Hamilton - Lagrange formalism for Kasevich - Chu atom interferometer

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Kasevich - Chu interferometer: representation-free approach

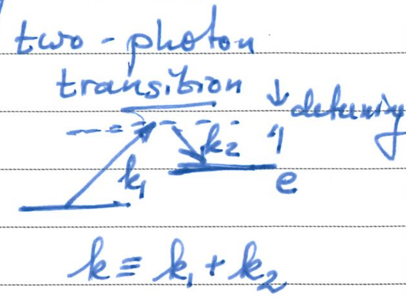
most elementary model no gravity gradients!

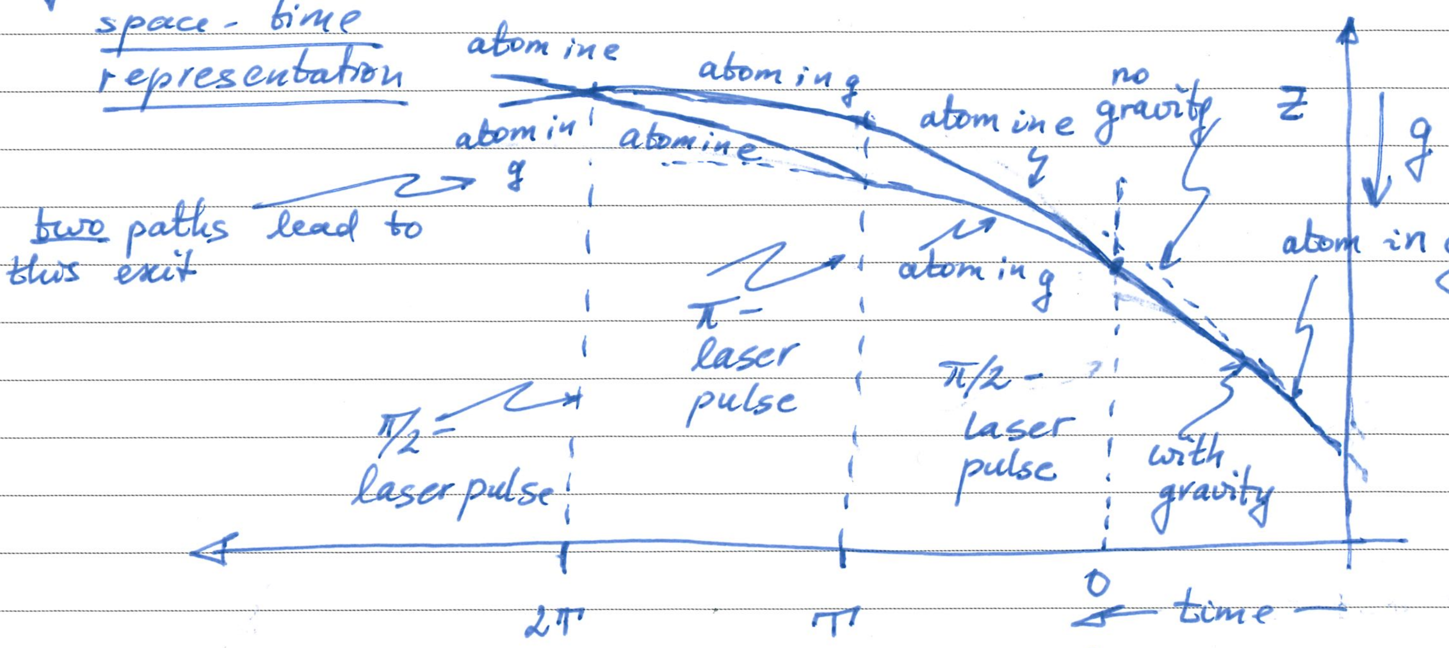
g ↓ z ↑

↓ laser fields

• ← atoms ≡ three-Level atom

↑ laser fields





beam splitter (momentum transfer $\hbar k$)   mirror   beam splitter (momentum transfer $\hbar k$)

effect of numerical factors

$$|\psi_g^{(u)}\rangle = \mathcal{N} \; e^{-i\hat{H}_f T/\hbar} \; e^{-ik\hat{z}} \; e^{-i\hat{H}_f T/\hbar} \; e^{+ik\hat{z}} \; |\psi(0)\rangle \qquad \text{upper path}$$

no laser pulse

$$|\psi_g^{(l)}\rangle = \mathcal{N} \; e^{-ik\hat{z}} \; e^{-i\hat{H}_f T/\hbar} \; e^{+ik\hat{z}} \; e^{-i\hat{H}_f T/\hbar} \; |\psi(0)\rangle \qquad \text{lower path}$$

