Ch. 4 Time-domain Control

Recall: state vector $x(t)$ $\qquad \dot{x} = f(x,u) \quad y = h(x,u)$

$\dot{x} = Ax + Bu, \quad y = Cx + Du \qquad A = \partial_x f|_{x_0,u_0}, \quad B = \cdots$

Why study the time domain?

1. easily handle multiple inputs + multiple outputs ("MIMO")
2. handle noise in a natural way
3. leads to more sophisticated view of dynamics, control
4. generalize to time varying and nonlinear cases

To solve $\dot{x} = Ax + Bu$, let $z = e^{-At} x$

Matrix exponential: $e^{At} = \mathbb{I} + tA + \frac{1}{2!} t^2 A^2 + \cdots$

$$A^n = (RDR^{-1})^n = RD^nR^{-1} \quad = \quad R \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} R^{-1}$$

$$\dot{z} = \frac{d}{dt}\left(e^{-At}x\right) = -Ae^{-At}x + e^{-At}\overbrace{(Ax+Bu)}^{\dot{x}} = e^{-At}Bu(t)$$

$$\Rightarrow \quad z(t) = z(0) + \int_0^{\infty} dt'\, e^{-At'} Bu(t')$$

$$x(t) = e^{At}x(0) + \int_0^t dt'\, e^{A(t-t')} Bu(t')$$

Ex: $\ddot{y} + y = u \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}}_{R^{-1}}$

$$e^{At} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

— rotation matrix

Converting between time and frequency domains

state space $\longrightarrow$ transfer function

$\dot{x} = Ax + Bu \qquad y = Cx + Du$

$sX = AX + Bu \qquad y(s) = CX(s) + Du(s)$

$\hookrightarrow \quad (s\mathbb{I} - A)X = Bu \quad \Rightarrow \quad X = (s\mathbb{I} - A)^{-1} Bu$

$$\Rightarrow \quad G(s) = C(s\mathbb{I} - A)^{-1} B + D$$

$\rightarrow$ **Poles of transfer func. G given by eigs of A.**

[Recall: $Av = \lambda v \implies (\lambda \mathbb{I} - A)v = 0 \implies \det(\lambda \mathbb{I} - A) = 0$

the determinant gives the n^{th}-order characteristic poly. of A, whose n roots $\lambda_1 \dots \lambda_n$ are the n eigs of A.]

MIMO case: $\qquad G =$ $p \times m$ matrix of transfer funcs.
G_{ij} = transf. function between input j and output i

Ex: Oscillator, observe position only.

$$G(s) = \underset{C}{(1\ 0)} \left[s \underset{\mathbb{I}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} - \underset{A}{\begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix}} \right]^{-1} \underset{B}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \frac{1}{1 + 2\zeta s + s^2}$$

Observe pos. and velocity: $\quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad$ not $(1\ 1)$!!

$$\rightarrow \quad G(s) = \frac{1}{1 + 2\zeta s + s^2} \begin{pmatrix} 1 \\ s \end{pmatrix}$$ $\qquad$ 2×1 matrix of tf's.

Changing coords

$\dot{x} = Ax + Bu \qquad y = Cx$

let $\quad x' = Tx \quad$ or $\quad x = T^{-1}x'$

$\Rightarrow \quad T^{-1}\dot{x}' = A(T^{-1}x') + Bu, \qquad y = CT^{-1}x'$

$$\dot{x}' = (TAT^{-1})x' + TBu$$

$\Rightarrow \quad A' = TAT^{-1}, \quad B' = TB, \quad C' = CT^{-1}$

so that $\quad \dot{x}' = A'x' + B'u \qquad y = C'x'$

- $G(s)$ invariant since A', A have same eigs $\quad A' = TR\Lambda R^{-1}T^{-1} = \tilde{R}\Lambda\tilde{R}^{-1}$
- state x is "abstract"; many equiv representations

Time-domain response

- **Impulse** response $\quad u = \delta(t)$

$y(s) = G(s)\,u(s) \quad \Rightarrow \quad y = G * u = G * \delta = G(t)$

$\Rightarrow \quad G(t) =$ impulse response function [Green function in physics]

states: $\quad x_{imp}(t) = 0 + \int_0^t dt'\, e^{A(t-t')} B\,\delta(t') = e^{At}B$

