical or quantum scattering approaches, Tutti-Frutti,…) has started to bring new results of interest for GW detection and must be pursued vigorously.

Henri Poincaré

«Il n’y a pas de problèmes résolus,
il y a seulement des problèmes
plus ou moins résolus »



«There are no solved problems,
there are only
more or less solved problems »