↑ factors are identical in the two paths   ↑ no laser pulse

Hamiltonians: atom exits in g

upper path: $\hat{H}_u = \dfrac{\hat{p}^2}{2m} + mg\hat{z} - \hbar k\hat{z}\left[\delta(t) - \delta(t-T)\right]$

lower path: $\hat{H}_l = \dfrac{\hat{p}^2}{2m} + mg\hat{z} - \hbar k\hat{z}\left[\delta(t-T) - \delta(t-2T)\right]$

constant acceleration

↑ linear gravity   no gravity gradient

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microscopic derivation follows from interaction of atom with the two fields in second order perturbation theory

why do we have $-\hbar k z\,\delta(t) \equiv V_{ep}$ as an effective potential?

Hamilton equations of motion (classical)

$p \equiv$ momentum in z-direction

$$\dot{z} = \frac{\partial H}{\partial p} \;; \qquad \dot{p} = -\frac{\partial H}{\partial z}$$

minus sign

$$\dot{z} = \frac{p}{m} \;; \qquad \dot{p} = -mg + \hbar k\,\delta(t)$$

due to minus sign in Hamiltonian we find transfer of momentum $\hbar k$

small positive time interval

$$p(\Delta t) = p(-\Delta t) + \hbar k$$

$\Rightarrow$ momentum increase by $\hbar k$

momentum shortly after δ-pulse

momentum shortly before δ-pulse

contribution due to gravity $\equiv -mg\,\Delta t$ goes to zero for $\Delta t \to 0$

Time evolution in quantum mechanics.

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H}\,|\Psi(t)\rangle$$

total time derivative

remark:

Hamiltonian is time-dependent due to delta-functions. However, due to the delta-function we can neglect the other terms such as the

Hamiltonian

$$\hat{H}_f \equiv \frac{\hat{p}^2}{2m} + mg\hat{z}$$

in the gravitational field; still time-ordering is necessary;

<u>solution</u>:

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\Psi(0)\rangle$$

any Hamiltonian

initial state (not necessarily at time zero)

System considered here consists of <u>two</u> quantum degrees

(i) internal states $|g\rangle$ and $|e\rangle$

(ii) center-of-mass motion $|\psi\rangle$

Hence, we deal with $|\Psi\rangle = \sum_{j=g,e} \psi_j\, |j\rangle |\psi_j\rangle$

state of combined system

atomic state

state of center-of mass motion in state $|j\rangle$

We now follow one given trajectory (upper or lower path) (see below Fig 1) starting from the initial state

$$|\Psi(0)\rangle \equiv |g\rangle|\psi_g(0)\rangle \equiv |g\rangle\,|\psi(0)\rangle$$

means shortly <u>before</u> laser pulse

there are also factors such as $\frac{1}{\sqrt{2}}$ due to beam-splitter effect and "i" due to Schrödinger evolution (Rabi oscillation) of atom; we ~~due~~ not consider them here

$$|\psi_g^{(u)}\rangle = \mathcal{N}\, e^{-i\hat{H}_f t/\hbar} \left(e^{-ik\hat{z}}\, e^{-i\hat{H}_f t/\hbar}\, e^{ik\hat{z}} \right) |\psi(0)\rangle$$

$$|\psi_g^{(l)}\rangle = \mathcal{N} \left(e^{-ik\hat{z}}\, e^{-i\hat{H}_f t/\hbar}\, e^{ik\hat{z}} \right) e^{-i\hat{H}_f t/\hbar} |\psi(0)\rangle$$

total state at exit with atom in $|g\rangle$ is

$$|\Psi\rangle = |g\rangle \left[|\psi_g^{(u)}\rangle + |\psi_g^{(l)}\rangle \right]$$