- **Step** response $\quad u(t) = \theta(t)$

Controllability and observability

Two "technical" questions:

1) Given input(s) $u(t)$, can you "control" all elements of x?
   $\rightarrow$ controllability

2) Given observation(s) $y(t)$, can you "infer" all elements of x?
   $\rightarrow$ Observability

I. Controllability

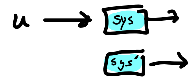
- begin w/ SISO case (scalar u, y; $x \in \mathbb{R}^n$)

Controllable solution: $\exists\, u(t)$ makes sys evolve from $x(0) = x_0 \rightarrow x(\tau) = x_\tau$ in finite time τ

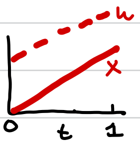
Controllable set: set of all x_τ that can be reached from x_0 in time τ

Controllable system: For any x_0, can reach any $x_\tau \in \mathbb{R}^n$ for some τ and $u(t)$ $\quad \exists u: x_0 \rightarrow x_\tau$

note: sometimes this property is called "reachability" and "controllability" is used when you start at x_0 and end at $x_\tau = 0$. [Diff. is that $e^{At} x_0$ makes you fall into $x = 0$ if sys. is stable.]



Ex 1: $\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} = \begin{pmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix} u$ or: $\dot x_1 = -\lambda_1 x_1 + u$, $\dot x_2 = -\lambda_2 x_2$



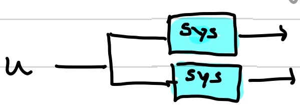
$X_2(t)$ is completely unaffected by $u(t)$ → not controllable

- If we only had the x_1 equation ($\dot x = -\lambda x + u$) then it is controllable

$x_d = t$ $(0 \to 1)$
$u = 1 + t$
$(\lambda = 1)$

- actually even stronger $x = x_d(t) \Rightarrow u(t) = \dot x_d + \lambda x_d(t)$
  can specify whole trajectory! "invert" the dynamics
- does not always work;

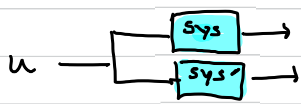
Ex 2:



same input to two identical sys. ⇒ not cont.

Ex 3: $\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} = \begin{pmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}1\\ 1\end{pmatrix} u(t)$ → Q?

yes! See Prob 4.1



Test for Controllability

"the recipe"

1) Define controllability matrix $W_c \equiv (B \;\; AB \;\; A^2B \; \cdots \; A^{n-1}B)$
   SISO → B a column matrix ⇒ W_c is $n \times n$

2) W_c invertible ($\det W_c \neq 0$) ⇒ $\{A, B\}$ controllable

Ex 1: $A = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $AB = \begin{pmatrix} -\lambda_1 \\ 0 \end{pmatrix}$ $W_c = \begin{pmatrix} 1 & -\lambda_1 \\ 0 & 0 \end{pmatrix}$

$\det W_c = 0 \Rightarrow$ not controllable

Ex 2: $A = \begin{pmatrix} -\lambda_1 & \\ & -\lambda_2 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $AB = \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \end{pmatrix}$ $W_c = \begin{pmatrix} 1 & -\lambda_1 \\ 1 & -\lambda_2 \end{pmatrix}$

$\det W_c = \lambda_1 - \lambda_2 \Rightarrow$ controllable if $\lambda_1 \neq \lambda_2$

Why it works

1) Can set $x_0 = 0$ (by defining $x_\tau \to x_\tau - e^{A\tau} x_0$)

2) **Cayley-Hamilton**: matrix A obeys its n^{th}-order characteristic equation

$\Rightarrow$ A^l can be re-expressed as $O(n-1)$ polynomial

$\Rightarrow$ $e^{At} = \sum_{j=0}^{n-1} \alpha_j(t) A^j$

3) **impulse response** $x_{imp} = e^{At} B = \sum_{j=0}^{n-1} \alpha_j(t) A^j B$

controllability $\Rightarrow$ impulse response spans $\mathbb{R}^n$

$\Rightarrow$ set of n vectors $B, AB, A^2B, \dots A^{n-1}B$ are all linearly independent $\Rightarrow$ $\det W_c \neq 0$

Notes

1) MIMO: B a matrix W_c needs to have rank $= n$ "full rank"

2) Controllability $\not\Rightarrow$ prescribed trajectory
   - sometimes can invert: $u = \dot{x} + \lambda x$ sometimes no (see problem)

3) Can get to x_τ at τ but may not be able to hold x_τ

4) Might be good enough to control only some states or to control in a region only

5) Control trajectories can be nonlocal 

6) linearity lost when input saturates need to reconsider...
   - might not "reach" desired x in time τ $\Rightarrow$ "reachable set"

7) For nonlinear dynamics, more complicated and no general "formula" exists
   - but linearization often works...

