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main theme of today's lecture:

<u>how can we tell a phenomenon is</u>
<u>quantum</u>?

naive answer: $\hbar$ enters!

<u>counter example:</u> non-linear Schrödinger equation of <u>two</u> classical equations!

but still WKB

<u>topics:</u>

motivation of Schrödinger equation

Stefan Boltzmann law of black-body radiation

<u>references:</u>

papers with
- M.O. Scully, D.M. Greenberger and D. Kobe
  PNAS, Varenna & Physica Scripta
- H. Paul, D.M. Greenberger, S.T. Stenholm
  Physica Scripta

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Schrödinger equation: introductory and historical remarks

- Hermann Weyl 1918 : Streckenfaktor $\oint dx_\mu A^\mu$
- Erwin Schrödinger 1922 : application of Weyl to Bohr orbit
  appearance of "i"
- de Broglie 1924 : matter waves $e^{imc^2\tau}$ (τ: proper time)
- Heisenberg, Born Jordan (1925) Matrizenmechanik
- Schrödinger Wellenmechanik : Kolloquium Debye
- Weyl Gauge Theory :
- Feynman path integral
  Gregor Wenzel "Zur Quantenoptik"

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Schrödinger equation :

mathematical identity : $\mathcal{Z} = \mathcal{Z}(t,\vec{r}) \equiv A(t,\vec{r})\, e^{i\Theta(t,\vec{r})}$

$$i\frac{\partial}{\partial t}\mathcal{Z} + \beta \vec{\nabla}^2 \mathcal{Z} = \left(\mathcal{V} + i\frac{1}{2A^2}\mathcal{C}\right)\mathcal{Z}$$

with

$$\mathcal{V} \equiv -\frac{\partial\Theta}{\partial t} - \beta\left(\vec{\nabla}\Theta\right)^2 + \beta\frac{\vec{\nabla}^2 A}{A}$$

and

$$\mathcal{C} \equiv \frac{\partial}{\partial t}A^2 + 2\beta\,\vec{\nabla}\cdot\left(A^2\vec{\nabla}\Theta\right)$$

importance of "i" connection to phase space

classical statistical physics: "WKB physics"

Hamilton-Jacobi equation $\quad -\frac{\partial S}{\partial t} = \frac{(\vec{\nabla}S)^2}{2m} + V$

$$S^{(cl)} = S^{(cl)}(t,\vec{r};\vec{\alpha})$$

three-dimensional $\quad$ constant of motion

Van Vleck determinant $\quad D \equiv \left\| \frac{\partial^2 S}{\partial x_j\,\partial\alpha_k} \right\| \quad j,k = 1,2,3$

determinant not absolute value

continuity equation $\quad \frac{\partial}{\partial t}D + \vec{\nabla}\cdot\left(D\frac{\vec{\nabla}S^{(cl)}}{m}\right) = 0$

is a consequence of Hamilton-Jacobi + definition of van Vleck determinant

in 1-D : $\quad S^{(cl)} = S^{(cl)}(t,z;E)$

one space dimension $\quad$ constant of integration

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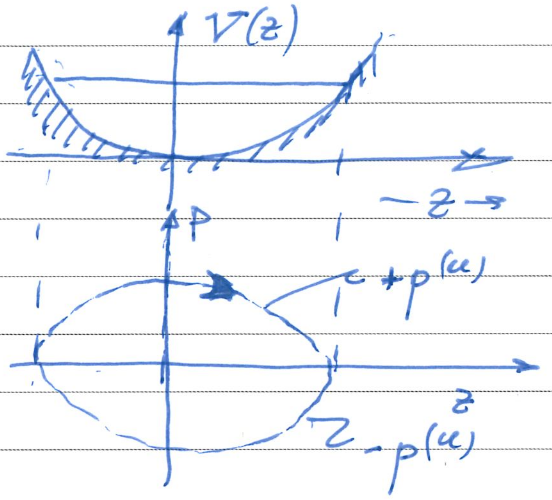
classical mechanics in conservative potential $V = V(z)$

$$E = \frac{p^{(cl)\,2}}{2m} + V(z) \Rightarrow p^{(cl)}(z;E) = \pm\sqrt{2m\left[E - V(z)\right]}$$

different motions



solution of Hamilton - Jacobi equation :

$$S^{(cl)}(t,z;E) = -E \cdot t + \int_{z_0}^{z} dz'\, p^{(cl)}(z';E) + S'$$

constant

constant of motion $\alpha_z = E$

$$\underline{\frac{\partial S^{(cl)}}{\partial t} = -E} \qquad \underline{\frac{\partial S^{(cl)}}{\partial z} = p^{(cl)}}$$

$$\Rightarrow \frac{1}{2m}\left(\frac{\partial S^{(cl)}}{\partial z}\right)^2 + V = \frac{\left(p^{(cl)}\right)^2}{2m} + V = E = -\frac{\partial S'}{\partial t}$$