- important result: two paths differ in the order of the operators

calculate: $e^{-ik\hat{z}}\, e^{-i\hat{H}_f t/\hbar}\, e^{ik\hat{z}} =$

$$= e^{-ik\hat{z}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-it}{\hbar}\right)^n \hat{H}_f^n\, e^{ik\hat{z}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-it}{\hbar}\right)^n e^{-ik\hat{z}}\, \hat{H}_f \cdot \hat{H}_f \cdot \ldots \hat{H}_f\, e^{ik\hat{z}}$$

$$\mathbb{1} = e^{+ik\hat{z}}\, e^{-ik\hat{z}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-it}{\hbar}\right)^n e^{-ik\hat{z}} \left(\frac{\hat{p}^2}{2m} + mg\hat{z} \right) e^{+ikz} \cdot e^{-ikz} \left(\frac{\hat{p}^2}{2m} + mgz \right) e^{ikz}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-it}{\hbar}\right)^n \left[\frac{(p+\hbar k)^2}{2m} + mg\hat{z} \right] \left[\frac{(p+\hbar k)^2}{2m} + mg\hat{z} \right] \ldots$$

$$= e^{-i\hat{H}_e t/\hbar} \, ;$$

$$\hat{H}_e \equiv \frac{1}{2m} (\hat{p} + \hbar k)^2 + mg\hat{z} = \frac{1}{2m} \hat{p}^2 + mg\hat{z} + \frac{\hbar k}{m} \hat{p} + \frac{(\hbar k)^2}{2m}$$

$$= \hat{H}_g + \frac{\hbar k}{m} \hat{p} + \frac{(\hbar k)^2}{2m}$$

$$\hat{H}_g \equiv \frac{1}{2m}\hat{p}^2 + mg\hat{z}$$

$$|\psi_g^{(u)}\rangle = \mathcal{N}\, e^{-i\hat{H}_g T/\hbar}\, e^{-i\hat{H}_e T/\hbar}\, |\psi(0)\rangle$$

$$|\psi_g^{(\ell)}\rangle = \mathcal{N}\, e^{-i\hat{H}_e T/\hbar}\, e^{-i\hat{H}_g T/\hbar}\, |\psi(0)\rangle$$

The two paths differ in the order of the two Hamiltonians

They do not commute!

$$[\hat{H}_e, \hat{H}_g] = [\underbrace{\hat{H}_g + \frac{\hbar k}{m}\hat{p} + \overbrace{\frac{(\hbar k)^2}{2m}, \hat{H}_g}^{\text{commute}}}_{\text{commute}}] = \frac{\hbar k}{m}[\hat{p}, \hat{H}_g]$$

$$= \frac{\hbar k}{m}\left[\hat{p}, \frac{1}{2m}\hat{p}^2 + mg\hat{z}\right] = \frac{\hbar k}{m} mg [\hat{p}, \hat{z}]$$

$$= \hbar k g \frac{\hbar}{i} = \underline{-i\hbar^2 kg};$$

$[\hat{p}, \hat{z}] = \frac{\hbar}{i}$ commutator in grave of Max Born

<u>Heisenberg equation of motion:</u>

$$\hat{f}(t) \equiv e^{+i\hat{H}t/\hbar}\hat{f}_0\, e^{-i\hat{H}t/\hbar} \qquad \text{Heisenberg picture}$$

$$\frac{d}{dt}\hat{f} = \frac{i}{\hbar}[\hat{H}\hat{f} - \hat{f}\hat{H}] = \frac{i}{\hbar}[\hat{H}, \hat{f}]$$

$$[\hat{H}_e, \hat{H}_g] = \frac{\hbar k}{m}[\hat{p}, \hat{H}_g] = \frac{\hbar k}{m}\left(-\frac{\hbar}{i}\right)\frac{d\hat{p}}{dt} = \hbar k g \frac{\hbar}{i}$$

this commutator is proportional to time rate of change of momentum = acceleration

$$\underline{\frac{d}{dt}\hat{p} = -mg}\, ; \quad \text{constant gravitational force}$$

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what is the phase shift in the interferometer?

Baker – Cambell – Hausdorf relation:

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$

if

$$[\hat{A},[\hat{A},\hat{B}]] = [B,[\hat{A},\hat{B}]] = 0$$

$$|\psi_g^{(u)}\rangle = \mathcal{N} e^{-i(\hat{H}_g+\hat{H}_e)T/\hbar} e^{\frac{1}{2}[\hat{H}_g,\hat{H}_e]T^2/\hbar^2} |\psi(0)\rangle$$

$$|\psi_g^{(l)}\rangle = \mathcal{N} e^{-i(\hat{H}_e+\hat{H}_g)T/\hbar} e^{\frac{1}{2}[\hat{H}_e,\hat{H}_g]T^2/\hbar^2} |\psi(0)\rangle$$

$$\frac{1}{2}[\hat{H}_g,\hat{H}_e]T^2/\hbar^2 = \frac{1}{2}(-i\hbar^2 kg)T^2/\hbar^2 = -\frac{i}{2}kgT^2 = -\frac{i}{2}\varphi$$

$$\varphi \equiv kgT^2$$

$$H_g + H_e = \left[\frac{1}{2m}\left(p+\frac{1}{2}\hbar k\right)^2 + mgz + \frac{(\hbar k)^2}{8m}\right]2 \equiv \bar{H}\cdot 2$$

$$= H_e + H_g \qquad \bar{H} \equiv \frac{1}{2m}\left(p+\frac{1}{2}\hbar k\right)^2 + mgz + \frac{(\hbar k)^2}{8m}$$

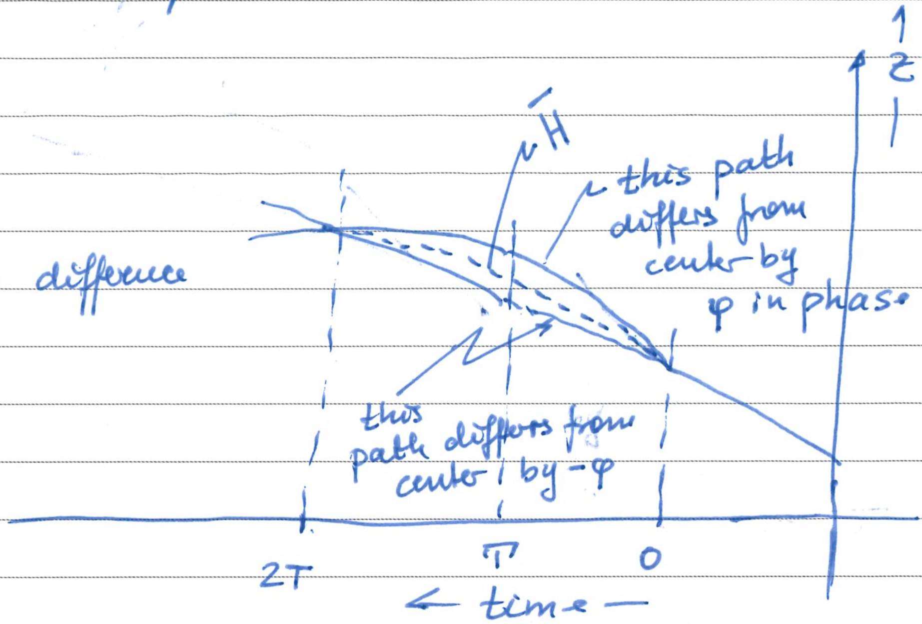
$$|\psi_g^{(u)}\rangle = \mathcal{N} e$$

$$|\psi_g^{(u)}\rangle = \mathcal{N} e^{+i\varphi/2} e^{-i\bar{H}(2T)/\hbar} |\psi(0)\rangle$$

$$|\psi_g^{(l)}\rangle = \mathcal{N} e^{-i\varphi/2} e^{-i\bar{H}(2T)/\hbar} |\psi(0)\rangle$$

geometric interpretation :

total path difference is 2φ



key result : non-commutivity of the time evolutions of the two paths is the observable !

question : how does this feature manifests itself in Lagrangian formulation ?

return to the roots, that is de Broglie :

matter wave $\sim e^{imc^2\tau/\hbar}$ ← proper time

line element → $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \equiv c^2 d\tau^2$ (metric: $g_{\mu\nu}$)

$x \equiv (x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) \equiv (ct, x, y, z)$ $\begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

weak field - small velocity limit $\qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ← small correction

$$d\tau = \frac{1}{c}\sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\; dt$$

(dt: coordinate time)

only Newton correction in g_{00}

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$h_{ii} = 0 \qquad i = 1,2,3$

$$\tau = \frac{1}{c}\int dt \sqrt{\underset{1+\frac{2}{c^2}\Phi}{g_{00}}\, c^2 - \vec{v}^{\,2}}$$