8) Even for linear systems, hard problem numerically for $N \gg 1$ cond(W_c) ...

9) Inverse problem: Given A, find B so $\{A, B\}$ controll.
   - need to try 2^N combos for $x \in \mathbb{R}^N$
   - Barabasi et al. $A \rightarrow$ network
     $\rightarrow$ graph-theory techniques $\rightarrow$ poly. time ...

II. Observability

controllability : input u "coupled" to entire state vec. x
observability : output y " "

- to control need to set u(t) over interval
- to observe, need to measure y(t) over interval

eg (B=0):

$$y = Cx$$
$$\dot{y} = C\dot{x} = CAx$$
$$\ddot{y} = C\ddot{x} = CA^2 x$$
$$\vdots$$
$$y^{(n-1)} = CA^{n-1} x$$

$$\Rightarrow \begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} X \equiv W_0 X$$

$Y = W_0 X$ If W_0 is invertible, $X = W_0^{-1} Y$

- To know $Y(t)$ can estimate $\frac{dy}{dt} \approx \frac{y(t) - y(t-\Delta t)}{\Delta t}$
  - lousy in practice (noisy!) but OK in principle
- Adding u(t) does not affect observability (just alters x(t))

Ex 1 : $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $y = (1\ 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ → unobservable

- observing y tells you nothing about x_2
- just x_1 eq. would be OK: $\dot{x}_1 = x_1$ $y = x_1$ is observable
  - but unstable

Ex. 2 $\ddot{x}+x=0$, $y=x$ $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$ $y=(1\ 0)\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$CA = (1\ 0)\begin{pmatrix}0&1\\-1&0\end{pmatrix} = (0\ 1)$ $W_0 = \begin{pmatrix}C\\CA\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ ✓

ie observe position over time; can infer velocity..

What about reverse? $C=(0\ 1)$
→ Measure velocity, not position

$CA = (-1\ 0)$ $W_0 = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$ ✓

Why is it observable? Recall, for harmonic osc.

$$\ddot{x}+x=0 \quad\Rightarrow\quad \begin{aligned} x(t) &= x_0\cos t + V_0 \sin t\\ v(t) &= V_0\cos t - x_0 \sin t\end{aligned}$$

So we are asking, if we measure just the velocity over an interval, do we know the entire state? Yes!

For example $V(0)=V_0$ $V(\pi/2)=-x_0$
So measuring the velocity at two times implies we know x_0, V_0 But then we know $\begin{pmatrix}x(t)\\v(t)\end{pmatrix}$

Ex 2a: $\ddot{x} = 0$ $\quad y = x$ or $\dot{x}$ $\qquad$ We have $x(t) = x_0 + v_0 t$

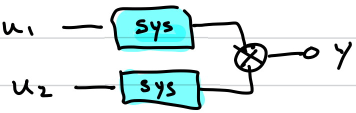
So if we measure $x(t)$, we know x_0, v_0

But if we measure $v(t) = v_0$, we do not know x_0!

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad c = (1\ 0)$ $\Rightarrow W_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$\quad c = (0\ 1)$ $\quad W_o = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ✗

Ex 3.



not observable

e.g. $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad y = (1\ 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad W_o = \begin{pmatrix} 1 & 1 \\ -\lambda & -\lambda \end{pmatrix}$ ✗

→ **Duality**

$W_c = (B \ AB \ A^2B \cdots A^{n-1}B)$ is same as W_o if $A \to A^T$, $B \to C^T$

Notice: $\dot{x} = -A^T x + C^T u$ $\qquad y = B^T x$

has $W_c = (C^T \ A^T C^T \ (A^T)^2 C^T \cdots) = W_o^T$

$u(t)$: affects state $x(t)$ at future times

$y(t)$: determines $x(t)$ based on past obs.