Van Vleck determinant :

$$D = \frac{\partial^2 S}{\partial E\,\partial z} = \frac{\partial}{\partial E}\, p^{(cl)} = \frac{\partial}{\partial E}\sqrt{2m(E - V)} = \frac{1}{2}\,\frac{1}{p^{(cl)}}\, 2m$$

$$\underline{D = \frac{m}{p^{(cl)}}}$$

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$$\underline{\frac{\partial}{\partial t} D} = \frac{\partial}{\partial t}\left(\frac{\partial^2 S}{\partial z \partial E}\right) = \frac{\partial}{\partial z \partial E}\left(\frac{\partial S}{\partial t}\right) =$$

$$= \frac{\partial^2}{\partial z \partial E}\left[-\frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 - V\right] =$$

does not depend on E

$$= \frac{\partial}{\partial z}\left(-\frac{1}{m}\frac{\partial S}{\partial z}\frac{\partial^2 S}{\partial E \partial z}\right)$$

$$= -\frac{\partial}{\partial z}\left(\frac{\partial^2 S}{\partial E \partial z} \; \frac{\frac{\partial S}{\partial z}}{m}\right)$$

$$\underline{= -\frac{\partial}{\partial z}\left(D \, \frac{\frac{\partial S}{\partial z}}{m}\right)};$$

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ansatz!

$$\psi^{(cl)} \equiv D^{1/2} e^{i S^{(cl)}/\hbar}$$

reduced Planck's constant to make it dimensionless

$$\uparrow \qquad \Theta \equiv \frac{S^{(cl)}}{\hbar}$$

$$A^2 \equiv D$$

Van Vleck continuity equation

$$\mathcal{C} = \frac{\partial}{\partial t} D^2 + \vec{\nabla} \cdot \left(D \frac{\vec{\nabla} S^{(cl)}}{m} \underbrace{\beta \frac{2m}{\hbar}}_{\Rightarrow 1} \right) = 0$$

$$\underline{\beta = \frac{\hbar}{2m}} \quad ; \quad \underline{\mathcal{C} = 0}$$

$$\nu = -\frac{1}{\hbar} \frac{\partial S^{(cl)}}{\partial t} - \frac{\hbar}{2m} \frac{1}{\hbar^2} \left(\vec{\nabla} S^{(cl)} \right)^2 + \frac{\hbar}{2m} \frac{\vec{\nabla}^2 D^{1/2}}{D^{1/2}}$$

$$\hbar \nu = -\frac{\partial S^{(cl)}}{\partial t} - \frac{1}{2m} \left(\vec{\nabla} S^{(cl)} \right)^2 + \frac{\hbar^2}{2m} \frac{\vec{\nabla}^2 D^{1/2}}{D^{1/2}}$$

classical Hamilton Jacobi equation

$$\downarrow$$

$$= V + \frac{\hbar^2}{2m} \frac{\vec{\nabla}^2 D^{1/2}}{D^{1/2}}$$

"Schrödinger equation" : multiply mathematical identity by $\hbar$ and use $\beta = \frac{\hbar}{2m}$ and $\mathcal{C} = 0$

$$i\hbar \frac{\partial}{\partial t} \psi^{(cl)} = \left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + V + \frac{\hbar^2}{2m} \frac{\vec{\nabla}^2 D^{1/2}}{D^{1/2}} \right] \psi^{(cl)}$$

non-linear Schrödinger equation : $D^{1/2} \equiv |\psi^{(cl)}|$

$$i\hbar \frac{\partial}{\partial t} \psi^{(cl)} = \left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + V + \frac{\hbar^2}{2m} \frac{\vec{\nabla}^2 |\psi^{(cl)}|}{|\psi^{(cl)}|} \right] \psi^{(cl)}$$

$\uparrow$ quantum potential of Madelung & Bohm

Wenzel - Kramers - Brillouin
↓
meaning of quantum potential in WKB: in 1-D

$$D = \frac{m}{p^{(cl)}} \qquad D^{1/2} = \sqrt{\frac{m}{p^{cl}}} = \frac{\sqrt{m}}{\sqrt{p^{(cl)}}}$$

$$\psi^{(cl)} = \frac{\sqrt{m}}{\sqrt{p^{(cl)}}} e^{iS^{(cl)}/\hbar} = \frac{\sqrt{m}}{-\sqrt{p^{(cl)}}} e^{-\frac{i}{\hbar}\left[Et - \int_{z_0}^{z} dz\, p^{(cl)}(z;E)\right]}$$

deviation from stationary solution of Schrödinger equation ≡ quantum potential

WKB waves satisfy Schrödinger equation with quantum potential $\sim \sqrt{p^{(cl)}} \cdot \frac{d^2}{dz^2}\left(\frac{1}{\sqrt{p^{cl}}}\right)$ ~ change of amplitude of wave (second order)
⇒ quantum reflections

concluding remarks:

- $\psi^{(cl)}$ combines two real-valued function in a single complex one
- none involves $\hbar$
- equation for $\psi^{(cl)}$ is non-linear
- $\hbar$ cannot be fixed; any quantity with units "action" would do
- amplitud equation couples to phase equation but not phase to amplitude

quantum mechanics: $\quad \psi^{(q)} \equiv A^{(q)} e^{iS^{(q)}/\hbar}$

postulate conservation law: $0 = \frac{\partial}{\partial t} A^{(q)2} + \vec{\nabla} \cdot \left[\left(A^{(q)}\right)^2 \frac{\vec{\nabla} S^{(q)}}{m}\right]$
is not guaranteed

quantum Hamilton-Jacobi:

$$\hbar \mathcal{V} \equiv V = -\frac{\partial S^{(q)}}{\partial t} - \frac{1}{2m}\left(\vec{\nabla} S^{(q)}\right)^2 + \frac{\hbar^2}{2m} \frac{\vec{\nabla}^2 A^{(q)}}{A^{(q)}}$$

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resulting equation for $\psi^{(q)}$ is linear Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi^{(q)} = \left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V\right]\psi^{(q)}$$

quantum Hamilton-Jacobi equation

$$-\frac{\partial}{\partial t}S^{(q)} = \frac{1}{2m}\left(\vec{\nabla}S^{(q)}\right)^2 + V - \frac{\hbar^2}{2m}\frac{\vec{\nabla}^2 A^{(q)}}{A^{(q)}}$$

different sign than in non-linear Schrödinger equation

conclusions: • two coupled equations yield linear equation
• symmetric coupling

remark as to $\hbar$: Stefan-Boltzmann law

two classical laws give a quantum: black-body radiation universal radiation

ingredients: thermodynamic relation $\left(\frac{\partial E}{\partial V}\right)_T = T^2\frac{\partial}{\partial T}\left(\frac{p}{T}\right)$

pressure formula $p = \frac{1}{3}u$

since blackbody radiation is universal:

$$\Rightarrow \quad u = a \cdot T^4$$

$\Rightarrow$ "a" can only contain universal constants such as c, k_B

energy density — temperature

due to thermodynamics

$$[a] = \frac{[u]}{[T^4]} = \frac{[\text{Energy}]}{[\text{Volume}]}\frac{k_B^4}{[k_B T]^4} = \frac{k_B^4}{[\text{Volume}][\text{Energy}]^3}$$

due to electrodynamics $\rightarrow [c] = \frac{[\text{Length}]}{[\text{Time}]}$ $\quad [\text{Volume}] = c^3[\text{Time}]^3$

$$[a] = \frac{k_B^4}{c^3[\text{action}]^3} = \frac{k_B^4}{c^3 h^3} \equiv a, \; h \equiv \text{Plancks constant}$$

quantum mechanics provides only numerical factor

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$$\alpha = \frac{8\pi^5}{15}\frac{k_B^4}{h^3c^3} = \frac{k_B^4}{(N\hbar)^3 c^3} \qquad N \equiv \left(\frac{15}{8\pi^5}\right)^{1/3} = 0.183$$

standard approach in text books: mixed quantum + classical

<u>classical</u> <u>quantum</u>

mode density of electromagnetic field → Planck distribution ← average energy of quantized oscillator

Planck distribution ↓ integration over frequency $\int_0^\infty d\nu$

thermodynamical relation → Stefan - Boltzmann law including numerical factor

pressure $= \frac{1}{3}$ (energy density) → Stefan - Boltzmann law

do not lead numerical value of a but dimensions

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derivation of Boltzmann

$$p = \frac{1}{3} u \quad ; \quad u = \frac{E}{V}$$

p: pressure of radiation

u: energy density

$$\left(\frac{\partial E}{\partial V}\right)_T = T^2 \frac{\partial}{\partial T}\left(\frac{P}{T}\right);$$

thermodynamic relation from second law

$$\left(\frac{\partial}{\partial V}(uV)\right)_T = T^2 \frac{\partial}{\partial T}\left(\frac{1}{3}\frac{u}{T}\right) = \frac{1}{3}T^2\left(\frac{1}{T}\frac{\partial u}{\partial T} - \frac{1}{T^2}u\right)$$

$$\Rightarrow u = \frac{1}{3}\left[T\frac{\partial u}{\partial T} - u\right]$$

$$4u = T\frac{\partial u}{\partial T} \qquad u = a \cdot T^4$$

differential equation of first order → one constant of motion

$$T \cdot \frac{\partial u}{\partial T} = T\left(a \cdot 4T^3\right) = 4a \cdot T^4 = 4u;$$