Φ ← Newton potential

$$= \int dt \sqrt{1 + \frac{2}{c^2}\Phi - (\vec{v}/c)^2} \cong$$

$$= \int dt \left[1 + \frac{1}{c^2}\left(\Phi - \frac{1}{2}\vec{v}^{\,2}\right)\right]$$

$$mc^2\tau = mc^2 t - \int dt\, L$$

$$\underline{L \equiv \frac{1}{2}m\vec{v}^{\,2} - m\Phi}$$

for Kasevich-Chu interferometer, there is a need to include effect of laser pulses.

connection to Hamiltonian formulation via Legendre transformation

$$L = \underset{\substack{\uparrow \\ \text{canonical momentum}}}{p} \cdot \overset{\substack{\text{velocity} \\ \downarrow}}{v} - H \qquad v \equiv \frac{dz}{dt}$$

but how is it done in quantum mechanics:

<u>Feynman path integral:</u>

infinitesimal propagator $\Delta t \to 0$

$$\equiv \langle z_f |\, e^{-i\hat{H}\Delta t/\hbar}\, | z_i \rangle \equiv G(z_f, \Delta t | z_i, 0)$$

$$\hat{H} \equiv \frac{1}{2m}(\hat{p} + \hbar k)^2 + V(\hat{z})$$

in the limit Δt we separate two exponentials

$$G = \langle z_f |\, e^{-i\frac{1}{2m}(p+\hbar k)^2\Delta t/\hbar}\, | z_i \rangle\, e^{-iV(z_i)\Delta t/\hbar}$$

insert momentum states

$$= \int_{-\infty}^{\infty} dp\, \langle z_f |\, e^{-i\frac{1}{2m}(p+\hbar k)^2\Delta t/\hbar}\, |p\rangle\langle p | z_i \rangle\, e^{-iV(z_i)\Delta t/\hbar}$$

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Legendre transformation for Hamiltonian

$$H = \frac{1}{2m}\left(p + \hbar k\right)^2 + mgz$$

$$v = \dot{z} = \frac{\partial H}{\partial p} = \frac{1}{m}\left(p + \hbar k\right) \Rightarrow \underline{p = mv - \hbar k}$$

$$L = p \cdot v - H =$$

$$= \left(m \cdot v - \hbar k\right) \cdot v - \frac{1}{2m}\left(m v\right)^2 - mgz =$$

$$\underline{L = \frac{1}{2} m v^2 - mgz - \hbar k \cdot v}$$

$$= \int_{-\infty}^{\infty} dp \underbrace{\langle z_f | p \rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{ip z_f/\hbar}} e^{-i \frac{1}{2m}(p+\hbar k)^2 \Delta t/\hbar} \underbrace{\langle p | z_i \rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{-ipz_i/\hbar}} e^{-i V(z_i)\Delta t/\hbar}$$

$$G = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \, \exp\left\{\frac{i}{\hbar}\Delta t \left[\overbrace{p \cdot \frac{z_f - z_i}{\Delta t}}^{p \cdot v} - \overbrace{\left(\frac{1}{2m}(p+\hbar k)^2 + V(z_i)\right)}^{= H}\right]\right\}$$

integral quadratic in p $\equiv$ method of stationary phase is exact

$$\mathcal{L} \equiv p \cdot \frac{z_f - z_i}{\Delta t} - \left[\frac{1}{2m}(p+\hbar k)^2 + V(z_i)\right]$$

$$\frac{\partial \mathcal{L}}{\partial p} = \frac{z_f - z_i}{\Delta t} - \frac{1}{m}(p+\hbar k) \equiv 0$$

this condition determines point of stationary phase in p, that is p_s

$$p_s \equiv m \frac{z_f - z_i}{\Delta t} - \hbar k \, ;$$

analogous to classical physics

$$\frac{\partial^2 \mathcal{L}}{\partial p^2} = -\frac{1}{m} \, ;$$

$$G = \exp\left\{\frac{i}{\hbar}\Delta t \, \mathcal{L}\Big|_{p=p_s}\right\} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \, \exp\left\{-i \frac{\Delta t}{2m\hbar}(p-p_s)^2\right\}$$

$$= \exp\left\{\frac{i}{\hbar}\Delta t \, L\right\} \cdot \frac{1}{2\pi\hbar} \sqrt{\frac{2\pi m\hbar}{i\,\Delta t}}$$

$$G = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \, e^{\frac{i}{\hbar} L \Delta t} \, ;$$

Fresnel integral: $\displaystyle\int_{-\infty}^{\infty} dp \, e^{-i\alpha p^2} = \sqrt{\frac{\pi}{i\alpha}}$