"We can know the past but not control it;
we can control the future but do not know it."

~ Shannon, 1959

Part 2: Observers, controller design

4.2 Control based on the state

Divide control into two problems
1) assume we know $x(t)$; find $u[x]$
2) estimate $\hat{x}(t)$ from observations $\rightarrow u[\hat{x}]$

Claim: If $\{A, B\}$ is controllable, then the closed-loop dyn. based on $u = -Kx$ $\left[\dot{x} = (A - BK)x \equiv A' x\right]$ will have n eigenvalues (poles) that can be chosen freely using the n gains in row vector K [SISO case!]

Sketch of proof:

"Controller canonical form"

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \ldots, \quad \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots a_1 x_1 + u$$

Let $u = -Kx = -(k_1 \; k_2 \; \cdots \; k_n)$ $\qquad BK = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ k_1 & \cdots & k_n \end{pmatrix}$

$$A' = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ & & 1 & \\ & & & \ddots \\ -a_n - k_1 & -a_{n-1} - k_2 & & 1 \end{pmatrix} \qquad a_i \rightarrow a_i - k_{n-i+1}$$

Thus we see that the dynamics of A' are set K
$\hookrightarrow$ arb. eigs!

Ex: Harmonic osc. $\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} = \begin{pmatrix}0&1\\-1&-2\zeta\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}u$

$$u = -(k_1\ \ k_2)\begin{pmatrix}x_1\\x_2\end{pmatrix} = -k_1x_1 - k_2x_2$$

$$BK = \begin{pmatrix}0\\1\end{pmatrix}(k_1\ \ k_2) = \begin{pmatrix}0&0\\k_1&k_2\end{pmatrix} \qquad A' = A - BK = \begin{pmatrix}0&1\\-1-k_1&-2\zeta-k_2\end{pmatrix}$$

eigs: $|sI - A'| = \begin{vmatrix} s & -1\\ 1+k_1 & s+2\zeta+k_2\end{vmatrix} = s^2+(2\zeta+k_2)s+(1+k_1)=0$

choose k_1, k_2 to put eigs anywhere

eg closed-loop poles at $s=(-2,-2)$

$(s+2)^2 = s^2+4s+4 \Rightarrow k_2 = 4-2\zeta \quad k_1=3$

or $s=(-a,-a) \Rightarrow k_1=a^2-1 \quad k_2 = 2(a-\zeta)$

Notice how gains increase with a (how far you move the poles)

- In general: use control software to find gain K given $\{A, B,$ desired pole positions$\}$

- N poles is a lot of choice! ☹

1) high gains inject noise (and u may be too big)

2) if sys is unstable, you must move RHP poles → LHP

So: Move as few poles as little as possible to stabilize, get ω_c high enough

(order by ζ...)

4.3 Output control

- if $\{A, C\}$ is observable, then $y(t \le t_0) \to \hat{x}(t_0)$
- naive differentiation too noisy

Observer $y \to \hat{x} \to u(\hat{x})$ "synchronize"

Naive obs.

$$\dot{x} = Ax + Bu \qquad e \equiv x - \hat{x} \text{ (estimation err.)}$$

$$\dot{\hat{x}} = A\hat{x} + Bu \qquad \text{assume we know } A, B!$$

subtract $\Rightarrow$ $\dot{e} = Ae$ inputs cancel!

No good ☹ i) if A is unstable, $e \to \infty$

ii) need $e \to 0$ faster than A dynamics

Observer (for real!) use feedback on deviations between observations y and predictions $\hat{y} = C\hat{x}$

$$\dot{x} = Ax + Bu \qquad y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \quad \Rightarrow \quad \dot{e} = (A - LC)e$$

L = observer gains $\longleftrightarrow$ $A' = A - LC$ arb. poles.

Ex: Harmonic osc. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $C = (1 \ 0)$

$L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$ $LC = \begin{pmatrix} l_1 & 0 \\ l_2 & 0 \end{pmatrix}$ $A' = A - LC = \begin{pmatrix} -l_1 & 1 \\ -1-l_2 & 0 \end{pmatrix}$

eigs $\rightarrow$ $s^2 + l_1 s + (l_2 + 1) = 0$

eg $s = (-2, -2) \leftrightarrow L = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$\hookrightarrow$ should be faster here, for visualization

Observer - controller $u = -K\hat{x} + k_r r$

$$\dot{x} = Ax + B(-K\hat{x} + k_r r), \qquad y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x} + k_r r) + L(y - C\hat{x})$$

$$\dot{e} = Ae - LCe = (A - LC)e$$

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \underbrace{\begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}}_{M} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} B k_r \\ 0 \end{pmatrix} r$$

$\hat{x} = x - e$

M is block triangular $\Rightarrow$ eigs given by

$$\det(s\mathbb{I} - A + BK)\,\det(s\mathbb{I} - A + LC) = 0$$

$\longrightarrow$ **Separation principle** (design K, L sep.) - only for lin sys.

Two ways to use this:

1) with hardware: y = input to observer $\hat{x}$ = input to control.

2) simulate both physical sys. + observer

$$\frac{d}{dt}\begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A-BK-LC \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} Bk_r \\ Bk_r \end{pmatrix} r + \begin{pmatrix} B \\ 0 \end{pmatrix} d$$

input disturb.

actually, should write $\begin{pmatrix} Bk_r & B \\ Bk_r & 0 \end{pmatrix}\begin{pmatrix} r \\ d \end{pmatrix}$

Ex: Harmonic osc. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $C = (1\ 0)$

$K = (k_1\ k_2)$ $L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$

$$A - LC - BK = \begin{pmatrix} -l_1 & 1 \\ -(1+k_1+l_2) & -k_2 \end{pmatrix}$$

- undamped osc. has time scale 1
- closed-loop control a bit faster (+damping)
- observer 5-10X faster than closed-loop time scale

Disturbance cancellation

remove effect of $d(t)$

– assume $d(t)$ generated from known dynamics

$$\dot{x} = Ax + B(u+d) \qquad y = Cx$$
$$\dot{x}_d = A_d x_d \qquad d = C_d x_d$$

$$\frac{d}{dt}\begin{pmatrix} x \\ x_d \end{pmatrix} = \begin{pmatrix} A & BC_d \\ 0 & A_d \end{pmatrix}\begin{pmatrix} x \\ x_d \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \qquad y = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x \\ x_d \end{pmatrix}$$

Note: $W_o = \begin{pmatrix} \text{"}C\text{"} \\ \text{"}CA\text{"} \end{pmatrix} = \begin{pmatrix} C & 0 \\ CA & CBC_d \end{pmatrix}$ ✓

$W_c = \begin{pmatrix} \text{"}B\text{"} & \text{"}AB\text{"} \end{pmatrix} = \begin{pmatrix} B & AB \\ 0 & 0 \end{pmatrix}$ ✗

makes sense...
can observe $d(t)$ but not control it!

Have two observers, $\hat{x}$, $\hat{x}_d$

$$u = -K\hat{x} - \hat{d} = -\begin{pmatrix} K & C_d \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix}$$

Observer eqs:

$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x} - \hat{d} + \hat{d}) + L(y - \hat{y}) \qquad \hat{y} = C\hat{x}$$
$$\dot{\hat{x}}_d = A_d \hat{x}_d + L_d(y - \hat{y})$$

All together:

$$\dot{x} = Ax - BK\hat{x} - BC_d\hat{x}_d + BC_d x_d$$
$$\dot{x}_d = A_d x_d$$
$$\dot{\hat{x}} = A\hat{x} - BK\hat{x} + L(Cx - C\hat{x})$$
$$\dot{\hat{x}}_d = A_d\hat{x}_d + L_d(Cx - C\hat{x})$$

$$\frac{d}{dt}\begin{pmatrix} x \\ x_d \\ e_x \\ e_d \end{pmatrix} = \underbrace{\left(\begin{array}{cc|cc} A-BK & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & A_d & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & A-LC & BC_d \\ & & -L_dC & A_d \end{array}\right)}_{M}\begin{pmatrix} x \\ x_d \\ e_x \\ e_d \end{pmatrix}$$

still have sep. principle!

notice: eigs of M → factor of $(sI - A_d)$

→ internal model principle

To track a reference or eliminate a disturb., controller needs a model of dynamics that generates the ref. / disturb.